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Differential formulas for simploidal Bernstein polynomials

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Abstract

In a previous report, we developed formulas for basic operations on simploidal Bernstein polynomials. Here we extend that work with formulas for partial and directional derivatives of such polynomials.

Symbol	Description	Section
t, x, y, z	*real numbers*	
F	*real valued functions*	
i, j, k, p, q, r, s	*natural numbers*	
$\kappa, \lambda, \mu, \nu, \rho, \sigma$	*multi-indices*	[2] 2.1
$\mathbb{I}, \mathbb{I}_d, \mathbb{I}_d^m$	*sets of multi-indices*	[2] 2.1
g, h	*degrees*	[2] 2.1
α, β, γ	*multi-degrees*	[2] 2.1
$\mathbb{M}_m, \mathbb{M}_{m,\delta}$	*sets of irregular matrices*	[2] 2.1
Λ, Ω	*hyper-indices*	[2] 2.1
$\mathbb{H}, \mathbb{H}_d, \mathbb{H}_d^m, \mathbb{H}_d^{\alpha,\kappa}$	*sets of hyper-indices*	[2] 2.1
$\Lambda!, \kappa!$	*(multi-),hyper-factorial*	[2] 2.1
d, e	*dimensions*	[2] 2.2
δ, ε	*multi-dimensions*	[2] 2.2
$\mathbb{A}^n, \mathbb{A}^\delta$	*(multi-)affine spaces*	[2] 2.2
u, v, w	*points of affine spaces*	[2] 2.2
U, V	*points of multi-affine spaces*	[2] 2.2
ξ	*vector of an affine vector space*	2.2
Ξ	*vector of a multi-affine vector space*	2.3
B	*univariate and tensorial Bernstein polynomials*	[2] 2.5
$\mathcal{B}$	*simplicial and simploidal Bernstein polynomials*	[2] 2.5

Table 1: Summary of notation

1 Introduction

In a previous report [2], we developed formulas for basic operations on simploidal Bernstein polynomials. Here we extend that work with formulas for partial and directional derivatives of such polynomials.

2 Notation and Basic concepts

2.1 Functions

If $\mathbb{X}$ is a set and $\mathcal{E}$ is an expression that involves some free variable x, we write

$$(x : \mathbb{X} \to \mathcal{E})$$

for the function with domain $\mathbb{X}$ that maps each x in $\mathbb{X}$ to the corresponding value of $\mathcal{E}$.

2.2 Canonical affine vector space

We define $\mathbb{V}^d$, the *canonical affine vector space of dimension* $d \in \mathbb{N}$, as the set

$$\mathbb{V}^d = \{\xi \in \mathbb{R}^{d+1} \mid \sum_{i=0}^{d} u_i = 0\}. \tag{2.1}$$

together with the *vector sum* and *scalar multiplication* operations as defined in $\mathbb{R}^{d+1}$.

Note that $\mathbb{V}^d$ is the natural tangent space of the *canonical affine space* $\mathbb{A}^d$ defined in the previous report [2]. Therefore, the difference $u - v$ for any $u, v \in \mathbb{A}^d$ is a vector $\xi \in \mathbb{V}^d$, and, conversely, the sum of a vector $\xi \in \mathbb{V}^d$ with a point $u \in \mathbb{A}^d$ is a point $u + \xi = \xi + u$ of $\mathbb{A}^d$. For more details on tangent spaces, the reader should refer to [1]

2.3 Canonical multi-affine vector space

For any multi-index $\delta \in \mathbb{I}_m$, we define $\mathbb{V}^\delta$, the *multi-affine vector space of dimension* δ, as the Cartesian product

$$\mathbb{V}^{\delta_0} \times \cdots \times \mathbb{V}^{\delta_m} \tag{2.2}$$

Each element of $\mathbb{V}^\delta$ is an elementof $\mathbb{M}_{m,\delta}$, that is an irregular matrix with $m+1$ rows, where the row i has δ_i elements for each i. Indeed, $\mathbb{V}^\delta$ is a linar subspace of $\mathbb{M}_{m,\delta}$, consisting of those matrices for which the sum of each row is zero.

As in section 2.2, $\mathbb{V}^\delta$ is the natural tangent space of the *canonical multi-affine space* $\mathbb{A}^\delta$ [2, sec. 2.2]. Therefore, the difference $U - V$ for any $U, V \in \mathbb{A}^\delta$ is a vector $\Xi \in \mathbb{V}^\delta$, and, conversely, the sum of a vector $\Xi \in \mathbb{V}^\delta$ with a point $U \in \mathbb{A}^\delta$ is a point $U + \Xi 2 = \Xi + U$ of $\mathbb{A}^\delta$.

2.4 Derivatives

If F is a function from $\mathbb{R}$ to $\mathbb{R}$ (or to any vector space), we will denote by $\partial^r F$ the r-th derivative of F.

For any $d \in \mathbb{N}$, the derivative ∂F of a function F from $\mathbb{R}$ to $\mathbb{A}^d$ is a function from $\mathbb{R}$ to $\mathbb{V}^d$. Similarly, for any $\delta \in \mathbb{I}_d$ the derivative ∂F of a function F from $\mathbb{R}$ to $\mathbb{A}^\delta$ is a function from $\mathbb{R}$ to $\mathbb{V}^\delta$

2.5 Partial derivatives

If F is a function from $\mathbb{R}^d$ to $\mathbb{R}$ (or to any vector space), we will denote by $\partial_i^r F$ its r-th partial derivative with respect to coordinate i of the argument. Moreover, for any multi-index $\kappa \in \mathbb{I}_{d-1}$ [2] we define the *mixed derivative of F of order κ* as

$$\partial^\kappa F = \partial_0^{\kappa_0} \partial_1^{\kappa_1} \cdots \partial_{d-1}^{\kappa_{d-1}} F$$

2.6 Gradient

The *gradient* of a function F from $\mathbb{A}^d$ tp $\mathbb{R}$ is the function ∂F from $\mathbb{A}^d$ to $\mathbb{V}^d$ defined by

$$(\partial F)_i = \partial_i F$$

for all i. Similarly, the gradient of a function F from $\mathbb{A}^\delta$ to $\mathbb{R}$ is the function ∂F from $\mathbb{A}^\delta$ to $\mathbb{V}^\delta$ defined by

$$(\partial F)_{i,j} = \partial_{ij} F.$$

2.7 Leibnitz formula

In this report, we will need Leibnitz's formula for the derivative of the product of functions. This formula can be written in a compact form with the use of multi-indices. Let $F_0(t), \cdots, F_d(t)$ be $d+1$ functions from $\mathbb{R}$ to $\mathbb{R}$. Then

$$\partial^r \left(\prod_{i=0}^{d} F_i \right) = \sum_{\kappa \in \mathbb{I}_d^r} \frac{r!}{\kappa!} \prod_{i=0}^{d} (\partial^{\kappa_i} F_i) \tag{2.3}$$

3 Derivatives of Bernstein polynomials

3.1 Derivatives of univariate polynomials

For any $g, i \in \mathbb{N}$, with $i \leq g$, let B_i^g denote the univariate Bernstein polynomial of degree g and index i [2, sec. 2.5]. The r-th derivative of B_i^g is

$$\partial^r B_i^g(z) = \partial^r \left(\frac{g!}{i!(g-i)!} z^i (1-z)^{g-i} \right) \tag{3.1}$$

Using the Leibnitz formula (2.3), we get

$$\partial^r B_i^g(z) = \frac{g!}{i!(g-i)!} \sum_{j=0}^{r} \binom{r}{j} \left(\partial^j z^i\right) \left(\partial^{r-j}(1-z)^{g-i}\right)$$

Note that the inner term of the summation is zero whenever $j > i$ or $r - j > g - i$, since the j-th derivative of a degree i polynomial is zero. Therefore

$$\begin{aligned}
\partial^r B_i^g(z) &= \frac{g!}{i!(g-i)!} \sum_{j=r-g+i}^{i} \binom{r}{j} \frac{i!}{(i-j)!} z^{i-j} \frac{(g-i)!}{(g-i-r+j)!} (1-z)^{g-i-r+j} (-1)^{r-j} \\
&= \frac{g!}{(g-r)!} \sum_{j=r-g+i}^{i} \binom{r}{j} (-1)^{r-j} \binom{g-r}{i-j} z^{i-j} (1-z)^{g-i-r+j} \\
&= \sum_{j=r-g+i}^{i} \frac{g!}{(g-r)!} \binom{r}{j} (-1)^{r-j} B_{i-j}^{g-r}(z) \qquad (3.2)
\end{aligned}$$

3.2 Partial derivatives of tensorial polynomials

For any $d \in \mathbb{N}$, and any $\alpha, \kappa \in \mathbb{I}_{d-1}$, with $\kappa \leq \alpha$, let B_κ^α denote the d-dimensional tensorial Bernstein polynomial of multi-degree α and multi-index κ [2, sec. 2.5] defined on $\mathbb{R}^d$. The r-th partial derivative of B_κ^α with respect to the i-th coordinate of the argument is

$$\begin{aligned}
\partial_i^r B_\kappa^\alpha(x) &= \partial_i^r \left(\prod_{j=0}^{d-1} B_{\kappa_j}^{\alpha_j}(x_j) \right) \\
&= \partial_i^r \left(B_{\kappa_i}^{\alpha_i}(x_i) \prod_{\substack{j=0 \\ j \neq i}}^{d-1} B_{\kappa_j}^{\alpha_j}(x_j) \right) \\
&= \left(\sum_{p=r-\alpha_i+\kappa_i}^{\kappa_i} \frac{\alpha_i!}{(\alpha_i - r)!} \binom{r}{p} (-1)^{r-p} B_{\kappa_i - p}^{\alpha_i - r}(x_i) \right) \prod_{\substack{j=0 \\ j \neq i}}^{d} B_{\kappa_j}^{\alpha_j}(x_j)
\end{aligned}$$

Therefore, for any $\lambda \in \mathbb{I}_d$ the mixed derivative of order λ is

$$\partial^\lambda B_\kappa^\alpha(x) = \sum_{\substack{\mu \in \mathbb{I}_d \\ \mu \geq \lambda - \alpha + \kappa \\ \mu \leq \kappa}} \frac{\alpha!}{(\alpha - \lambda)!} \binom{\lambda}{\mu} (-1)^{r-|\mu|} B_{\kappa-\mu}^{\alpha-\lambda}(x) \qquad (3.3)$$

4 Directional derivatives

If F is a function from a given domain $\mathbb{D}$ to $\mathbb{R}$ (or to any vector space), and ξ is a vector in the tangent space of $\mathbb{D}$, we denote by $D_\xi^r F$ *the* r*-th directional derivative of* F *in the* *direction* ξ is the function from $\mathbb{D}$ to $\mathbb{R}$ defined by

$$D_\xi^r F(u) = (\partial^r (t : \mathbb{R} \to F(u + t\xi))(0)).$$

for any $u \in \mathbb{D}$. In the particular case that $\mathbb{D} = \mathbb{A}^\delta$ and $r = 1$, the directional derivative can be obtained from the gradient by the formula

$$\begin{aligned} D_\xi f &= (u : \mathbb{A}^\delta \to (\partial(t : \mathbb{R} \to f(u + t\xi))(0))) \\ &= (u : \mathbb{A}^\delta \to \sum_{i,j} (\partial_{ij} f)(u)\xi_{ij}) \\ &= \sum_{i,j} \xi_{ij} (\partial f)_{ij} \end{aligned} \tag{4.1}$$

4.1 Directional derivatives of tensorial polynomials

The r-th derivative of the tensorial Bernstein polynomial B_κ^α in the direction $z \in \mathbb{R}^d$, is

$$D_z^r B_\kappa^\alpha(x) = \partial^r \left(t : \mathbb{R} \to \prod_{i=0}^{d-1} B_{\kappa_i}^{\alpha_i}(x_i + tz_i) \right)(0) \tag{4.2}$$

for any $x \in \mathbb{R}^d$. Using the Leibnitz formula (2.3), we get

$$\begin{aligned} D_z^r B_\kappa^\alpha(x) &= \sum_{\lambda \in \mathbb{I}_{d-1}^r} \frac{r!}{\lambda!} \prod_{i=0}^{d-1} \partial^{\lambda_i} \left(t : \mathbb{R} \to B_{\kappa_i}^{\alpha_i}(x_i + tz_i) \right)(0) \\ &= \sum_{\lambda \in \mathbb{I}_{d-1}^r} \frac{r!}{\lambda!} \prod_{i=0}^{d-1} z_i^{\lambda_i} (\partial^{\lambda_i} B_{\kappa_i}^{\alpha_i})(x_i) \\ &= \sum_{\lambda \in \mathbb{I}_{d-1}^r} \frac{r!}{\lambda!} z^\lambda (\partial^\lambda B_\kappa^\alpha)(x) \end{aligned} \tag{4.3}$$

4.2 Directional derivatives of simplicial polynomials

For any $\xi \in \mathbb{V}^d$, the r-th derivative of the simplicial Bernstein polynomial $\mathcal{B}_\kappa^g$ in the direction ξ is given by the formula

$$D_\xi^r \mathcal{B}_\kappa^g(u) = \partial^r \left(t : \mathbb{R} \to \frac{g!}{\kappa!}(u + t\xi)^\kappa \right)(0) \tag{4.4}$$

for any $u \in \mathbb{A}^d$. Using the Leibnitz formula (2.3), we get

$$D_\xi^r \mathcal{B}_\kappa^g(u) = \frac{g!}{\kappa!} \sum_{\lambda \in \mathbb{I}_d^r} \frac{r!}{\lambda!} \prod_{i=0}^{d} \partial^{\lambda_i} \left(t : \mathbb{R} \to (u_i + t\xi_i)^{\kappa_i} \right)(0) \tag{4.5}$$

Note that the inner term of the summation equals zero whenever $\lambda_i > \kappa_i$ for any i, since the λ_i-th derivative of a κ_i-degree polynomial is zero. Using the chain rule we get

$$\begin{aligned} D_\xi^r \mathcal{B}_\kappa^g(u) &= \frac{g!}{\kappa!} \sum_{\substack{\lambda \in \mathbb{I}_d^r \\ \lambda \leq \kappa}} \frac{r!}{\lambda!} \prod_{i=0}^{d} \frac{\kappa_i!}{(\kappa_i - \lambda_i)!} u_i^{\kappa_i - \lambda_i} \xi_i^{\lambda_i} \\ &= \frac{g!}{\kappa!} \sum_{\substack{\lambda \in \mathbb{I}_d^r \\ \lambda \leq \kappa}} \frac{r!}{\lambda!} \frac{\kappa!}{(\kappa - \lambda)!} u^{\kappa - \lambda} \xi^\lambda \\ &= \frac{g!}{(g-r)!} \sum_{\substack{\lambda \in \mathbb{I}_d^r \\ \lambda \leq \kappa}} \frac{r!}{\lambda!} \xi^\lambda \frac{(g-r)!}{(\kappa - \lambda)!} u^{\kappa - \lambda} \\ &= \frac{g!}{(g-r)!} \sum_{\substack{\lambda \in \mathbb{I}_d^r \\ \lambda \leq \kappa}} \frac{r!}{\lambda!} \xi^\lambda \mathcal{B}_{\kappa-\lambda}^{g-r}(u) \end{aligned}$$

If we consider the defining formula of $\mathcal{B}_\kappa^g$

$$\mathcal{B}_\kappa^g(u) = \frac{g!}{\kappa!} u^\kappa$$

to be valid over the whole $\mathbb{R}^{d+1}$, instead of just $\mathbb{A}^d$, we can write

$$D_\xi^r \mathcal{B}_\kappa^g(u) = \frac{g!}{(g-r)!} \sum_{\substack{\lambda \in \mathbb{I}_d^r \\ \lambda \leq \kappa}} \mathcal{B}_\lambda^r(\xi) \;\; \mathcal{B}_{\kappa-\lambda}^{g-r}(u) \tag{4.6}$$

4.3 Directional derivatives of simploidal polynomials

For any multi-dimension $\delta \in \mathbb{I}_m$ and any $\Xi \in \mathbb{V}^\delta$, the r-th derivative of the simploidal Bernstein polynomial $\mathcal{B}_\Lambda^\alpha$ in the direction Ξ is

$$D_\Xi^r \mathcal{B}_\Lambda^\alpha(U) = \partial^r \left(t : \mathbb{R} \to \prod_{i=0}^{m} \mathcal{B}_{\Lambda_i}^{\alpha_i}(U_i + t\,\Xi_i) \right)(0) \tag{4.7}$$

for any $U \in \mathbb{A}^\delta$. Using the Leibnitz formula (2.3), we get

$$D_\Xi^r \mathcal{B}_\Lambda^\alpha(U) = \sum_{\kappa \in \mathbb{I}_m^r} \frac{r!}{\kappa!} \prod_{i=0}^{m} \partial^{\kappa_i} \left(t : \mathbb{R} \to \mathcal{B}_{\Lambda_i}^{\alpha_i}(U_i + t\,\Xi_i) \right)$$

As before, whenever $\kappa_i > \alpha_i$ for any i, the inner term of the summation equals zero. Using the formula (4.6) we get

$$\begin{aligned} D^r_{\Xi}\mathcal{B}^{\alpha}_{\Lambda}(U) &= \sum_{\substack{\kappa\in\mathbb{I}^r_m\\ \kappa\leq\alpha}} \frac{r!}{\kappa!} \prod_{i=0}^{m} \frac{\alpha_i!}{(\alpha_i-\kappa_i)!} \sum_{\substack{\lambda\in\mathbb{I}^{\kappa_i}_{\delta_i}\\ \lambda\leq\Lambda_i}} \mathcal{B}^{\kappa_i}_{\lambda}(\Xi_i)\ \mathcal{B}^{\alpha_i-\kappa_i}_{\Lambda_i-\lambda}(U_i) \\ &= \sum_{\substack{\kappa\in\mathbb{I}^r_m\\ \kappa\leq\alpha}} \frac{r!}{\kappa!} \frac{\alpha!}{(\alpha-\kappa)!} \sum_{\substack{\Omega\in\mathbb{H}^{\kappa}_{m,\delta}\\ \Omega\leq\Lambda}} \prod_{i=0}^{m} \mathcal{B}^{\kappa_i}_{\Omega_i}(\Xi_i)\ \mathcal{B}^{\alpha_i-\kappa_i}_{\Lambda_i-\Omega_i}(U_i) \end{aligned}$$

As in section 4.2, if we consider the defining formula of $\mathcal{B}^{\alpha}_{\Lambda}$

$$\mathcal{B}^{\alpha}_{\Lambda}(U) = \prod_{i=0}^{m} \frac{\alpha_i!}{\Lambda_i!} U_i^{\Lambda_i}$$

to be valid over the whole $\mathbb{M}_{m,\delta}$, instead of just $\mathbb{A}^{\delta}$, we can write

$$D^r_{\Xi}\mathcal{B}^{\alpha}_{\Lambda}(U) = \sum_{\substack{\kappa\in\mathbb{I}^r_m\\ \kappa\leq\alpha}} \frac{r!}{\kappa!} \frac{\alpha!}{(\alpha-\kappa)!} \sum_{\substack{\Omega\in\mathbb{H}^{\kappa}_{m,\delta}\\ \Omega\leq\Lambda}} \mathcal{B}^{\kappa}_{\Omega}(\Xi)\ \mathcal{B}^{\alpha-\kappa}_{\Lambda-\Omega}(U) \tag{4.8}$$

References

[1] C.T.J. Dodson and T. Poston. *Tensor Geometry*. Springer-Verlag, 2 edition, 1990.

[2] L. Freitas and J. Stolfi. Conversion formulas for simploidal Bernstein polynomials. Technical Report IC-08-12, May 2008.