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Generating Simple Bricks and Braces

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Dedicated to Lászlo Lovász on the occasion of his sixtieth birthday

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Abstract

The *bicontraction* of a vertex of degree two in a graph consists of contracting both the edges incident with that vertex. The *retract* of a graph is the graph obtained from it by bicontracting all its vertices of degree two. An edge e of a brick G is *thin* if the retract of $G-e$ is a brick, and is *strictly thin* if that retract is a simple brick. Thin and strictly thin edges in braces on six or more vertices are similarly defined.

We showed in [3] that every brick distinct from K_4, $\overline{C_6}$ and the Petersen graph has a thin edge. In Section 2, we shall show that every brace has a thin edge.

McCuaig [7] showed that every brace, which is not a biwheel or a prism or a Möbius ladder, has a strictly thin edge. Analogously, Norine and Thomas [8] showed that every brick, which is different from the Petersen graph and is not in any one of five well-defined infinite families of graphs, has a strictly thin edge. These theorems yield procedures for generating simple braces and bricks, respectively.

In Sections 4 and 5, we shall show that the results of McCuaig [7] on braces, and of Norine and Thomas [8] on bricks, may be deduced fairly easily from ours mentioned above. The proofs of these results are so remarkably similar that we are able to present them simultaneously.

1 Introduction

Graphs considered here are loopless, but they may have multiple edges. To avoid repetitive use of phrases such as 'up to isomorphism', we shall simply say that two graphs are the same if they are isomorphic to each other. The notation and terminology we use is essentially that of Bondy and Murty [1].

Bricks and braces, which are the objects of this study, are important classes of 3-connected matching covered graphs. Two basic theorems about 3-connected graphs serve as prototypes for the results concerning bricks and braces mentioned in the abstract. The first theorem is due to Thomassen [9] (also see [1], Section 9.4).

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Theorem 1.1
Every 3-connected graph distinct from K_4 contains an edge such that either its deletion or its contraction from the graph results in a 3-connected graph. □

The above theorem yields a procedure for generating 3-connected graphs. Suppose that H is a 3-connected graph with a vertex x of degree greater than three, and let H' be the graph obtained from H by splitting x into two vertices x_1 and x_2 so that each of x_1 and x_2 has at least two neighbours in H'. Then the graph G obtained from H' by adding an edge e joining x_1 and x_2 is also 3-connected. We say that G is obtained from H by *expanding* vertex x. (Since $H = G/e$, expansion of a vertex may be regarded as a reversal of the operation of contraction.) This observation implies the following procedure for generating 3-connected graphs (see [1], Chapter 9).

Corollary 1.2
Given any 3-connected graph G there exists a sequence $G_1, G_2, \ldots, G_k$ of 3-connected graphs such that (i) $G_1 = K_4$, and $G_k = G$, and (ii) for $2 \leq i \leq k$, G_i is obtained from G_{i-1} by either adding an edge or by expanding a vertex. □

The above corollary is no longer valid if G is simple and we insist that all graphs in the sequence $G_1, G_2, \ldots, G_k$ also be simple. For example, this is the case for any wheel; neither the deletion nor the contraction of an edge in a wheel results in a simple 3-connected graph. However, Tutte [11] proved that the wheels are the only exceptions:

Theorem 1.3
Every simple 3-connected graph which is not a wheel contains an edge such that either its deletion or its contraction from the graph results in a simple 3-connected graph. □

Corollary 1.4
Given any simple 3-connected graph G there exists a sequence $G_1, G_2, \ldots, G_k$ of 3-connected simple graphs such that (i) G_1 is a wheel, and $G_k = G$, and (ii) for $2 \leq i \leq k$, G_i is obtained from G_{i-1} by either adding an edge or by expanding a vertex. □

Theorem 1.3 may be deduced easily from Theorem 1.1 (see Thomassen [10]). The idea of our work is to follow this example and derive the theorems concerning the existence of strictly thin edges in bricks and braces (due, respectively, to Norine and Thomas [8] and McCuaig [7]) from our results asserting the existence of thin edges in bricks and braces (proved, respectively, in [3] and Section 2 of this paper). As mentioned in the abstract, our deduction of their theorems from ours are so similar that we are able to present them together. Hopefully, this unified approach leads to a deeper appreciation of their important results.

It should be noted that [7] and [8] contain more general results than the ones presented here. A description of those results will be given at the end of the paper.

For the convenience of the reader, we begin with a brief review of the relevant terminology, definitions and results from the theory of matching covered graphs.

1.1 Matching Covered Graphs

A graph G is *matching covered* if it is connected, has at least two vertices and each edge lies in a perfect matching. Some authors refer to matching covered graphs as *1-extendable* graphs. Every 2-edge-connected cubic graph is matching covered. Three important examples of matching covered cubic graphs are shown in Figure 1. The treatise by Lovász and Plummer [6] contains the basic theory of matching covered graphs.

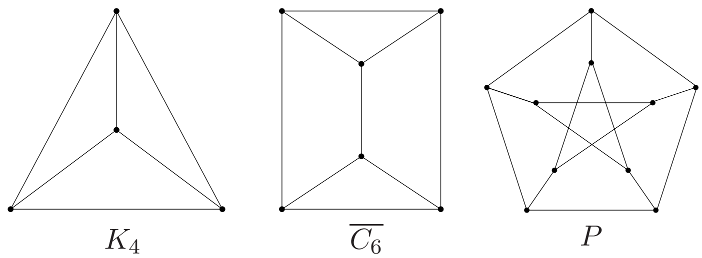


Figure 1: Three basic matching covered graphs

Tight Cuts

Let X be a set of vertices of a graph G. We denote by $\partial(X)$ the set of all edges with one end in X and one end in $\overline{X} = V \setminus X$. Clearly, $\partial(X)$ is an edge cut of G; we shall simply refer to such sets of edges as *cuts*. If G is connected and $\partial(X) = \partial(Y)$, then either $Y = X$ or $Y = \overline{X}$; these two sets are then referred to as the *shores* of $\partial(X)$. A cut is *trivial* if it has a shore consisting of exactly one vertex.

Given any cut $C := \partial(X)$ in a connected graph G, one may obtain two other graphs, namely G/X and $G/\overline{X}$, by contracting the shores of C to single vertices. These two graphs are called the *C-contractions* of G. When it is necessary to name the *contraction vertices* (that is, the vertices resulting from the contractions of shores), we shall use an alternative notation to represent C-contractions. Thus, $G/X \to x$ and $G/\overline{X} \to \overline{x}$ denote G/X and $G/\overline{X}$, respectively, where x and $\overline{x}$ are the corresponding contraction vertices.

Now let G be a matching covered graph. A cut C of G is *t*ight if $|C \cap M| = 1$, for every perfect matching M of G. A basic fact concerning matching covered graphs is that, if C is a tight cut of G, then both the C-contractions of G are also matching covered.

Every tight cut in a matching covered graph is a bond. In any connected graph, both the shores of a bond induce connected subgraphs. We thus have the following simple proposition.

Proposition 1.5
Let $\partial(X)$ be a tight cut in a matching covered graph. Then the subgraphs of G induced by X and $\overline{X}$ are both connected. □

Bricks and Braces

Simplest examples of tight cuts are the trivial cuts. If a matching covered graph is free of nontrivial tight cuts then it is a *brace* if it is bipartite, a *brick* otherwise. The three graphs shown in Figure 1 are bricks. The complete bipartite graphs K_2, C_4 and $K_{3,3}$ are the unique simple braces on two, four and six vertices, respectively.

Barriers

Let G be a graph with a perfect matching. Then a subset B of V is called a *barrier* if $o(G - B) = |B|$, where $o(G - B)$ denotes the number of odd components of $G - B$. A barrier is *nontrivial* if it has at least two elements. Now suppose that G is matching covered, B a nontrivial barrier of G, and K an odd component of $G - B$. Then it is easy to see that $\partial(V(K))$ is a tight cut of G.

Bricks and braces may be characterized in terms of barriers. A deep theorem of Edmonds, Lovász and Pulleyblank [4] states that a 3-connected matching covered graph is a brick if and only if it is free of nontrivial barriers. On the other hand, let G be any bipartite matching covered graph with bipartition (A, B). Then, clearly, both A and B are barriers of G. But it may have other barriers besides these two. It can be shown that G is a brace if and only if A and B are its only nontrivial barriers.

Bicontractions and Retracts

Let G be a matching covered graph and let v be a vertex of degree two, with two distinct neighbours v_1 and v_2. The *bicontraction* of v is the operation of contracting the two edges vv_1 and vv_2 incident with v. Note that $\{v_1, v_2\}$ is a barrier of G and the graph resulting from the bicontraction of v is the same as the graph G/X, where $X := \{v, v_1, v_2\}$. Since $\partial(X)$ is a tight cut, it follows that the bicontraction of v results in a matching covered graph.

The *retract* of G is the graph obtained from G by bicontracting all its vertices of degree two. (Here, we apply this notion only to graphs with either just one vertex of degree two, or two nonadjacent vertices of degree two.) It follows from the above observation that retracts of matching covered graphs are also matching covered. The retract of the graph $P - e$, the graph obtained from the Petersen graph by deleting an edge, is shown in Figure 2.

Uncrossing Tight Cuts

Let G be a matching covered graph. Two cuts $C := \partial(X)$ and $D := \partial(Y)$ of G *cross* if each of the four sets $X \cap Y$, $X \setminus Y$, $Y \setminus X$ and $\overline{X \cup Y}$ is non-null. A collection $\mathcal{C}$ of cuts of G is *laminar* if no two of its cuts cross.

The following result is a fundamental property of cuts in graphs.

Lemma 1.6 (Submodularity)
Let G be a matching covered graph, $C := \partial(X)$ and $D := \partial(Y)$ two cuts of G. Let L denote the set of edges that joins a vertex of $X \setminus Y$ to a vertex of $Y \setminus X$. Let $I :=$

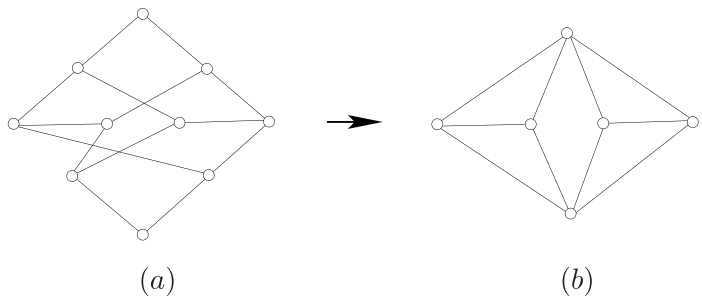


Figure 2: (a) $P - e$, (b) the retract of $P - e$.

$\partial(X \cap Y)$, $U := \partial(\overline{X \cup Y})$. *For any set* M *of edges of* G*, the following equality holds:* $|M \cap C| + |M \cap D| = |M \cap I| + |M \cap U| + 2|M \cap L|$. □

Corollary 1.7 (Modularity)
Let G *be a matching covered graph,* $C := \partial(X)$ *and* $D := \partial(Y)$ *two tight cuts of* G*. If* $|X \cap Y|$ *is odd then each of* $\partial(X \cap Y)$ *and* $\partial(\overline{X \cup Y})$ *is tight and no edge of* G *joins a vertex of* $X \setminus Y$ *to a vertex of* $Y \setminus X$*.* □

Tight Cut Decompositions

Let G be a matching covered graph. We may apply to G a *tight cut decomposition procedure*, which produces a list of bricks and braces, called a *tight cut decomposition* of G. If G itself is a brick or a brace then the list is the singleton containing the graph G. Otherwise, let C be any nontrivial tight cut of G. Then, both C-contractions of G are matching covered. Recursively apply the tight cut decomposition procedure to each C-contraction of G. The resulting lists are then combined to produce the tight cut decomposition of G. We remark that, associated with a tight cut decomposition of G there is a maximal laminar collection $\mathcal{C}$ of nontrivial tight cuts of G. Based on the modularity property (Corollary 1.7), Lovász [5] proved the following remarkable result on tight cut decompositions.

Theorem 1.8
Any two applications of the tight cut decomposition procedure to a matching covered graph produce the same list of bricks and braces, up to multiple edges. □

In particular, the numbers of bricks and braces are numerical invariants of matching covered graphs.

1.2 Thin and Strictly Thin Edges

We denote the subgraph of a graph G obtained by deleting an edge e by $G - e$ (in [1] it is denoted by $G \setminus e$.) An edge e in a matching covered graph G is *removable* if $G - e$ is also matching covered. The following theorem concerning braces is easy to prove. (It can be deduced, for example, from Lemma 2.6.)

Theorem 1.9
In a brace on six or more vertices, every edge is removable. □

Analogous result for bricks does not hold. However, Lovász [5] showed:

Theorem 1.10
Every brick distinct from K_4 *and* $\overline{C_6}$ *has a removable edge.* □

The deletion of a removable edge from a brick need not be a brick; in fact, the deletion may even result in a matching covered graph with more than one brick. For example, any edge in the Petersen graph is removable, but the matching covered graph obtained by deleting it has two bricks (see Figure 2). A removable edge e of a brick G is *b-invariant* if $G - e$ has precisely one brick. Resolving a conjecture of Lászlo Lovász, we proved the following result in [2].

Theorem 1.11
Every brick distinct from K_4 *and* $\overline{C_6}$ *and the Petersen graph has a b-invariant removable edge.* □

Motivated by the problem of recursively generating bricks, we were led to the notion of thin edges. (Our definition of a thin edge in [3] was phrased in terms of sizes of barriers, but is equivalent to the one given here.) Let G be a brace on six or more vertices or a brick. An edge e of G is *thin* if the retract of G is a brick when G is a brick, and is a brace when G is a brace. Using Theorem 1.11, we proved in [3] the following result.

Theorem 1.12
Every brick distinct from K_4, $\overline{C_6}$ *and the Petersen graph has a thin edge.* □

In Section 2, we shall prove the following analogue of the above theorem for braces.

Theorem 1.13
Every brace on six or more vertices has a thin edge.

Figure 3 illustrates thin edges and non-thin edges of a brace.

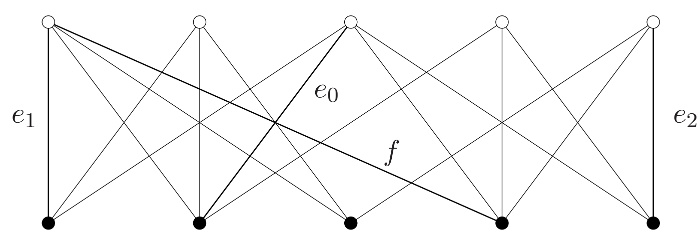


Figure 3: Edges e_0, e_1, and e_2 are thin, but f is not.

In order to establish recursive procedures for generating simple bricks and braces, one needs the notion of a strictly thin edge. Let G be either a simple brick or a simple brace on six or more vertices. An edge e of G is *strictly thin* if e is thin and the retract of $G - e$ is simple.

The Index of a Thin Edge

We associate with each thin edge a number called its index, as defined below. Let G be a brick (or a brace on six or more vertices), and let e be a thin edge of G. Then the retract of $G - e$ is a brick (respectively, a brace). The *index* of e is:

- *zero*, if both ends of e have degree four or more in G;
- *one*, if exactly one end of e has degree three in G;
- *two*, if both ends of e have degree three in G and edge e does not lie in a triangle;
- *three*, if both ends of e have degree three in G and edge e lies in a triangle.

The three edges e_0, e_1 and e_2 in Figure 3 are, respectively, thin edges of index zero, one and two in that brace.

Note that thin edges of index three are contained in triangles. Thus, only bricks can contain such edges. Examples of thin edges of indices one, two, and three are indicated by solid lines in the three bricks, respectively, shown in Figure 4. The thin edge of index three in the third brick is also strictly thin, but the indicated thin edges in the first two bricks are not. In fact, the first two bricks do not contain any strictly thin edges. Infinite families of bricks and braces which do not contain any strictly thin edges are described in Section 3.

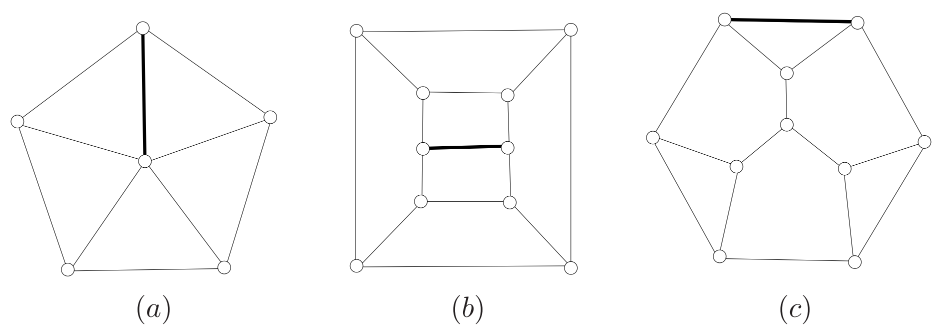


Figure 4: Thin edges of indices one, two, and three in bricks

2 Existence of Thin Edges in Braces

The objective of this section is to present a proof of Theorem 1.13. Before proceeding to the proof, we describe a number of preliminary results. Most of these are either well-known or are easily proved.

2.1 Bipartite Matching Covered Graphs

Let G be a matching covered graph with bipartition (A, B), and let X be a set of vertices of G such that $|X|$ is odd. Then $|X \cap A|$ and $|X \cap B|$ are distinct; one with smaller cardinality is called the *minority part* and is denoted X_-, and the other, with larger cardinality, is called the *majority part* of X and is denoted X_+.

Tight Cuts in Bipartite Matching Covered Graphs

The following property, which is easily proved, gives a description of tight cuts in bipartite matching covered graphs.

Lemma 2.1 (See Lemma 1.4 in [5])
Let G be a bipartite matching covered graph, $C := \partial(X)$ a cut of G. Then, C is tight if and only if (i) $|X_+| = |X_-| + 1$ and (ii) no edge of C is incident with a vertex of X_-. □

Corollary 2.2
Every tight cut decomposition of a bipartite matching covered graph consists solely of braces. □

In our illustrations in this section, we shall represent vertices in A by hollow discs and the vertices in B by black discs, and refer to vertices of A and B, respectively, as white and black vertices. In view of Lemma 2.1, all edges in a tight cut $C := \partial(X)$ must emanate from vertices of the same colour in X.

The next lemma plays a crucial role in the proof of Theorem 1.13. If S is any set and e is an element of S, we shall simply write $S - e$ for the set $S \setminus \{e\}$.

Lemma 2.3
Let G be a brace, e be an edge incident with a vertex u of G, and let $C := \partial(X)$ and $D := \partial(Y)$ be two nontrivial cuts of G such that $C - e$ and $D - e$ are both tight in $G - e$. If u lies in $X \cap Y$ then the cuts $\partial(X \cap Y) - e$ and $\partial(\overline{X \cup Y}) - e$ are both nontrivial and tight in $G - e$.

Proof: Let (A, B) denote the bipartition of G. By hypothesis, u is in $X \cap Y$. Assume without loss of generality that u lies in A. As the cuts $C - e$ and $D - e$ are both nontrivial and tight in $G - e$, it follows that $X_- = X \cap A$ and $Y_- = Y \cap A$. (Thus, the edges of $C - e$ and $D - e$ emanate from black vertices in X and Y, respectively.) Let v denote the end of e in B. Then, v lies in $\overline{X \cup Y}$. Now consider the quadrants $X \setminus Y$ and $Y \setminus X$. If either of them is empty, then one of X and Y is a subset of the other. In this case, the assertion holds immediately. We may thus assume that C and D cross.

As $C - e$ is nontrivial and tight in $G - e$, by Proposition 1.5, the subgraph of $G - e$ induced by X is connected. Hence $D - e$ has an edge that joins a vertex b in $X \cap Y$ to a vertex a in $X \setminus Y$. As the edge ba belongs to the cut $D - e$, vertex b lies in Y_+ which a subset of B. Thus vertex a lies in A. Likewise, $C - e$ has an edge that joins a vertex b' in $B \cap (X \setminus Y)$ to a vertex a' in $A \cap \overline{X \cup Y}$. See Figure 5.

Let us first observe that $|X \setminus Y|$ cannot be odd. If it were, by Corollary 1.7, $\partial(X \setminus Y) - e$ and $\partial(Y \setminus X) - e$ would be tight cuts of $G - e$. But the cut $\partial(X \setminus Y) - e$ contains edges ab and $b'a'$ which emanate from vertices of different colours. It follows that $|X \setminus Y|$ is even, and hence that $|X \cap Y|$ is odd.

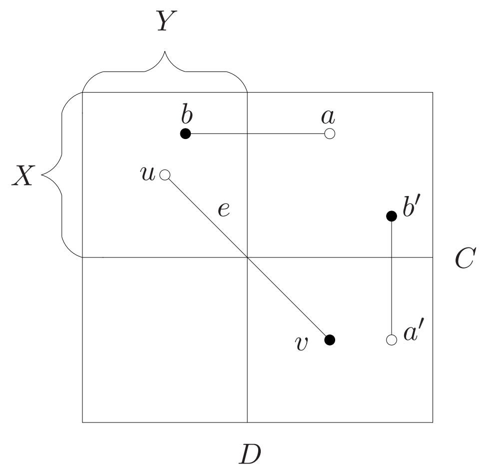


Figure 5: The four quadrants in Lemma 2.3.

Now, by Corollary 1.7, it follows that both $\partial(X \cap Y) - e$ and $\partial(\overline{X \cup Y}) - e$ are tight cuts of $G - e$. They are both nontrivial because u and b belong to $X \cap Y$ and v and a' belong to $\overline{X \cup Y}$. □

A Characterization of Braces

The following lemma provides a characterization of bipartite matching covered graphs. It follows immediately from Theorem 4.1.1 in Lovász and Plummer's book [6].

Lemma 2.4
Let G be a graph with bipartition (A, B) and at least four vertices. Assume that G has a perfect matching. Then, the following assertions are equivalent:

(i) *Graph G is matching covered.*

(ii) *For every partition (A_0, A_1) of A and every partition (B_0, B_1) of B such that $|A_0| = |B_0|$, graph G has at least one edge that joins A_0 to B_1.*

(iii) *For every vertex a of A and every vertex b of B, graph $G - a - b$ has a perfect matching.*

□

The next result, due to Lovász [5], characterizes braces.

Lemma 2.5
Let G be a matching covered graph with bipartition (A, B). Then, G is a brace if and only if $G - a_1 - a_2 - b_1 - b_2$ has a perfect matching, for any two vertices a_1 and a_2 in A and any two vertices b_1 and b_2 in B. □

Recall that an edge e of a matching covered graph G is removable if $G - e$ is matching covered. Lemma 2.4 implies the following useful result concerning nonremovable edges in bipartite matching covered graphs.

LEMMA 2.6
Let G be a bipartite matching covered graph with bipartition (A, B), and let ab, with $a \in A$ and $b \in B$, be a nonremovable edge of G. Then there exist partitions (A_0, A_1) of A and (B_0, B_1) of B such that $|A_0| = |B_0|$ and ab is the only edge with one end in A_0 and one end in B_1. □

The following lemma establishes a useful connectivity property of braces. For a set X of vertices of a graph G, we denote by $N(X)$ the set of vertices of G that are adjacent to some vertex of X.

LEMMA 2.7
Let G be a brace on six vertices or more. Then G is 3-connected. Moreover, for any set X of three vertices of G that meets both parts of the bipartition of G, the graph $G - X$ is connected.

Proof: Let (A, B) denote the bipartition of G. Let X be a set of vertices of G such that either (i) X has two or fewer vertices or (ii) X has precisely three vertices and meets both A and B. We shall prove that $G - X$ is connected.

Consider first the case in which X has a vertex, v, in A, and a vertex, w, in B. By Lemmas 2.5 and 2.4, graph $G - v - w$ is matching covered. Moreover, $G - v - w$ has four or more vertices, whence it is 2-connected. We deduce that $G - X$ is connected in this case, even if $|X| = 3$.

Consider next the case in which X does not contain vertices in both A and B. By hypothesis, $|X| \leq 2$. If $|X| < 2$ then $G - X$ is connected, because G, a matching covered graph on more than two vertices, is 2-connected. We may thus assume that X consists of precisely two vertices, v and w, both in A, say. Let x be any vertex in B. By the previous case, graph $G - X - x$ is connected. Vertex x must be adjacent to two or more vertices, as G is 2-connected. If x is adjacent to only two vertices then the set $N(x) \cup \{x\}$ is the shore of a nontrivial tight cut, as G has six or more vertices. This is a contradiction. We deduce that x is adjacent to three or more vertices. One of the vertices adjacent to x does not lie in X. As $G - X - x$ is connected, then $G - X$ is also connected. □

2.2 Graphs Obtained by Deleting an Edge from a Brace

Assume that G is a bipartite and matching covered graph. By Corollary 2.2, the tight cut decomposition of G produces only braces. By Theorem 1.8, all tight cut decompositions of G produce the same family of braces, up to multiple edges. Here denote by $b(G)$ the number of braces in any tight cut decomposition of G.

Let G be a brace on six or more vertices. In trying to show that G has a thin edge, we select an edge e of G according to some criterion (to be described in the next section). By Theorem 1.9, $G - e$ is matching covered. If $G - e$ is a brace, then e is a thin edge. Assuming that it is not, we must arrive at a contradiction. Motivated by this, we proceed to explore the structure of matching covered graphs which are obtained from braces by the deletion of one edge.

LEMMA 2.8
Let G be a brace on six or more vertices, and let e be an edge incident with a vertex u of G. Let $\mathcal{C}$ be a laminar family of nontrivial cuts of G such that, for each cut $C \in \mathcal{C}$, the cut $C - e$ is tight in $G - e$. Let $\mathcal{S}$ denote the set of shores of the cuts in $\mathcal{C}$ which contain the end u of e. Then, set inclusion is a total order on $\mathcal{S}$.

Proof: Let v denote the end of e distinct from u. Let $C := \partial(X)$ be a cut in $\mathcal{C}$. Adjust notation, by interchanging X with $\overline{X}$ if necessary, so that u lies in X. By definition, $C - e$ is a nontrivial tight cut of $G - e$. Then, no edge of $C - e$ is incident with a vertex in X_-. As G is a brace, e is incident with a vertex in X_-. Thus, u lies in X_-. Likewise, v lies in $\overline{X}_-$. These conclusions hold for each cut C in $\mathcal{C}$.

Let $D := \partial(Y)$ denote a cut in $\mathcal{C} - C$. Adjust notation so that u lies in Y. Then, u lies in $X \cap Y$ and v lies in $\overline{X \cup Y}$. As $\mathcal{C}$ is laminar, cuts C and D do not cross. Thus, one of $X \setminus Y$ and $Y \setminus X$ is empty. We deduce that either $X \subset Y$ or $Y \subset X$, where the inclusion is proper, because C and D are distinct. □

The above lemma implies the following simple structural description of tight cut decompositions of graphs obtained by deleting an edge from a brace. *The notation established here is used throughout the rest of this section.*

THEOREM 2.9
Let G be a brace on six or more vertices with bipartition (A, B), and let e be an edge of G. Let $\ell := b(G - e)$ be the number of braces of $G - e$. Suppose that $\mathcal{C}$ is a maximal laminar family $\mathcal{C}$ of nontrivial tight cuts, and $\mathcal{G}$ the corresponding tight cut decomposition of $G - e$. Then, there exists a nested sequence $X_1 \subset X_2 \subset \cdots \subset X_{\ell-1}$ of subsets of V such that the $\ell - 1$ cuts $C_1, C_2, \ldots, C_{\ell-1}$ in $\mathcal{C}$, and the ℓ braces $G_1, G_2, \ldots, G_\ell$ in $\mathcal{G}$ are related to these shores as follows:

$$\left.\begin{array}{rcl} C_i &=& \partial(X_i), \text{ for } i = 1, 2, \ldots, \ell - 1, \\ G_1 &=& (G - e)/\overline{X_1}, \\ G_i &=& ((G - e)/X_{i-1})/\overline{X_i}, \text{ for } 1 < i < \ell, \\ G_\ell &=& (G - e)/X_{\ell-1}. \end{array}\right\} \qquad (1)$$

Furthermore, there exist partitions $(A_1, A_2, \ldots, A_\ell)$ of A, and $(B_1, B_2, \ldots, B_\ell)$ of B, such that, for $i = 1, 2 \ldots, \ell - 1$,

$$X_i := (A_1 \cup B_1) \cup (A_2 \cup B_2) \cup \cdots \cup (A_i \cup B_i),$$

and the following properties hold:

$$\begin{array}{rcl} |A_1| &=& |B_1| - 1 \\ |A_i| &=& |B_i|, \quad \text{for } 1 < i < \ell \\ |A_\ell| &=& |B_\ell| + 1 \end{array}$$

(Figure 6 shows a brace in which $G - e$ has four braces.)

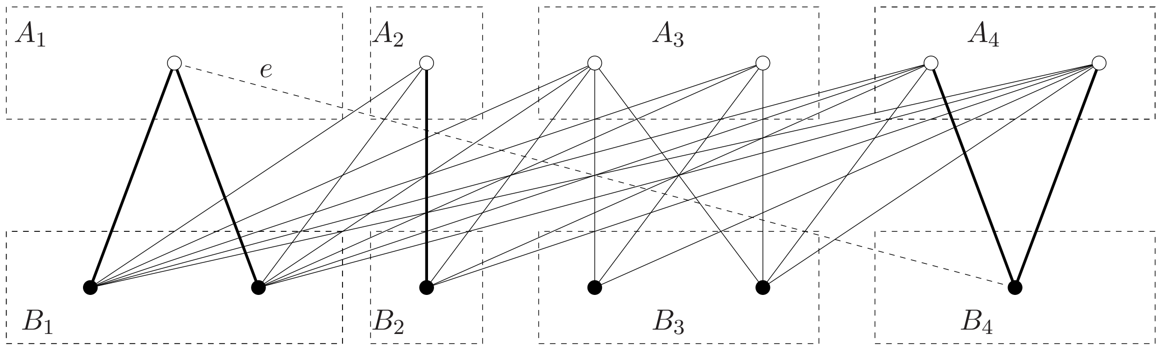


Figure 6: A tight cut decomposition of $G - e$.

Proof: Every tight cut in $\mathcal{C}$ has a shore that contains the end of e in A. Let $\mathcal{S}$ denote the set of these shores. By Lemma 2.8, set inclusion is a total order on $\mathcal{S}$. Thus, we may enumerate the shores in $\mathcal{S}$ in a sequence as $X_1 \subset X_2 \subset \ldots \subset X_{\ell-1}$. Let $X_0 := \emptyset$ and $X_b := V(G)$. Now, for $i = 1, 2, \ldots, \ell$, let $A_i := A \cap (X_i \setminus X_{i-1})$, and $B_i := B \cap (X_i \setminus X_{i-1})$. The assertion holds, by Lemma 2.1. □

COROLLARY 2.10
Let G be a brace on six or more vertices, and e be an edge of G. Then, any brace H obtained from $G - e$ by a tight cut decomposition of $G - e$ has at most two contraction vertices. □

The following lemma characterizes edges which are not removable in $G - e$.

LEMMA 2.11
Let G be a brace on six or more vertices, e an edge of G, such that $G - e$ is not a brace. Consider any tight cut decomposition $\mathcal{G}$ of $G - e$, and adopt the corresponding notation described in the assertion of Theorem 2.9. Let f be an edge which is not removable in $G - e$. Then, one of the following alternatives holds:

(i) *either edges e and f are adjacent and their common end is adjacent to only two vertices in $G - e$, or*

(ii) *$|A_i| = |B_i| = 1$, for some i, $1 < i < \ell$, and f joins the vertex in A_i to the vertex in B_i.*

(The non-removable edges in the graph $G - e$ shown in Figure 6 are indicated by thick lines.)

Proof: It is easy to see that if the two C-contractions of a graph are matching covered then the graph itself is matching covered. Thus, an edge of $G - e$ is removable in $G - e$ if and only if it is removable in each brace in $\mathcal{G}$ that contains the edge. Let us thus determine the non-removable edges of each brace in $\mathcal{G}$. Let G_i, $1 \le i \le \ell$, be one of the braces of $\mathcal{G}$. If G_i

has six or more vertices then every edge of G_i is removable in G_i. Assume thus that G_i has only four vertices.

Consider first the case in which $i = 1$. As G_1 has only four vertices, it follows that A_1 is a singleton, its only vertex, a_1, is an end of e, and it is adjacent in $G - e$ only to the two vertices of B_1. Let x denote the contraction vertex of G_1. Each vertex of B_1 is adjacent to three or more vertices of G, as G is a brace on six or more vertices. Thus, each vertex of B_1 is joined in G_1 to x by two or more edges. We deduce that every edge of G_1 incident with x is a multiple edge, and hence it is removable. Consequently, the non-removable edges of G_1 are incident with a_1. We conclude that the non-removable edges of G_1 are adjacent to e and the common end a_1 is adjacent in $G - e$ only to the two vertices of B_1. A similar argument applies to the brace B_ℓ.

Consider now the case in which $1 < i < \ell$. Then, $G_i = ((G - e)/X_{i-1})/\overline{X_i}$ has two contraction vertices, x (resulting from the contraction of X_i) and x' (resulting from the contraction of $\overline{X_i}$).

As G_i has only four vertices, $|A_i| = |B_i| = 1$, say $A_i = \{a_i\}$ and $B_i = \{b_i\}$. Then $(\{x, b_i\}, \{x', a_i\})$ is the bipartition of G_i. Let us show that every edge incident with x or with x' is removable in G_i. Vertex a_i is adjacent in G to three or more vertices. Moreover, it is not an end of e. Therefore, a_i is joined in G_i to x by two or more edges. Consequently, those edges are removable in G_i, as they are multiple edges. Similarly, the edges joining b_i to x' in G_i are also multiple edges. Now consider the vertices x and x'. They are adjacent in G_i, as G_i is C_4, up to multiple edges. Let $e' = b'a'$ be an edge of G with $b' \in B_1 \cup \cdots \cup B_{i-1}$ and $a' \in A_{i+1} \cup \cdots \cup A_\ell$. Let u be the end of the edge e in A. Now consider the graph $G - u - a_i - b'$. By Lemma 2.7, this graph is connected. Therefore, there must be another edge, besides e', which joins a vertex in $B_1 \cup \cdots \cup B_{i-1}$ to a vertex in $A_{i+1} \cup \cdots \cup A_\ell$. We may thus conclude that x and x' are joined by multiple edges in G_i. Thus, if G_i has a non-removable edge, it is the edge joining a_i to b_i.

Let $\mathcal{C}$ denote the laminar collection of nontrivial tight cuts of G used in obtaining the tight cut decomposition $\mathcal{G}$ of $G - e$. We have seen that every edge of $G - e$ that lies in a cut of $\mathcal{C}$ is removable in each brace in $\mathcal{G}$ that contains the edge. Therefore, every edge of $G - e$ that lies in some cut of $\mathcal{C}$ is removable in $G - e$. Thus, a non-removable edge f does not lie in any such cut. If f lies in G_1 or if f lies in G_ℓ then e and f are adjacent, their common end is adjacent to only two vertices in $G - e$. Alternatively, if f lies in G_i, $1 < i < \ell$, then $A_i \cup B_i$ is a doubleton that consists precisely of the two ends of f. □

The following corollary, which will play a key role in our proof, may be deduced easily from the above lemma.

Corollary 2.12

Let G be a brace on six or more vertices, and let e be an edge of G. If the degree of a vertex w in $G - e$ is three or more, then at most one edge of $G - e$ incident with w is not removable in $G - e$. □

2.3 Proof of Theorem 1.13

Our objective is to show that every brace on six or more vertices has a thin edge. Let G be a brace on six or more vertices, (A, B) its bipartition. Let e be any edge of G. By Theorem 1.9, edge e is removable. If $b(G-e) = 1$, that is, if $G-e$ is a brace, then e is thin of index zero. We may thus assume that $b(G-e) > 1$. Consider a tight cut decomposition $\mathcal{G}$ of $G-e$. We define the *rank* of e, denoted $r(e)$, to be the maximum of the numbers of vertices of the braces of $G-e$. More precisely, $r(e) := \max\{|V(G_i)| : G_i \in \mathcal{G}\}$. The value $r(e)$ is well-defined, by Theorem 1.8. We define now the following invariant of G:

$$r^\star := \max\{r(e) : e \in E(G)\}.$$

We have assumed that $b(G-e) > 1$ for each edge e of G. Every multiple edge of G is thin of index zero. Therefore, we may assume that G is simple. The only simple brace on six vertices is $K_{3,3}$. Every edge of $K_{3,3}$ is thin of index two. We may thus assume that G has eight or more vertices. The following lemma shows that the if $r^\star = 4$, then G is the cube. (The cube is the only cubic brace on eight vertices.)

Lemma 2.13
Let G be a brace on eight or more vertices such that, for any edge e of G, all the braces of $G-e$ have exactly four vertices. Then G is the cube, and each of its edges is thin.

Proof: Let e be any edge of G. Consider a tight cut decomposition $\mathcal{G}$ of $G-e$, adopt the notation used in the statement of Theorem 2.9. As all the braces of $G-e$ have precisely four vertices, it follows that $A_1, A_2, \ldots, A_{\ell-1}$ and $B_2, B_3 \ldots, B_\ell$ are singletons, whereas B_1 and A_ℓ are both doubletons. Consequently,

$$\ell = |V(G)|/2 - 1. \tag{2}$$

As G is simple, and A_1 and B_ℓ are both singletons, it follows that both ends of e have degree three. This conclusion holds for each edge e of G. We deduce that G is cubic.

By hypothesis, G has eight vertices or more. It follows from (2) that $\ell \geq 3$. Sets A_2 and B_2 are both singletons. Let v denote the vertex of A_1, let v' denote the vertex of A_2. All the neighbours of v' in G lie in $B_1 \cup B_2$. The three neighbours of v' are then the three vertices of $B_1 \cup B_2$. We deduce that the two edges of $\partial(v) - e$ lie in a quadrilateral. This conclusion holds for each edge e incident with v. In other words, any two vertices of G that lie in $N(v)$ have two common adjacent vertices, one of which is v.

Denote the neighbour set $N(v)$ of v by S. We claim that there is no vertex u, other than v, such that $N(u) = S$. Because, if this were the case, as G is cubic and $|V(G)| \geq 8$, $\partial(S \cup \{u, v\})$ would be a nontrivial tight cut of the brace G, which is absurd. Hence v is the only vertex of G that is adjacent to each vertex of $N(v)$. Now, as G is cubic, and each pair of vertices of S have two adjacent vertices in common, it follows that $|N(S)| = 4$. Consequently, cut $D := \partial(S \cup N(S))$ is tight in G. If G has ten vertices, or more, then D would be nontrivial, which is also absurd. We deduce that G has fewer than ten vertices. Consequently, G has exactly eight vertices. We conclude that G is the cube. It is easy to verify that each edge of the cube is thin. □

Choosing the edge e: In view of the above theorem, we may assume that $r^\star \geq 6$. We may also assume that G has eight vertices or more, and is simple. Let $R := \{e \in E(G) : r(e) = r^\star\}$. Let e be an edge of R, $\mathcal{G}$ be a tight cut decomposition of $G - e$, and $G^\star$ be a brace in $\mathcal{G}$ on $r^\star$ vertices.

If possible, choose $e \in R$, $\mathcal{G}$ and $G^\star$ so that $G^\star$ has two contraction vertices.

Adopt the notation in the statement of Theorem 2.9. Then, $G^\star = G_j$, where $1 \leq j \leq \ell$. We may adjust notation so that $j \neq 1$, by interchanging A and B, if necessary. Let C denote the cut C_{j-1} in $\mathcal{C}$, and let X denote the shore X_{j-1} of C whose contraction yields a contraction vertex of G_j. If G_j has two contraction vertices, that is, if $j < \ell$, let C' denote the cut C_j, and let X' denote the shore $\overline{X_j}$ of C' whose contraction yields the second contraction vertex of G_j. In addition, if G_j has two contraction vertices, adjust notation so that $|X| \geq |X'|$, by interchanging A and B if necessary. (Figure 7 illustrates the case in which G_j has two contraction vertices.) A vertex of $G^\star$ is *internal* if it is not a contraction vertex of G_j. We denote the set of internal vertices of $G^\star$ by I.

The following upper bound on $r^\star$ is obvious:

$$r^\star = r(e) \leq 1 + |\overline{X}|, \tag{3}$$

with equality only if $G^\star = G_j$ has precisely one contraction vertex, in which case $j = \ell$.

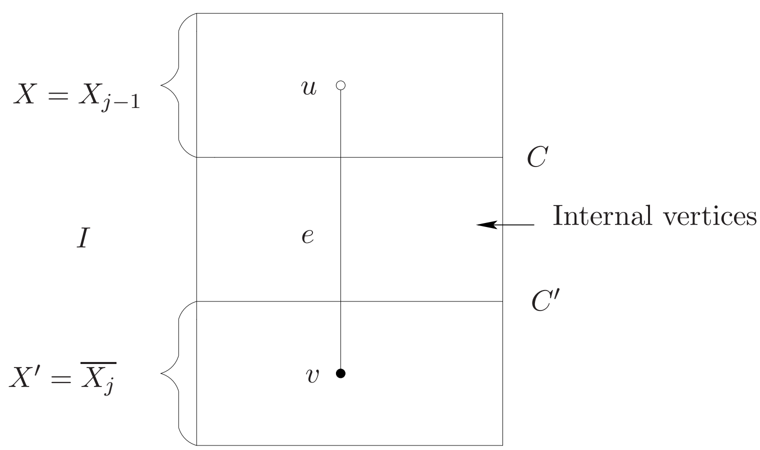


Figure 7: The cuts that define brace $G^\star = G_j$.

It now suffices to show that $|X| < 5$. To see this, assume that $|X| < 5$. Note first that as C is nontrivial, it follows that $|X| = 3$. If G_j has only one contraction vertex then $\ell = 2$, and one of the braces of $\mathcal{G}$ has only four vertices. Consequently, edge e is thin. If G_j has two contraction vertices then, as C' is nontrivial and $|X'| \leq |X|$, it follows that $|X'| = 3$. Consequently, $\ell = 3$ and braces G_1 and G_3 have only four vertices, implying that e is thin in this case as well.

To prove that $|X| < 5$, assume the contrary. In the remaining part of the proof we deduce a contradiction to the definition of edge e. For this, note that edge e has one end in the minority parts of each of X and $\overline{X}$. Let u denote the end of e in X_-, v its other end. Recall that (A, B) denotes the bipartition of G. Adjust notation so that u lies in A. As $|X| \geq 5$, we have that X_- contains two or more vertices. Let $s \neq v$ be a vertex in X_-. As G has more than four vertices, vertex s has degree three or more in G. By Corollary 2.12, at least two edges incident with s are removable in $G - e$. Let f and g denote any two such edges. Let t and w denote the ends of f and g in B. The vertices of G adjacent to s in G lie all in X_+. Thus, both t and w lie in X_+ (see Figure 8).

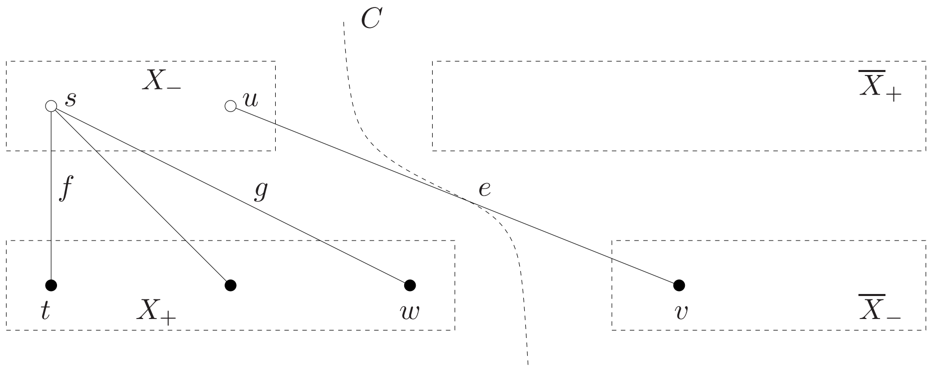


Figure 8: Edges e, f and g.

The strategy now is to use the fact that neither f nor g can be of higher rank than e to deduce a condition which violates the property of braces stated in Lemma 2.5. We shall show that there exist nontrivial tight cuts $\partial(Y) - f$ of $G - f$ and $\partial(Z) - g$ of $G - g$, and two black vertices b_1 and b_2 which lie in $Y_+ \cap Z_+$, and two white vertices a_1 and a_2 which lie in $\overline{Y}_+ \cap \overline{Z}_+$. It is then a simple matter to check that every perfect matching of $G - a_1 - a_2 - b_1 - b_2$ must contain both f and g, which is impossible. The search for the appropriate cuts $\partial(Y)$ and $\partial(Z)$ involves making crucial use of the fact that f and g are removable in both G and $G - e$, and that their rank is no larger than the rank of e. We shall start with an examination of a tight cut decomposition of $G - f$.

Denote by m the number of braces of $G - f$. Let $\mathcal{H} := \{H_1, H_2, \ldots, H_m\}$ be a tight cut decomposition of $G - f$. Let $\mathcal{D} := \{D_1, D_2, \ldots, D_{m-1}\}$ denote the collection of nontrivial cuts used to obtain $\mathcal{H}$. As $m > 1$, we may apply Theorem 2.9, with f playing the role of e. Thus, there exists a nested sequence $Y_1 \subset Y_2 \subset \cdots \subset Y_{m-1}$ of subsets of V, all containing the end s of f, such that:

$$\left.\begin{array}{rcl} D_i & = & \partial(Y_i),\ \text{for } i = 1, 2, \ldots, m-1, \\ H_1 & = & (G - f)/\overline{Y_1}, \\ H_i & = & ((G - f)/Y_{i-1})/\overline{Y_i},\ \text{for } 1 < i < m, \\ H_m & = & (G - f)/Y_{m-1}. \end{array}\right\} \qquad (4)$$

Moreover, edge f has the end s in $A \cap Y_1$, and the end t in $B \cap \overline{Y_{m-1}}$.

By definition, f is removable in $G - e$. We may thus obtain a tight cut decomposition of $G - e - f$ by replacing each brace G_i in $\mathcal{G}$ by a tight cut decomposition of $G_i - f$. Edge f

does not lie in G_j. Therefore, $r(G-e-f) = |V(G_j)| = r^\star$. We may also obtain a tight cut decomposition of $G-e-f$ by replacing each brace H_i in $\mathcal{H}$ by a tight cut decomposition of $H_i - e$. We deduce that $r(f) \geq r(G-e-f) = r^\star$, implying that $r(f) = r^\star$. This condition imposes constraints on the ways in which the cuts in $\mathcal{D}$ interact with the cut C. The following lemma considers the case in which a cut in $\mathcal{D}$ crosses the cut C. Its proof has many features in common with that of Lemma 2.3.

Two of the four quadrants determined by the shores of C and D play prominent roles; they are the Northwest quadrant $X \cap Y$ and the Southeast quadrant $\overline{X \cup Y}$. Thus, for brevity, we shall write $N := X \cap Y$, and $S := \overline{X \cup Y}$. See Figure 9.

Lemma 2.14

Let D be a cut of $\mathcal{D}$ that crosses cut C, let Y denote the shore of D that contains the end s of f in X_-. Then,

- *$|X \cap Y|$ is odd,*
- *cut $\partial(N) - e - f$ is nontrivial and tight in $G-e-f$, and*
- *cut $\partial(S) - e$ is tight in $G-e$.*

Proof: Cut $D-f$ is tight in $G-f$, and hence $D-e-f$ is tight in $G-e-f$. Therefore, by Proposition 1.5, the subgraph of $G-e-f$ induced by $\overline{Y}$ is connected. This implies that some edge of $G-e-f$ joins a vertex b_1 in $X \setminus Y$ to a vertex a_1 in S. Thus, b_1a_1 lies in C. As b_1a_1 is distinct from e, it follows that its end b_1 lies in X_+, which is a subset of B (Figure 9).

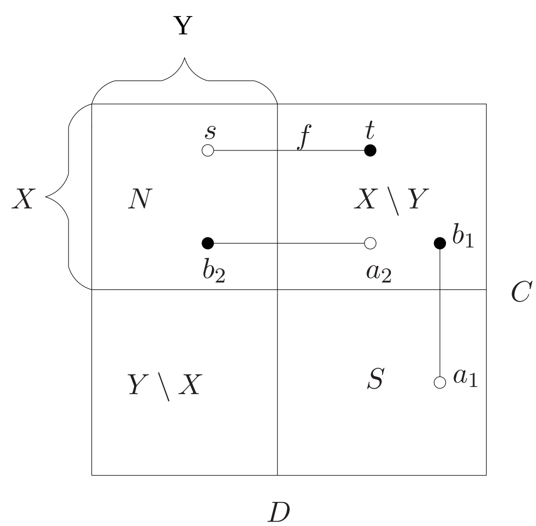


Figure 9: The four quadrants in Lemma 2.14.

Likewise, cut $C-e$ is tight in $G-e$, and hence $C-e-f$ is tight in $G-e-f$. Therefore, by Proposition 1.5, the subgraph of $G-e-f$ induced by X is connected. This implies that some edge of $G - e - f$ joins a vertex b_2 in N to a vertex a_2 in $X \setminus Y$. Thus, $b_2 a_2$ lies in D. As $b_2 a_2$ is distinct from f, it follows that a_2 lies in $\overline{Y}_+$, which a subset of A. Consequently, b_2 lies in B.

The cut $\partial(X \setminus Y)$ cannot be tight because there are edges in it emanating from vertices b_1 and a_2 of different colours. Therefore, by Corollary 1.7, it follows that $|X \cap Y| = |N|$ is odd, and that $\partial(N) - e - f$ and $\partial(S) - e - f$ are tight cuts in $G - e - f$. The cut $\partial(N) - e - f$ is nontrivial because both N, and its complement, have at least two vertices each. Now consider the cut $\partial(S) - e$. As $\partial(S) - e - f$ is tight in $G - e - f$, no edge of $G - e - f$ joins a vertex in S_- to a vertex in $\overline{S}_-$. But edge f has both its ends in X which is entirely contained in $\overline{S}$. Thus, the only edge of G which joins a vertex in S_- to a vertex in $\overline{S}_-$ is e. We conclude that $\partial(S) - e$ is tight in $G - e$. □

Recall that $G^\star = G_j$ is a brace in $\mathcal{G}$, the (fixed) tight cut decomposition of $G - e$, and that I denotes the set of internal vertices of $G^\star$ (see Figure 7). As $r^\star$ is at least six, we have that $|I| \geq 4$. Knowing where most of I lies in relation to the cuts in $\mathcal{D}$ will help us in comparing the orders of braces in $\mathcal{H}$ with the order of $G^\star$. In this connection, the following technical lemma plays a fundamental role.

Lemma 2.15
Let $D := \partial(Y)$ be a cut in $\mathcal{D}$, where Y contains vertex s. Then, either $|Y \cap I| = 0$ or $|Y \cap I| \geq 2$. Moreover, in the former case,

- *if D and C do not cross then $Y \subseteq X$,*
- *if C and D cross then $|S| \geq |I| + 3$.*

Proof: To prove the first part, it suffices to show that $|Y \cap I| = 0$ under the assumption that $|Y \cap I| \leq 1$. Thus, assume that Y and I have at most one vertex in common. Vertex s lies in Y, by hypothesis. Cut D lies in $\mathcal{D}$, also by hypothesis. Thus, t lies in $\overline{Y}$. As X contains both s and t, we have that the quadrants $X \cap Y$ and $X \setminus Y$ are nonempty. As I is a subset of $\overline{X}$, by our assumption I has at most one vertex in the quadrant $Y \setminus X$. Noting that $|I| \geq 4$, we conclude that $|S \cap I| \geq 3$, and hence that the quadrant S is nonempty. Thus, the only quadrant that could possibly be empty is $Y \setminus X$.

If $Y \setminus X = \emptyset$, then, clearly, $|Y \cap I| = 0$, C and D do not cross, and $Y \subseteq X$.

Consider now the case in which C and D cross. As $|S \cap I| \geq 3$, it follows that $\partial(S)$ is nontrivial. Moreover, by Lemma 2.14, cut $\partial(S) - e$ is tight in $G - e$. Thus, the end v of e in B lies in S.

On the other hand, $|\overline{X}|$ and $|S|$ are both odd, whence quadrant $Y \setminus X$ is even. As C and D cross, $|Y \setminus X| \geq 2$. By our assumption, at most one vertex of I lies in Y. Therefore, $\overline{X} \cap X'$ contains a vertex not in I. We deduce that the cut $C' = \partial(X')$ is nontrivial. Hence $G^\star$ has two contraction vertices and v lies in X'. (See Figure 10 for an illustration.) Cuts $\partial(S) - e$ and $C' - e$ are both nontrivial and tight in $G - e$. Moreover, v lies in $S \cap X'$. By Lemma 2.3, $\partial(S \cap X') - e$ is a nontrivial tight cut of $G - e$. Therefore, $|S \cap X'|$ is odd, implying that $|S \cap I|$ is even. We have seen that $G^\star$ has two contraction vertices, therefore $|I|$ is even. But $|Y \setminus I| \leq 1$ by our assumption. We deduce that $I \subseteq S$, implying that Y and

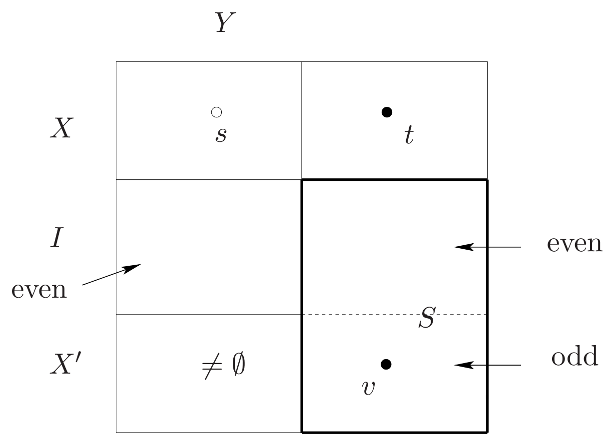


Figure 10: Parities of sets in Lemma 2.15.

I are disjoint. Finally, we have seen that $\partial(S \cap X')$ is nontrivial, therefore $S \cap X'$ contains three or more vertices. Thus three or more vertices of S do not lie in I. Consequently, $|S| \geq |I| + 3$, as asserted. □

Applying the above lemma to the cut $\partial(Y_{m-1})$ in $\mathcal{D}$, we obtain the following:

Corollary 2.16
$|Y_{m-1} \cap I| \geq 2$.

Proof: Let $Y := Y_{m-1}$. Assume, to the contrary, that $|Y \cap I| = 0$. Let D denote the cut $D_{m-1} = \partial(Y)$. We now apply Lemma 2.15 in order to obtain a contradiction.

Consider first the case in which C and D do not cross. Then, $Y \subset X$ (equivalently, $S = \overline{X}$). As $|X|$ and $|Y|$ are both odd, $|X \cap Y| = |Y|$ is odd, and $|X \setminus Y|$ is even. Now, since the vertex t lies in $X \setminus Y$, it follows that $|X \setminus Y| \geq 2$. By definition, $H_m = (G - f)/Y_{m-1}$. Thus, the vertex set of H_m consists of the contraction vertex, $X \setminus Y$, and S. Therefore:

$$r(f) \geq |V(H_m)| = 1 + |X \setminus Y| + |\overline{X}| \geq |\overline{X}| + 3 > r^\star,$$

where the strict inequality follows from (3). We deduce that $r(f) > r^\star$, a contradiction.

Consider next the case in which C and D cross. In this case, $|S| \geq |I| + 3$. Thus,

$$r(f) \geq |V(H_m)| \geq 1 + |S| \geq |I| + 4 > r^\star.$$

Again, a contradiction! □

Choosing the cut Y: In the rest of the proof, we work with a fixed cut Y in $\mathcal{D}$ selected using the following criterion:

> *Let k be the smallest integer i in the interval $(1, m-1)$ for which Y_i has a nonempty intersection with the set I of internal vertices of $G^\star$.*

By Corollary 2.16, k is well-defined. Let $D := D_k$, $Y := Y_k$, the shore of D that contains the end s of f. Note that, as before, $N = X \cap Y$ and $X \setminus Y$ are nonempty. In fact, using the fact that the subgraph of $G - e - f$ induced by X is connected, it is easy to see that $|N| \geq 2$. As I is a subset of $\overline{X}$, by the definition of k, the quadrant $Y \setminus X$ is nonempty. Thus, the only quadrant that is possibly empty is $S = \overline{X \cup Y}$. Let us first consider the special case in which $k = 1$ and show that, this quadrant has then at least two vertices.

Lemma 2.17
If $Y = Y_1$, then $|S| \geq 2$.

Proof: Assume to the contrary that $|S| \leq 1$. Then, $Y \setminus X$ has all but at most one vertex of $\overline{X}$. Thus, $|Y \setminus X| \geq |\overline{X}| - 1$. Also, as noted above, $|N| \geq 2$, and hence, $|Y| \geq |\overline{X}| + 1$. Now, since $H_1 = (G - f)/\overline{Y}$, we deduce that $r(f) \geq |V(H_1)| = |Y| + 1 \geq |\overline{X}| + 2 > r(e)$, where the last inequality follows from (3). Hence, $r(f) > r^\star$, a contradiction. □

The above lemma shows, in particular, that when C and D cross when $k = 1$. This is true more generally.

Lemma 2.18
Cuts C and D cross.

Proof: Assume, to the contrary, that C and D do not cross. Then, as observed earlier, $S = \overline{X \cup Y}$ is empty, This is not the case when $k = 1$ by Lemma 2.17. We may thus assume that $k > 1$.

Let $D' := D_{k-1}$, and let $Y' := Y_{k-1}$ be the shore of D' that contains vertex s. We now apply Lemma 2.15, with D' playing the role of D. Two cases arise depending on whether or not C and D' cross.

Case 1: C and D' cross. By Lemma 2.15, we have that

$$r(f) \geq |V(H_k)| > |\overline{X \cup Y'}| > |I| + 2 \geq r^\star.$$

A contradiction.

Case 2: C and D' do not cross. By the definition of k, $|Y' \cap I| = 0$, implying that $\overline{X}$ is a subset of $Y_k \setminus Y_{k-1}$. As $H_k = ((G - f)/Y')/\overline{Y}$, we now have $|V(H_k)| \geq 2 + |\overline{X}| > r^\star$, where the last inequality follows from (3). Thus, $r(f) > r^\star$,

Again, a contradiction. In all cases considered we derived a contradiction from the hypothesis that C and D do not cross. □

With the knowledge that cuts C and D cross, we now proceed to the final lemma. Shore Y of D contains the end s. Recall that $N := X \cap Y$, and $S := \overline{X \cup Y}$.

Lemma 2.19
Cut $\partial(S) - e$ is nontrivial and tight in $G - e$.

<u>Proof</u>: By Lemma 2.14, cut $\partial(S) - e$ is tight in $G - e$. By Lemma 2.17, when $k = 1$, $|S| \geq 2$. Thus we may restrict our attention to the case $k > 1$. Assume, if possible, that $\partial(S)$ is trivial and that x is the only vertex of S.

Let $D' := D_{k-1}$, and let $Y' := Y_{k-1}$ be the shore of D' that contains vertex s. By definition of k, Y' contains no vertex of I. We now apply Lemma 2.15 with D' playing the role of D. Two cases arise depending on whether or not C and D' cross.

Case 1: C and D' cross. In this case, $|\overline{X} \cap \overline{Y'}| \geq |I| + 3$. Thus,

$$r(f) \geq |V(H_k)| \geq 2 + |\overline{X} \cap \overline{Y'}| - 1 \geq |I| + 4 > r^\star,$$

which is absurd.

Before we move to the next case, let us observe that, up to this point, we have not made use of the proviso that e, $\mathcal{G}$, and $G^\star$ were to be selected so that, if possible, $G^\star$ had two contraction vertices subject to the condition that $r(e) = r^*$. It is only now, towards the tail end of our proof, we shall see that condition appearing, magically, to play a decisive role!

Case 2: C and D' do not cross. In this case, $Y' \subset X$, and $V(H_k)$ contains all the vertices of $\overline{X} - x$ and it also has two contraction vertices. Thus,

$$r(f) \geq |V(H_k)| \geq |\overline{X}| + 1 \geq r(e) = r^\star,$$

where the last inequality follows from (3). We deduce that equality holds throughout. In particular, $|V(H_k)| = r^\star = 1 + |\overline{X}|$. From (3), it follows that $G^\star$ has only one contraction vertex, whereas G_k has two contraction vertices. This contradicts the choice of e and $G^\star$.

In all cases considered we have derived a contradiction from the hypothesis that $\partial(S)$ is trivial. Thus, as asserted, $\partial(S) - e$ is a nontrivial tight cut of $G - e$. □

Since C and D cross, note that S is a proper subset of $\overline{X}$. By the above lemma it follows that $\partial(S) - e$ is a nontrivial tight cut of $(G - e)/X$. We may therefore conclude that $G^\star$ does indeed have two contraction vertices. Considering the interaction between the cuts C' and $\partial(S)$, we now derive property of the cut $\partial(Y)$ which will lead us to the final contradiction.

Corollary 2.20
$I \subset Y$.

<u>Proof</u>: Recall that by definition of k, and Lemma 2.15, $Y \setminus X$ contains two or more vertices in I, and $C' = \partial(X')$, where $G^\star = ((G - e)/X)/X'$.

Suppose that $\partial(S)$ and C' cross. Both $\partial(S) - e$ and $C' - e = \partial(X') - e$ are nontrivial and tight in $G - e$. Moreover, the end v of e lies in $X' \cap S$. Thus, by Lemma 2.3, $\partial(X' \cup S) - e$ is a nontrivial tight cut of $G - e$. If we let $\tilde{X}$ denote the complement of $X' \cup S$, we have that $X_{j-1} \subset \tilde{X} \subset X_j$. This implies that $\partial(\tilde{X} - e)$ is a nontrivial tight cut of the brace $G^\star = G_j$, which is absurd.

Thus, $C' = \partial(X')$ and $\partial(S)$ do not cross. The intersection of the two sets X' and S contains v, the complement of their union contains s. On the other hand, if X' is a proper subset of S, then $\partial(S) - e$ would be a nontrivial tight cut of $G^\star = ((G-e)/X)/X'$. Thus, the only possibility left is that S is a subset of X'. In this case, as X' does not contain any internal vertices of $G^\star$, we can conclude that S does not contain any internal vertices either. Therefore $I \subset Y$. □

In sum, we have shown that G has a cut $\partial(Y)$ such that $s \in Y$, $\partial(Y) - f$ is nontrivial and tight in $G-f$, $I \subset Y$ and cut $\partial(\overline{X \cup Y}) - e$ is nontrivial and tight in $G-e$. A conclusion similar to that holds for edge g: G has a cut $\partial(Z)$ such that $s \in Z$, $\partial(Z) - g$ is nontrivial and tight in $G-g$, $I \subset Z$ and cut $\partial(\overline{X \cup Z}) - e$ is nontrivial and tight in $G-e$.

Let $S := \overline{X \cup Y}$, and $T := \overline{X \cup Z}$. Both $\partial(S) - e$ and $\partial(T) - e$ are nontrivial and tight in $G-e$. Moreover, the end v of e lies in $S \cap T$. Now, let $J := S \cap T$. By Lemma 2.3, $\partial(J) - e$ is nontrivial and tight in $G-e$. As v, a vertex of B, lies in J, we deduce that $J_+ = J \cap A$. Since J is nontrivial, $|J_+| \geq 2$. Let a_1 and a_2 be any two vertices in J_+. Then,

$$\{a_1, a_2\} \subseteq \overline{Y} \cap \overline{Z} \cap A.$$

On the other hand, as $|I| \geq 4$. we may choose two vertices, b_1 and b_2, both in $I \cap B$. But I is a subset of each of Y and Z. Therefore, b_1 and b_2 both lie in $Y \cap Z \cap B$. That is,

$$\{b_1, b_2\} \subseteq Y \cap Z \cap B.$$

Since G is a brace, the graph $G - a_1 - a_2 - b_1 - b_2$ has a perfect matching, say M. As b_1 and b_2 are two vertices that lie in the majority part of Y, and since neither a_1 nor a_2 lies in Y, we deduce that edge f lies in M. (In fact, f is the only edge of M in D.) Likewise, b_1 and b_2 are two vertices that lie in the majority part of Z, and since neither a_1 nor a_2 lies in Z, we deduce that edge g also lies in M. Thus, M contains both edges f and g. This is a contradiction, because those two edges are incident with vertex s. As asserted, e is a thin edge of G. The proof of Theorem 1.13 is complete. Indeed, every brace on six or more vertices has a thin edge. □

3 Families of Graphs without Strictly Thin Edges

We shall now describe six infinite families of graphs. These families of graphs, together with the Petersen graph, include all the bricks and braces that do not contain strictly thin edges. McCuaig [7] proved this result for braces and Norine and Thomas [8] proved it for bricks. In Section 5, we shall present a combined proof of their theorems.

Odd Wheels

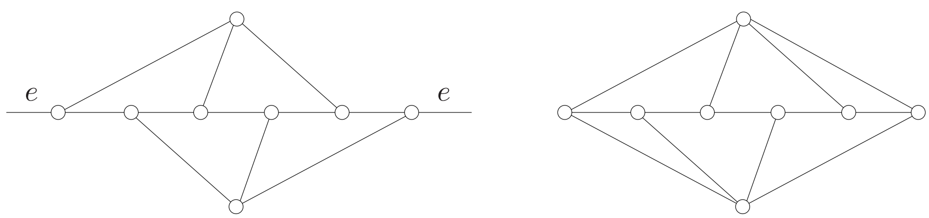
Let C_k be an odd cycle of length at least three. Then, the *odd wheel* W_k is defined to be the join of C_k and K_1. The smallest odd wheel is $W_3 \cong K_4$. The graph shown in Figure 4(a) is W_5. For $k \geq 5$, W_k has one vertex of degree k, called its *hub*; the remaining k vertices lie on a cycle which is referred to as the *rim*. Every odd wheel is a brick and none of its edges is strictly thin.

Biwheels

Let C_{2k} be an even cycle of length six or more with bipartition (X, X'), and let h and h' be two vertices (*hubs*) not on that cycle. The graph obtained by joining h to each vertex in X, and h' to each vertex in X', is known as a *biwheel* with h and h' as its hubs. Figure 11(a) shows a biwheel on eight vertices (the two half-edges labelled e are to be identified to complete the rim); it is isomorphic to the cube. Every biwheel is a brace and none of its edges is strictly thin

Truncated biwheels

Let $(v_1, v_2, \ldots, v_{2k})$ be a path of odd length, where $k \geq 2$, and let h and h' be two vertices (*hubs*) not on that path. We shall refer to the graph obtained by joining h to vertices in $\{v_1, v_3, \ldots, v_{2k-1}\} \cup \{v_{2k}\}$, and joining h' to vertices in $\{v_1\} \cup \{v_2, v_4, \ldots, v_{2k}\}$ as a *truncated biwheel*. The smallest truncated biwheel is isomorphic to $\overline{C_6}$. Figure 11(b) shows a truncated biwheel on eight vertices. Every truncated biwheel is a brick and none of its edges is strictly thin.



(a) (b)

Figure 11: (a) A biwheel, (b) a truncated biwheel.

Norine and Thomas [8] refer to truncated biwheels as prismoids. In our combined proof of the theorems of McCuaig [7] and Norine and Thomas [8], wheels, biwheels and this class of graphs arise together naturally. For this reason, we find the change of name to be appropriate.

Planar Ladders

Let $(u_1, u_2, \ldots, u_k, u_1)$ and $(v_1, v_2, \ldots, v_k, v_1)$ be two disjoint cycles of length at least three. The *planar ladder* on $2k$ vertices, also known as the *k-prism*, is the graph obtained from the union of these two cycles by joining u_i to v_i, for $1 \leq i \leq k$. (In other words, the k-prism is the Cartesian product of the k-cycle C_k and the complete graph K_2.) The 3-prism, also known as the *triangular prism* is isomorphic to $\overline{C_6}$, and the 4-prism is isomorphic to the cube. The graph shown in Figure 4(a) is the 5-prism, also known as the *pentagonal prism*. For every odd k, the k-prism is a brick and none of its edges is strictly thin. For every even k, the k-prism is a brace and none of its edges is strictly thin.

Möbius Ladders

Let $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_k)$ be two disjoint paths of length at least three. The graph obtained from the union of these two paths by joining u_i to v_i, for $1 \leq i \leq k$, and, in addition, joining u_1 to v_k, and u_k to v_1, is known as the *Möbius ladder* of order $2k$. The Möbius ladder of order six is isomorphic to $K_{3,3}$, and the Möbius ladder of order eight is shown in Figure 12(a) (this drawing is to be taken as an embedding on the Möbius strip). When k is odd, the Möbius ladder of order $2k$ is a brace and none of its edges is strictly thin. When k is even, the Möbius ladder of order $2k$ is a brick and none of its edges is strictly thin.

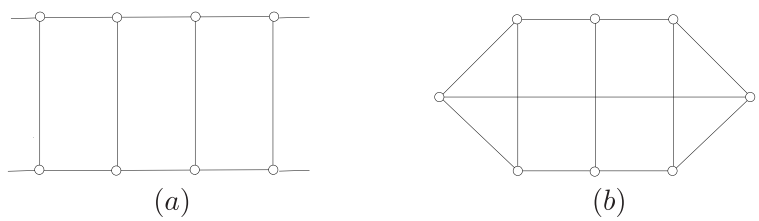


Figure 12: (a) a Möbius ladder, (b) a staircase.

Staircases

Let $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_k)$ be two disjoint paths of length at least two. The graph obtained from the union of these two paths by adjoining two new vertices x and y, and joining u_i to v_i, for $1 \leq i \leq k$, and, in addition joining x to u_1 and v_1, y to u_k and v_k, and x and y to each other, is referred to as a *staircase* by Norine and Thomas [8]. The staircase on six vertices is isomorphic to the triangular prism. The staircase on eight vertices, shown in Figure 12(b) is known to us as R_8 (see [2]); it is the only brick with precisely one removable edge. Every staircase is a brick and none of its edges is strictly thin.

4 Multiple Edges in Retracts

Let $\mathcal{G}$ denote the class of graphs consisting of the Petersen graph and members of the six infinite families described in Section 3. An *indecomposable graph* is a matching covered graph without nontrivial tight cuts. Thus, an indecomposable graph is either a brick or a brace. In the rest of this section, we shall let G denote an indecomposable graph which does not contain a strictly thin edge. Since multiple edges are strictly thin, G is simple. Our objective is to show G belongs to $\mathcal{G}$.

In the rest of this paper, we shall denote vertices which are known to be of degree three by solid dots.

As already observed, the three basic bricks K_4, $\overline{C_6}$ and P, and the brace $K_{3,3}$ are members of $\mathcal{G}$. Thus, by Theorems 1.12 and 1.13, we may assume that G has thin edges.

Let $e := uv$ be a a thin edge of G. Let H denote the retract of $G - e$. Since e is not a strictly thin edge, H must contain multiple edges. Let f and g be two multiple edges of H. Since G itself is simple, f and g must be incident in H with at least one contraction vertex (that is, one resulting from the bicontraction of an end of e).

Let us begin with a brief review of the conditions under which bicontractions of $G - e$ result in a graph with f and g as multiple edges. The simplest case arises when e is a thin edge of index one. Suppose that $e = uv$ is such as edge. Let vertex v be the end of e of degree three in G, and v_1 and v_2 be its neighbours distinct from u. Suppose that $f = v_1w$ and $g = v_2w$ are two edges incident with a common vertex w which belongs to $V(G) \setminus \{u, v_1, v_2\}$. Then f and g are multiple edges in the retract of $G - e$. See Figure 13. The degree of w is at least four in G, it is possible for u and w to be the same vertex.

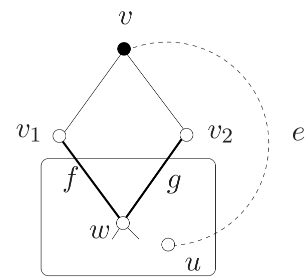


Figure 13: Multiple edges in the retract of $G - e$, index (e) =1.

Now consider the case in which $e = uv$ is a thin edge of index two. There are essentially three possible situations under which f and g become multiple edges in the retract of $G-e$. These three situations are illustrated in Figure 14. (The rectangles with rounded corners include all the non-contraction vertices in the retracts.)

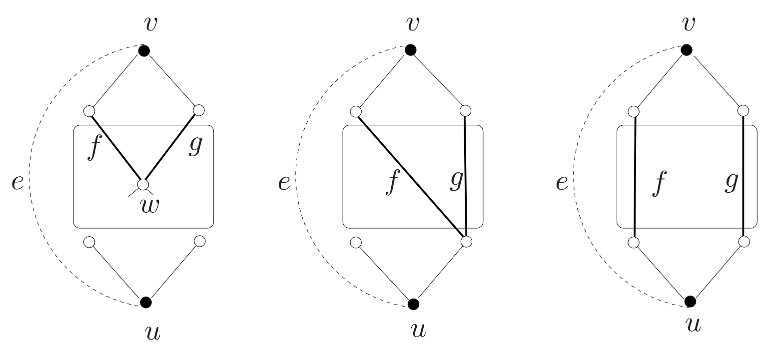


Figure 14: Multiple edges in the retract of $G - e$, index (e) =2.

Suppose now that e has a thin edge of index three, see Figure 15. If f and g are two edges which are not adjacent to e, but are incident with v_i and v_j, where $1 \le i < j \le 3$, then f and g are multiple edges in the retract of $G - e$. Two such edges are indicated by solid lines. Their common end w has degree at least four in G.

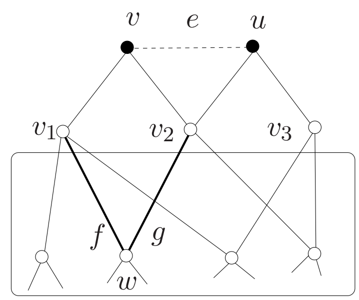


Figure 15: Multiple edges in the retract of $G - e$, index (e) =3.

Suppose now that e is a thin edge in G. What can one say about the multiple edges in the retract of $G - e$? If G is a brace, f and g are removable in G, but may or may not be thin. If G is a brick, neither f nor g may even be removable in G. We shall examine various possibilities and conclude that, in each case, the constraint that G has no strictly thin edges implies that G must contain certain configurations in the vicinity of the ends of e. Those local configurations must propogate themselves, and close up eventually, showing that G belongs to $\mathcal{G}$.

There are two lemmas which play key roles in analyzing the structure of G; one deals with the situation in which G is a brick and f is not removable in G, and the other deals with the case in which G is indecomposable (that is, either a brick or a brace), f is removable in G but is not thin. We begin by recalling a simple fact concerning bricks. An edge e of a graph G is said to *depend* on another edge f if every perfect matching containing e also contains f.

Proposition 4.1

Let G be a brick and e and f be two edges of G such that e depends on f. Then, there is a barrier B of $G - f$ such that e has both its ends in B and f has ends in different odd components of $G - f - B$. □

A stronger statement can be made in the particular case being discussed. Since $G-e-f$ has exactly one brick, if we take B to be a maximal barrier of $G - f$, then there is precisely one nontrivial odd component in $G - f - B$. Furthermore, if K denotes that nontrivial component, then $(G - e - f)/\overline{V(K)}$ is the unique brick H of $G - e$, up to multiple edges. (This follows from the fact that, when B is a maximal barrier all odd components of the graph obtained by deleting B are factor-critical.)

4.1 Key Lemmas

There is a certain type of subgraph which appears in the first lemma, which we call a house. Let F be an induced subgraph of G consisting of a pentagon with precisely one chord. Suppose that f is the chord and e is the edge of the pentagon disjoint from f. We shall refer to F as a *house* with e as the *floor* and f as the *ceiling* if the ends of both e and f have degree three in G.

Lemma 4.2

Let G be a simple brick, e a thin edge of G of index two or less, H the retract of $G-e$, f and g two multiple edges of H. If f is not removable in G then the following properties hold:

(i) *If* $\text{index}(e) = 1$ *then f is not adjacent to e, but has an end of degree three adjacent to both ends of e,*

(ii) *if* $\text{index}(e) = 2$ *then f is the ceiling of a house whose floor is e.*

Proof: Assume that f is not removable in G. As f is multiple in H, it is removable in H, and hence in $G-e$. We deduce that e is the only edge of G that depends on f. Therefore, by Proposition 4.1 and the observation following it, there exists a (maximal) barrier B of $G-f$ such that e has both ends in B and $G-f-B$ has precisely one nontrivial component K such that $(G-f-B)/\overline{(V(K)}$ is the retract H of $G-e$, up to multiple edges.

Brick G is simple, whereas its retract H has multiple edges. Therefore, the index of e is not zero. By hypothesis, it is two or less. Therefore, $\text{index}(e) \in \{1, 2\}$. If the index of e is one then K has $n-3$ vertices, whereas if the index of e is two then K has $n-5$ vertices.

Consider first the case in which $\text{index}(e) = 1$. In this case, K has $n-3$ vertices, and B has two vertices. Thus $B = \{u, v\}$, and f is clearly not adjacent to e. Furthermore, the end of f not in K has degree three and is adjacent to both ends of e. (See Figure 16(a).) Thus the assertion holds when the index of e is one.

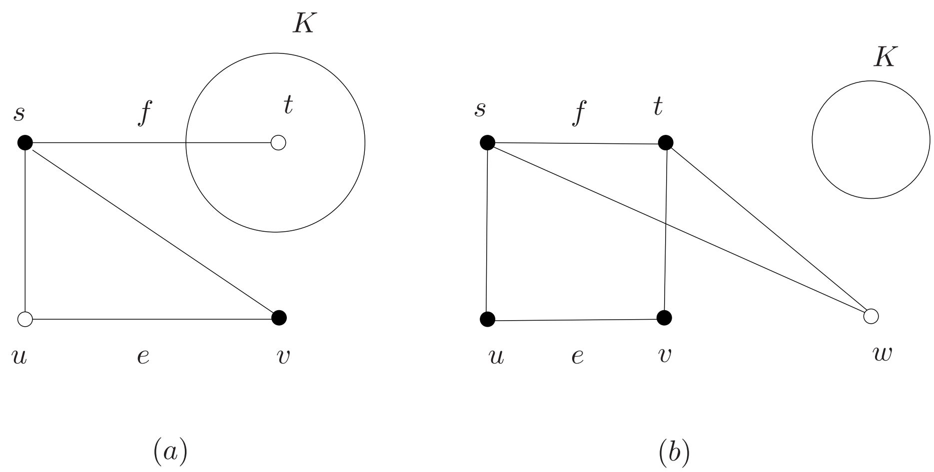


Figure 16: Lemma 4.2: (a) index of e is one, (b) index of e is two.

Consider next the case in which $\text{index}(e) = 2$. In this case, K has precisely $n-5$ vertices and B has precisely three vertices. Two of the vertices of B are the ends u and v of e, let w denote the third vertex of B. We claim that no end of f lies in K. To see this, assume the contrary. Let s denote the isolated vertex of $G-B$. Then, $N(s) = B$. In particular, s is adjacent to both ends of e (see Figure 16(a)). We have assumed e to have index two, but its ends have a common neighbour. This is not possible. Thus, as claimed, no end of f lies in K.

Let s and t denote the two ends of f (see Figure 16(b)). Vertex s has degree three or more, therefore s is adjacent to at least one end of e. The ends of e have no common neighbour, otherwise the index of e would be three. Therefore, s is adjacent to precisely one of u and v. Moreover, s has degree three and is adjacent to w. Likewise, t has degree three and is adjacent to w and to precisely one end of e. Adjust notation so that s is adjacent to u. If t is also adjacent to u, then the three neighbours of u would be s, t and v, implying that $\{v, w\}$ is a 2-vertex cut of the brick G. We deduce that t is adjacent to v. Consequently, the subgraph of G spanned by the three vertices of B and the two ends of f is a house in G with f as its ceiling and e as its floor, as asserted. □

Lemma 4.3

Let G be a simple brick on eight or more vertices, or a simple brace on ten or more vertices, $e = uv$ a thin edge of G, H the retract of $G - e$, f and g two multiple edges of H. Assume that f is removable in G but it is not thin. Then, the following properties hold:

(i) index$(e) = 2$,

(ii) *edges f and g have a common end that is not adjacent to any end of e,*

(iii) *edge g is thin in G.*

Proof: Since $e = uv$ is a thin edge which is not strictly thin, the index of e is either one, two or three. The number of vertices in the retract of $G - e$ is determined by the index of e. If that index is one, then H is either a brick or a brace on $n - 2$ vertices. Furthermore, any tight cut decomposition of $G - e$ has H and a C_4 (up to multiple edges) as its two members. If the index of e is two, then H is either a brick or is a brace on $n - 4$ vertices. Furthermore, any tight cut decomposition of $G - e$ consists of H and two C_4's (up to multiple edges), and H has two contraction vertices. Finally, when e has index three, H is a brick on $n - 4$ vertices. Any tight cut decomposition of $G - e$ has H and two C_4's (up to multiple edges), and H has only one contraction vertex. When the index of e is two or three, no end of e lies in H (see Figures 14 and 15). Also, regardless of the index of e, the retract H of $G - e$ is not C_4.

Since f and g are multiple edges in H, the edge f is removable in $G - e$ and, up to multiple edges, tight cut decompositions of $G - e$ and and $G - e - f$ yield the same graphs. By the hypothesis of the lemma, $G - f$ is matching covered. If $\partial(X) - f$ is a tight cut of $G - f$, then $\partial(X) - e - f$ is a tight cut of $G - e - f$, Thus, one may start with a tight cut decomposition of $G - f$ and refine it to obtain a tight cut decomposition of $G - e - f$. The upshot of these observations is that the members of any fixed tight cut decomposition $\mathcal{F}$ of $G - f$ satisfy one of the following properties:

(i) There is a member F of $\mathcal{F}$ which is of order n. In this case, $F = G - f$ is the only member of $\mathcal{F}$.

(ii) There is a member F of $\mathcal{F}$ which is of order $n - 2$. In this case, $\mathcal{F}$ will have just one more member and that is C_4 up to multiple edges.

(iii) There is a member F of $\mathcal{F}$, with two contraction vertices, which is of order $n-4$. In this case, apart from F, $\mathcal{F}$ will have two other members and they are both C_4's up to multiple edges.

(iv) There is a member F of $\mathcal{F}$, with just one contraction vertex, which is of order $n-4$. In this case, corresponding to that contraction vertex, there is a nontrivial cut $C := \partial(X)$ in G such that $C-f$ is a tight in $G-f$, where $F = (G-f)/\overline{X}$ and $F' := (G-f)/X \to x$ is a bipartite matching covered graph on six vertices.

In the first case, f is a thin edge of index zero, in the second it is a thin edge of index one, and in the third it is a thin edge of index two. Since, by hypothesis, f is not thin, we may eliminate the first three cases and focus just on the fourth. In that case, the retract H of $G-e$ also has $n-4$ vertices, implying that the index of $G-e$ is greater than one, and that no end of e is in H. Since one may refine any tight cut decomposition of $G-f$ to obtain a tight cut decomposition of $G-e-f$, it follows that F and H are the same up to multiple edges.

Now consider the bipartite matching covered graph $F' = (G-f)/X$ on six vertices. Let (A, B) denote the bipartition of that graph, where the contraction vertex x lies in A. The edge f must have both its ends in A; otherwise, B would be a nontrivial barrier of G. Also, as observed in the above paragraph, e is in F' and is not incident with the contraction vertex x. Then, edge e has one end, say u, in $A-x$, the other, say v, in B. As the index of e is greater than one, edges e and f are not adjacent. Consequently, f has one end, say v_1, in $A-u-x$, and the other end, say w, in X. Since $F' = (G-f)/X$ is bipartite, we note that $N(v_1)$ is a subset of $B \cup \{w\}$.

Graph F' has six vertices, therefore B has three vertices, all adjacent to u. Let s and t denote the two vertices of $B-v$. No vertex of G may thus be adjacent to both u and v. Consequently, index$(e) \neq 3$. We deduce that index$(e) = 2$, as asserted. See Figure 17.

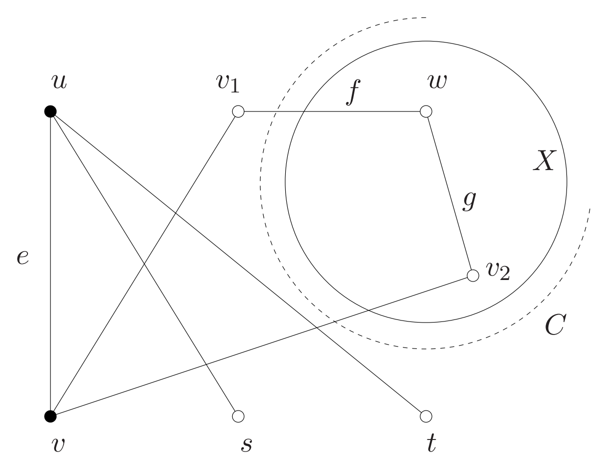


Figure 17: Edge f is removable but not thin in G.

The degree of v is three. Therefore, in F', v is adjacent to u, v_1, and the contraction vertex x. Thus, in G, v has a neighbour v_2 in X. Since f is not incident with any neighbour of u, the only way f and g can be multiple edges in H is for g to be incident with the v_2, and for f and g to have a common end not adjacent to an end of either u or v. That common end clearly has to be w.(See Figure 14.)

As both ends of g belong to X, there cannot be a house with e as the floor and g as the ceiling. Therefore, by Lemma 4.2, edge g is removable in G. Assume, to the contrary, that g is not thin. We may then apply to g the argument that we applied to f, and conclude that $N(v_2)$, just like $N(v_1)$, is a subset of $B \cup \{w\}$. In $G - B - w$, the vertices u, v_1 and v_2 are isolated, therefore $B \cup \{w\}$ is a barrier of G. This is a contradiction, because, no brick, or a brace on ten or more vertices, can have a barrier of size four. Thus, g is thin. □

The Parameter $\mathrm{index}_\star(G) = 3$

Thus far, we have discussed thin edges of indices one and two. It turns out that in proving the main theorem, thin edges of index three are easy to deal with. To explain the property which distinguishes them, it is convenient to introduce a new parameter. For G a brick or a brace having thin edges, let $\mathrm{index}_\star(G)$ denote the lowest index of a thin edge of G. Clearly, if $\mathrm{index}_\star(G) = 3$, then G is a brick. (The brick in Figure 4(c) is a brick with $\mathrm{index}_\star = 3$.)

LEMMA 4.4
Let G be a brick. If $\mathrm{index}_\star(G) = 3$ *then every thin edge of G is strictly thin.*

Proof: Let e be a thin edge of G of index three. Let H be the retract of $G - e$. Assume, to the contrary, that H is not simple. Let f and g denote two multiple edges of H. All the ends of e and its neighbours are contracted to a single vertex in the case where $\mathrm{index}(e) = 3$. Therefore, f and g have a common end, w, not adjacent to any end of e. Graph H is a brick, therefore vertex w is adjacent in H to three or more vertices. We deduce that w has degree four or more in G.

Let M be a perfect matching of G that contains edge e. As f and g are adjacent in G, it follows that at most one of f and g lies in M. Adjust notation so that f does not lie in M. As f is removable in H, it is also removable in $G - e$. Since the perfect matching M contains e but not f, we may conclude that, in fact, f is removable in G itself. By Lemma 4.3, edge f is thin in G. But the end w of f has degree four or more, therefore the index of f is one or less. This is a contradiction to the hypothesis that $\mathrm{index}_\star(G) = 3$. We conclude that e is strictly thin. This conclusion holds for each thin edge of G. □

In view of the above lemma, we may now restrict our attention to indecomposable graphs with $\mathrm{index}_\star = 1$ or 2 which have thin edges, but are free of strictly thin edges. The structure of such a graph neatly depends on the value of that parameter. We divide the proof accordingly.

4.2 Graphs G with $\mathrm{index}_\star(G) = 1$

In this case, odd wheels, biwheels and truncated biwheels emerge as the families of indecomposable graphs without strictly thin edges. The following lemma establishes the existence of a certain configuration in the vicinity of thin edges of index one in graphs with $\mathrm{index}_\star = 1$. It is a simple consequence of Lemmas 4.2 and 4.3. We shall state it in notation that is convenient for describing the iteration of that configuration.

Lemma 4.5
Let G be a brace on ten or more vertices, or a brick with $\mathrm{index}_\star(G) = 1$. Assume that G is free of strictly thin edges. Let $e := hv_2$ be a thin edge of index one of G, where v_2 is the end of e with degree three. Let v_1 and v_3 denote the neighbours of v_2 distinct from h. Then:

- *Vertices v_1 and v_3 have both degree three in G and there is a vertex h' of degree four or more, possibly equal to h, that is adjacent to both v_1 and v_3 (Figure 18). Moreover:*
  - *if $f := h'v_1$ is not removable in G, then h and h' are distinct and v_1 is adjacent to h.*
  - *if $f := h'v_1$ is removable in G, then it is thin of index one.*

(Similar statements also apply to $f' := h'v_3$.)

Proof: Let H denote the indecomposable graph obtained from $G - e$ by the bicontraction of v_2. By hypothesis, e is not strictly thin. Therefore, H has multiple edges. This implies that G has a vertex h', possibly equal to h, but distinct from v_2, that is adjacent to both v_1 and v_3.

Let us now prove that vertex h' has degree four or more in G. If G is a brick then h' is adjacent to three or more vertices in H. If G is a brace, then G has ten or more vertices, therefore H is a brace on six or more vertices. In both alternatives, h' is adjacent to three or more vertices in H. We conclude that h' has degree four or more in G.

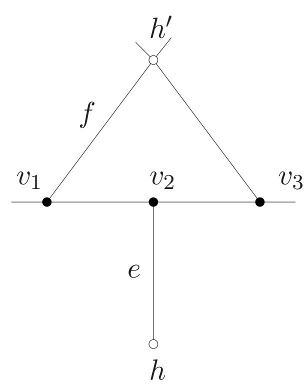


Figure 18: Illustration for Lemma 4.5.

To complete the proof, we consider separately two cases, depending on whether or not $f = h'v_1$ is removable in G. In the case in which f is not removable, the desired conclusion follows straightforwardly from Lemma 4.2. Thus, suppose that f is removable in G. By Lemma 4.3, if the edge f were not thin, index of e would have to be two. Thus, f is thin in G. By hypothesis, G is free of strictly thin edges. Therefore, index$(f) > 0$. One end of f, namely h', has degree four or more. We deduce that the other end v_1 of f has degree three, and that edge f has index one. By applying similar argument to the edge $f' = h'v_3$, we may conclude that v_3 also has degree three. □

Theorem 4.6

Let G be a brace on ten vertices or more, or be a brick. Suppose that G is free of strictly thin edges. If $\text{index}_\star(G) = 1$, *then G is either an odd wheel, a biwheel or a truncated biwheel.*

Proof: By hypothesis, G has a thin edge $e := hv_2$ of index one. Adjust notation so that v_2 has degree three. Then, h has degree four or more. Let v_1 and v_3 be the two neighbours of v_2 distinct from h. By Lemma 4.5, G has a vertex h' of degree four or more, possibly equal to h, that is adjacent to both vertices v_1 and v_3. Moreover, vertices v_1 and v_3 have both degree three.

Let thus $P := (v_1, \ldots, v_t)$, $t \geq 3$, be a path of maximum length in $G - h - h'$ that has the following properties (see Figure 19):

(i) Every vertex v_i of P has degree three, and is adjacent to h if i is even, and to h' if i is odd.

(ii) For every internal vertex v_i of P, the edge of G that joins v_i to one of h and h' is thin of index one.

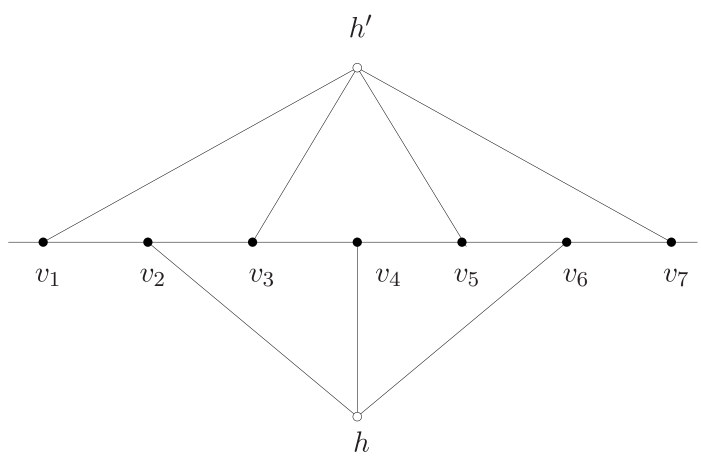


Figure 19: An illustration for Theorem 4.6.

Let $f := h'v_1$, and $f' := xv_t$, where $x = h'$ if t is odd and $x = h$ if t is even. To draw the desired conclusion, we shall now consider the status of the edges f and f'.

Consider first the case in which f is removable in G. By Lemma 4.5, we deduce that f is a thin edge of index one. Also, by Lemma 4.5, with f playing the role of e, we deduce that the neighbours of v_1, distinct from h', have degree three in G. Thus v_1 has a neighbour v_0 of degree three in G which is different from v_2. By the maximality of P, vertex v_0 lies in $V(P)$. All the vertices of P have degree three. Therefore, v_0 cannot be an internal vertex of P. We deduce that $v_0 = v_t$. Hence the subgraph of G induced by $V(P)$ is a cycle. Moreover, every vertex of P is adjacent only to vertices of $V(P) \cup \{h, h'\}$. Graph G, a brick or a brace on more than four vertices, is 3-connected. Therefore, $V(G) = V(P) \cup \{h, h'\}$. If t is odd then $h = h'$ and G is an odd wheel, on six or more vertices, with h as its hub. If t is even then h and h' are distinct and G is a biwheel, on ten or more vertices, with h and h' as its two hubs, In both alternatives, edge f' is also removable in G.

We may thus assume that neither f nor f' is removable in G. By applying Lemma 4.5, and using the fact that f is not removable, we deduce that h and h' are distinct and v_1 is joined to both h and h'. Now, let $e' = yv_{t-1}$, where $y = h$ when t is odd and $y = h'$ when t is even. Using the fact that f' is not removable and applying Lemma 4.5, with e' playing the role of e, we deduce that v_t is also joined to both h and h'. Moreover, every vertex of P is adjacent only to vertices of $V(P) \cup \{h, h'\}$. Graph G, a brick on more than four vertices, is 3-connected. Therefore, $V(G) = V(P) \cup \{h, h'\}$. As h and h' are distinct, it follows that t is even. Then, as the degrees of h and h' are equal and greater than three, it follows that $t \geq 6$. We deduce that G is a truncated biwheel. □

4.3 Thin Edges of Index Two

Now we turn to indecomposable graphs in which there are no strictly thin edges, and no thin edges of index one. Every such graph turns out to be either a prism, or a Möbius ladder, or a staircase. As in the last section, we start with a lemma which establishes the existence of a certain configuration in the vicinity of every thin edge of index two.

Lemma 4.7
Let G be a brace on ten vertices or more, or a brick. Suppose that $\text{index}_\star(G) = 2$ *and that G is free of strictly thin edges. Let $e := u_2v_2$ be a thin edge of index two. (Then u_2 and v_2 have degree three in G, and are at distance greater than two in $G - e$.) Let $N(u_2) := \{u_1, v_2, u_3\}$ and $N(v_2) := \{v_1, u_2, v_3\}$. Then, each vertex in $\{u_1, u_3, v_1, v_3\}$ has degree three. Moreover, there are precisely two edges joining vertices in $\{u_1, u_3\}$ to vertices in $\{v_1, v_3\}$ and those two edges are nonadjacent (see Figure 20(c)).*

<u>Proof</u>: Let H be the retract of $G - e$. By hypothesis, e is not strictly thin. Therefore, H has multiple edges. Let f and g denote two multiple edges of H. Then, both edges have ends that are adjacent to the same end of e. Adjust notation so that f is incident with vertex u_1, whereas g is incident with vertex u_3.

We assert that if f and g have a common end then that common end is adjacent to v_2. For this, assume the contrary. That is, assume that w is a common end of f and g that

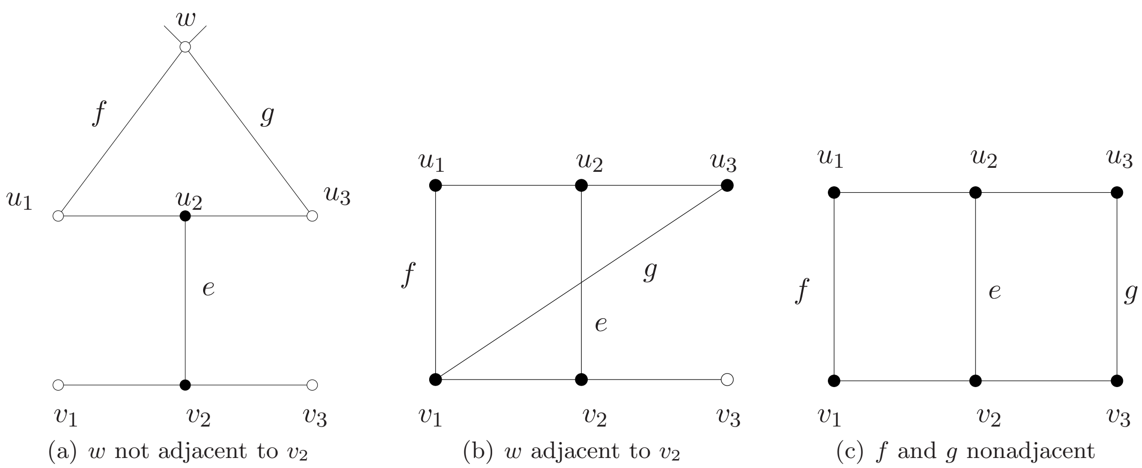


Figure 20: Illustration for Lemma 4.7.

is not adjacent to v_2 (see Figure 20(a)). Then, w is not adjacent to any end of e. In H, a brace on six or more vertices, or a brick, vertex w is adjacent to three or more vertices. Therefore, w has degree four or more in G. Let M be a perfect matching of G that contains edge e. As f and g are adjacent, it follows that M does not contain both edges. Adjust notation so that f does not lie in M, Then, f is removable in G. By Lemma 4.3, one of f and g is thin in G. But vertex w, the common end of those two edges, has degree four or more. Consequently, the index of whichever of f and g is thin is equal to one, or less. This is a contradiction to the hypothesis that $\text{index}_\star(G) = 2$. As asserted, a common end of f and g, if any, is adjacent to v_2.

Adjust notation so that f joins vertex u_1 to vertex v_1. Let us now prove that both ends of f have degree three. For this, consider first the case in which f is not removable in G. Since the index of e is two, no vertex of G is adjacent to both ends of e. By Lemma 4.2, both ends of f have degree three. Alternatively, if f is removable in G then, as both ends of f are adjacent to ends of e, it follows, by Lemma 4.3, that f is thin in G: in that case, as $\text{index}_\star(G) = 2$, it follows that both ends of f have degree three. In both alternatives, we deduce that both ends of f have degree three. Likewise, both ends of g have degree three.

Assume, to the contrary, that f and g are adjacent. Then, g joins u_3 to v_1 (see Figure 20(b)). In that case, as v_1 has degree three, it follows that $N(v_1) = \{u_1, v_2, u_3\} = N(u_2)$. Therefore, $N(u_2)$ is a barrier of G, where vertices v_1 and u_2 are isolated in $G - N(u_2)$. This conclusion implies that G is a brace on six vertices, a contradiction to the hypothesis that if G is a brace then it has ten or more vertices. Indeed, f and g are nonadjacent. Moreover, all their ends have degree three. This conclusion holds for each pair $\{f, g\}$ of multiple edges of H. Therefore, there is precisely one such pair, and the two edges are nonadjacent, as asserted. □

Theorem 4.8

Let G be a brace on ten or more vertices, or a brick. Assume also that G is free of strictly *thin edges. If* $\text{index}_\star(G) = 2$ *then* G *is either a prism, a Möbius ladder or a staircase.*

Proof: By hypothesis, G has a thin edge u_2v_2 of index two. Let u_1 and u_3 denote the two neighbours of u_2 distinct from v_2, let v_1 and v_3 denote the two neighbours of v_2 distinct from u_2. By hypothesis, e is not strictly thin. Therefore, by Lemma 4.7, the vertices in $\{u_1, u_3, v_1, v_3\}$ all have degree three. Moreover, there are precisely two edges that join u_1 and u_3 to v_1 and v_3, and the two edges are nonadjacent. Adjust notation so that u_1 is adjacent to v_1 and u_3 to v_3.

Let $P := (u_1, u_2, \ldots, u_t)$ and $Q := (v_1, v_2, \ldots, v_t)$, $t \geq 3$, be two disjoint paths in G of maximum length, such that for $i = 1, 2 \ldots, t$, the vertices u_i and v_i are adjacent and both have degree three, and, if $1 < i < t$ then edge u_iv_i is thin of index two (see Figure 21.).

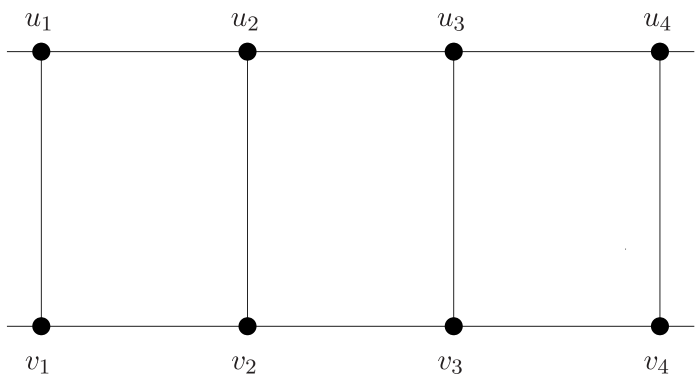


Figure 21: Illustration for Theorem 4.8.

Consider first the case in which at least one of u_1v_1 and u_tv_t is removable in G. Adjust notation so that u_1v_1 is removable in G. By Lemma 4.3, u_1v_1 is thin in G. The index of u_1v_1 is two or three, because $\text{index}_\star(G) = 2$. But the index of u_1v_1 cannot be three, otherwise there would be a vertex y adjacent to both u_1 and v_1, and consequently edge u_2v_2 would depend on edge u_1v_1, a contradiction. Indeed, u_1v_1 is thin of index two.

By Lemma 4.7, vertex u_1 has a neighbour u_0 distinct from u_2 and v_1, and v_1 has a neighbour v_0 distinct from u_1 and v_2, such that u_0 and v_0 are adjacent and have degree three. By the maximality of t, it follows that $\{u_0, v_0\} = \{u_t, v_t\}$. If $u_0 = u_t$ then $v_0 = v_t$ and G is a prism. If $u_0 = v_t$ then $v_0 = u_t$ and G is a Möbius ladder. In both alternatives, u_tv_t is also removable in G.

We may thus assume that neither u_1v_1 nor u_tv_t is removable in G. By Lemma 4.2, with u_2v_2 playing the role of e, we have that G has a vertex w_1 that, together with vertices u_1, u_2, v_1, v_2, span a house, where u_1v_1 is the ceiling and u_2v_2 the floor. Likewise, G has a vertex v_t which is adjacent to both u_t and v_t. Graph G, a brick or a brace, is 3-connected. Therefore, $V(G) = V(P) \cup V(Q) \cup \{w_1, w_t\}$. Moreover, as G has an even number of vertices, it follows that w_1 and w_t are distinct. Finally, w_1 and w_t must be adjacent, in order to have minimum degree three. We conclude that G is a staircase. □

5 The Main Result

The following result combines the results of McCuaig [7] and Norine and Thomas [8].

Theorem 5.1
Let G be a brick or a brace on six or more vertices. If G has no strictly thin edge then G is either the Petersen graph or is an odd wheel, a biwheel, a truncated biwheel, a prism, a Möbius ladder or a staircase.

Proof: If G has no thin edges then G is either K_2, C_4, K_4, $\overline{C_6}$ or the Petersen graph. The graph K_4 is an odd wheel and $\overline{C_6}$ a prism.

We may thus assume that G has thin edges. If $\text{index}_\star(G) = 3$ then, by Lemma 4.4, every thin edge of G is strictly thin. Thus, $\text{index}_\star(G) \leq 2$. As G has no strictly thin edges, its thin edges have positive index. Thus, $\text{index}_\star(G) \in \{1, 2\}$.

The only simple brace on six vertices is $K_{3,3}$, a Möbius ladder. Therefore we may assume that if G is a brace then it has eight or more vertices. Every brace on eight vertices other than the cube has a thin edge of index zero. The cube is a prism. We may thus assume that if G is a brace then G has ten vertices or more.

If $\text{index}_\star(G) = 1$ then, by Theorem 4.6, G is either an odd wheel, a biwheel or a truncated biwheel. If $\text{index}_\star(G) = 2$ then, by Theorem 4.8, G is either a prism, a Möbius ladder or a staircase. □

We conclude the paper with a brief description of generating procedures for simple bricks and braces. These involve expansion operations described in our paper [3] (for bricks), and McCuaig [7] (for braces).

Let H be a simple brick, and suppose that G is a brick with some strictly thin edge e of index i, such that H is the retract of $G - e$. Then G is said to be obtained from H by an *expansion of type i*. For example, an expansion of type zero simply involves adding to H an edge joining two nonadjacent vertices. An expansion of type one consists of first obtaining a graph H' from H by splitting a vertex x of degree four or more in H into two vertices x_1 and x_2, such that they each have at least two distinct neighbours in H', and then adding a new vertex y to H' and joining it to both x_1 and x_2 and to a vertex z in $H - x$. If G denotes the resulting graph and e the edge yz, then it is easy to verify that G is a brick, e is a strictly thin edge in G, and H is the retract of $G - e$. Detailed descriptions of expansions of bricks of type two and three (corresponding to thin edges of index two and three) can be found in [3].

Theorem 5.1 now implies the following result of Norine and Thomas [8].

Theorem 5.2
Given any simple brick G, there exists a sequence $G_1, G_2, \ldots, G_k$ of simple bricks such that: (i) G_1 is either an odd wheel, or a truncated biwheel, or a prism, or a Möbius ladder, or a staircase, or the Petersen graph, and $G_k = G$, and (ii) for $2 \leq i \leq k$, G_i is obtained from G_{i-1} by an expansion of type zero, one, two, or three.

Theorem 5.1 also implies the following result McCuaig [7] (descriptions of expansions of braces of types zero, one and two can be found in the same paper).

THEOREM 5.3
Given any simple brace G, there exists a sequence $G_1, G_2, \ldots, G_k$ of simple bricks such that: (i) G_1 is either an biwheel, or a prism, or a Möbius ladder, and $G_k = G$, and (ii) for $2 \leq i \leq k$, G_i is obtained from G_{i-1} by an expansion of type zero, one, or two. □

6 Splitter Versions

As noted earlier, Norine and Thomas [8] and McCuaig [7] proved generalizations of Theorems 5.2 and 5.3, respectively. They refer to those generalizations as 'splitter-versions', as they are motivated by a theorem on 3-connected graphs, due to P.D. Seymour, which known as the splitter theorem. For the sake of clarity, we shall restrict ourselves here to bricks.

A matching covered subgraph F of a matching covered graph G is a *conformal subgraph* of G if the graph $G - V(H)$ has a perfect matching. (By convention, the null graph has a perfect matching.) Conformal subgraphs are variously known in the literature as nice subgraphs ([6]), central subgraphs ([8]) or well-fitted subgraphs ([7]).

A brick H is a *matching minor* of another brick G if it can be obtained from a conformal subgraph of G by means of a sequence of bicontractions of vertices of degree two. For example, a well-known theorem of Lovász and Plummer [6] states that every brick contains either K_4 or $\overline{C_6}$ as a matching minor.

For stating the generalization of Theorem 5.2 proved by Norine and Thomas [8], we need to define a new class of graphs. A *prismoid* is either a truncated biwheel or one obtained from it by adding an edge joining its two hubs.

THEOREM 6.1
Let G be a simple brick which is not an odd wheel, a prismoid, planar ladder, Möbius ladder, a staircase, or the Petersen graph, and let $H \neq K_4$, $\overline{C_6}$ be any matching minor of G. Then there there exists a sequence $G_1, G_2, \ldots, G_k$ of simple bricks such that: (i) $G_1 = H$, and (ii) for $2 \leq i \leq k$, G_i is obtained from G_{i-1} by an expansion of type zero, one, two, or three.

An analogue of the above theorem for braces, due to McCuaig [7] is also described in [8]. the proofs of these theorems, given there are based on ear decompositions of matching covered graphs.

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References

[1] J. A. Bondy and U. S. R. Murty. *Graph Theory.* Springer, 2008.

[2] M. H. de Carvalho, C. L. Lucchesi, and U. S. R. Murty. On a conjecture of Lovász concerning bricks. II. Bricks of finite characteristic. *J. Combin. Theory Ser. B*, 85:137–180, 2002.

[3] M. H. de Carvalho, C. L. Lucchesi, and U. S. R. Murty. How to build a brick. *Discrete Math.*, 306:2383–2410, 2006.

[4] J. Edmonds, L. Lovász, and W. R. Pulleyblank. Brick decomposition and the matching rank of graphs. *Combinatorica*, 2:247–274, 1982.

[5] L. Lovász. Matching structure and the matching lattice. *J. Combin. Theory Ser. B*, 43:187–222, 1987.

[6] L. Lovász and M. D. Plummer. *Matching Theory*. Number 29 in Annals of Discrete Mathematics. Elsevier Science, 1986.

[7] W. McCuaig. Brace generation. *J. Graph Theory*, 38:124–169, 2001.

[8] S. Norine and R. Thomas. Generating bricks. *J. Combin. Theory Ser. B*, 97:769–817, 2007.

[9] C. Thomassen. Kuratowski's theorem. *J. Graph Theory*, 5:225–241, 1981.

[10] C. Thomassen. Plane representations of graphs. In J. A. Bondy and U. S. R. Murty, editors, *Progress in Graph Theory*, pages 46–69. Academic Press, 1984.

[11] W. T. Tutte. *Connectivity in Graphs*. Number 15 in Mathematical Expositions. University of Toronto Press, Toronto, 1961.