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Abstract

In this paper we introduce the concept of *k-flow-critical* graphs. These are graphs that do not admit a k-flow but such that any smaller graph obtained from it by contraction of edges or of sets of vertices is k-flowable. Throughout this paper we will refer to k-flow-critical graphs simply as *k-critical*.

Any minimum counterexample for Tutte's 3-Flow and 5-Flow Conjectures must be 3-critical and 5-critical, respectively. Thus, any progress towards establishing good characterizations of k-critical graphs can represent progress in the study of these conjectures. We present some interesting properties satisfied by k-critical graphs discovered recently.

1 Introduction

For a set E of edges of a graph G, the graph G/E is the graph obtained from G by contracting all edges in E. Similarly, $G - E$ is the graph obtained from G by removing all edges in E. When E contains one single edge e we simply denote by G/e or $G - e$ the graph resulting from contracting or removing edge e. For a pair of distinct vertices u and v of G, G_{uv} is the graph obtained from G by contracting the set $\{u, v\}$ to a single vertex. Graph $G + uv$ is the graph obtained from G by the addition of edge uv. All these definitions can be extended to a digraph D. In this case, graph $D + uv$ is an extension of D in which edge uv is directed from u to v.

Let k be an integer, $k > 2$. Let D be a digraph and f a mapping $f : E(D) \to \mathbb{Z}$. Let X be a set of vertices of D and the *cut of* X, denoted by ∂X, the set of edges of D having precisely one end in X. We define the *outflow at* X as the sum of the weights of the edges in ∂X leaving X minus the sum of the weights of the edges in ∂X entering X. In the particular case in which X has one single vertex v we say it is the *outflow at* v and denote it by $f(v)$. The outflow at X is equal to by the sum of the outflows at the vertices in X.

Mapping f is a *mod-k-flow* of D if the following properties are satisfied:

(i) For any edge e, $f(e)$ is not a multiple of k.

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(ii) Every vertex v is *balanced modulo* k, i.e., $f(v) \equiv 0 \pmod{k}$.

A mod-k-flow f is a *k-flow* of D if two more properties are satisfied:

(iii) For any edge e, $0 < f(e) < k$.

(iv) Every vertex is *balanced*, i.e., $f(v) = 0$.

Mapping f is a *near-mod-k-flow that misses u and v* if property (i) is satisfied and balance modulo k is achieved at every vertex, except precisely vertices u and v.

If a mapping f satisfies properties (i) and (iii) and all but possibly one vertex v are balanced, then f must be a k-flow. Recall that the outflow at v is the complement of the outflow at $X := V(D) - v$. The outflow at $X := V(D) - v$ is 0, then v must be balanced as well. Similarly, if a mapping f satisfies property (i) and all but possibly one vertex v are balanced modulo k, then f must be a mod-k-flow. Using a similar argument we conclude that when f is a near-mod-k-flow that misses u and v and the outflow at u is x, then the outflow at v must be $-x$. In order to specify the outflow at u (and consequently at v) we say f is a *near-mod-k-flow that misses u and v by x*.

Lemma 1.1 *Let D be a graph, f a near-mod-k-flow that misses two vertices u and v of D. Then, the graph $D + uv$ has a mod-k-flow.*

Proof: Extend f to a mod-k-flow f' of $D + uv$ by assigning to edge $e = uv$ the integer $f'(e)$ that balances u modulo k. That balances v modulo k as well. □

When f is either a k-flow or a mod-k-flow then, for every set X, the outflow at X is 0 or 0 mod k, respectively. Thus, D cannot have a cut-edge.

We say a graph G has a k-flow if there is a k-flow for some orientation D of G. Similarly, we say that G has a mod-k-flow if there is a mod-k-flow for some orientation D of G. The following result is well known. Proofs of Theorem 1.2, due to Tutte, can be found in papers by Younger [8] and Seymour [5], and in a book by Zhang [9].

Theorem 1.2 *A graph G has a k-flow if and only if G has a mod-k-flow.*

It should be noted, however, that there is a mod-k-flow f for some orientation D of G if and only if there is a mod-k-flow f' for every orientation D' of G. If D and D' differ on a set E' of edges, we obtain f' by simply stating that $f'(e) \equiv -f(e) \pmod{k}$ for all edges in E' and $f'(e) = f(e)$ for the remaining edges.

A graph G is *edge-k-critical* if it does not admit a k-flow but G/e admits a k-flow, for every edge e of G. Similarly, we say that graph G is *vertex-k-critical* if it does not admit a k-flow but G_{uv} admits a k-flow, for every pair of distinct vertices u and v of G. These definitions can be extended to digraphs. A digraph D is *edge-k-critical* or *vertex-k-critical*, respectively, if its underlying undirected graph is edge-k-critical or vertex-k-critical.

Every loopless vertex-k-critical graph must be edge-k-critical. A graph G that is edge-k-critical must have exactly one non-trivial connected component. This non-trivial component of G must be edge-k-critical itself. It is easy to see that every vertex-k-critical graph is connected. So, there are edge-k-critical graphs that are not vertex-k-critical, but all examples we know are disconnected.

Conjecture 1.3 *Every connected edge-k-critical graph is vertex-k-critical.*

Graph K_4 is an example of a vertex-3-critical graph. Actually, every odd wheel is a vertex-3-critical graph, as shown on Section 5. Other examples of vertex-3-critical graphs are shown on Figure 1. The Petersen graph is vertex-4-critical. Many other snarks are vertex-4-critical. Examples are Blanuša, Loupekhine, Celmins-Swart, double-star and Szekeres snarks, and flower-snarks J_n for n odd and $5 \leq n \leq 15$ [1]. We used a characterization of vertex-k-critical graphs demonstrated on Section 3 to implement a computer program that checks whether a graph is vertex-k-critical for $k = 3$ or $k = 4$. This program runs in exponential time.

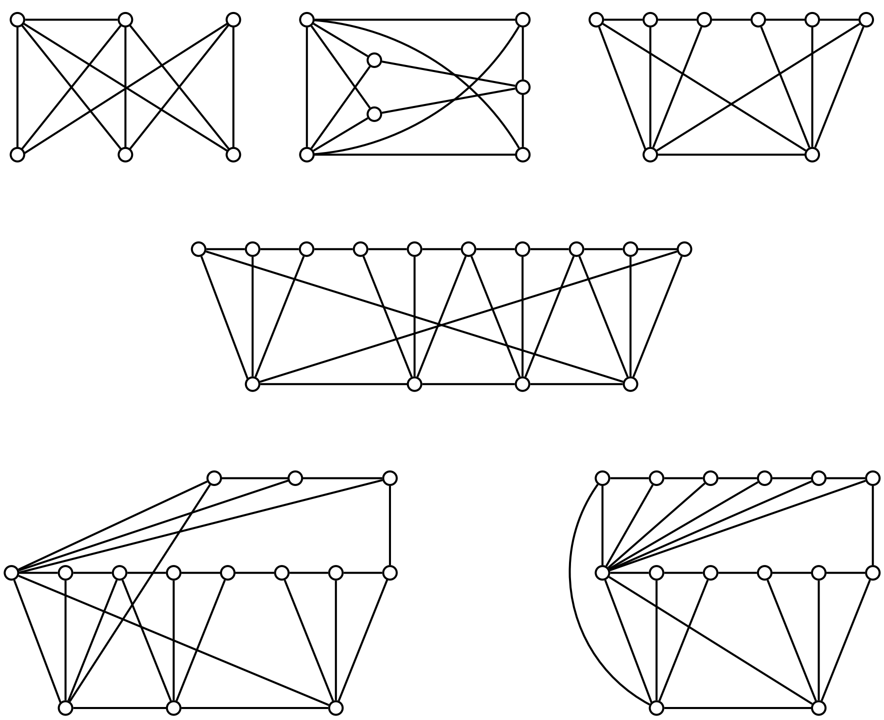
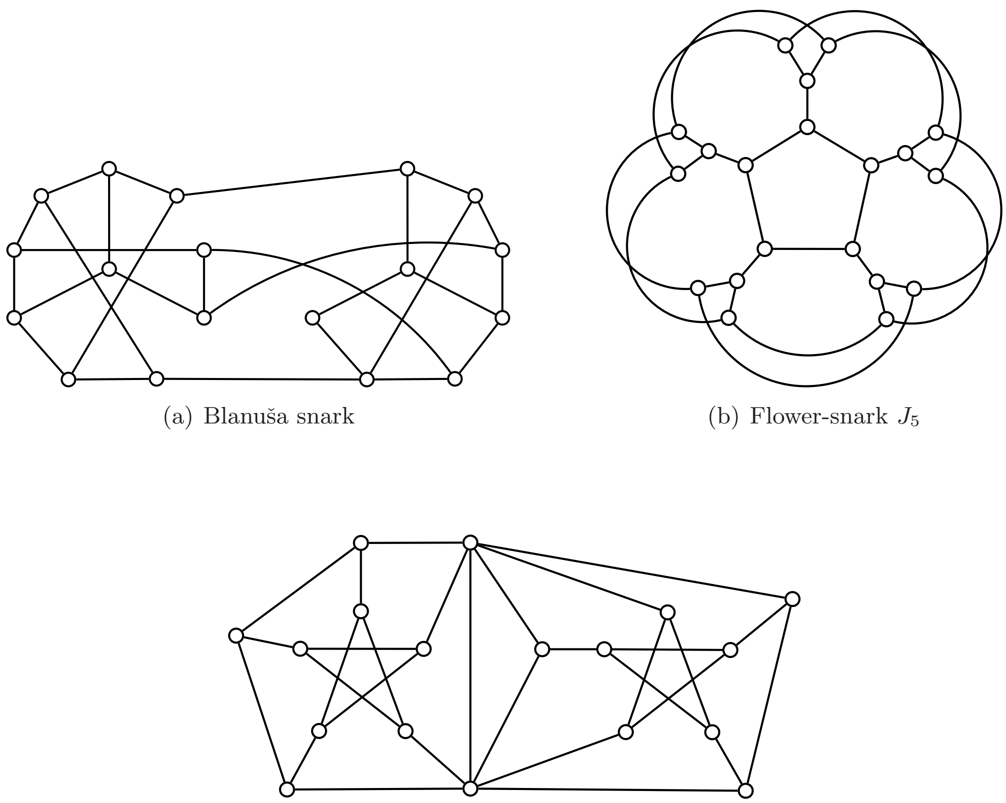


Figure 1: Examples of vertex-3-critical graphs

2 Motivation

Tutte proposed the following well known conjectures concerning 5-, 4- and 3-flows.

Conjecture 2.1 (5-Flow Conjecture) *Every 2-edge-connected graph admits a 5-flow.*



(a) Blanuša snark

(b) Flower-snark J_5

(c) Composition of two Petersen graphs

Figure 2: Examples of vertex-4-critical graphs

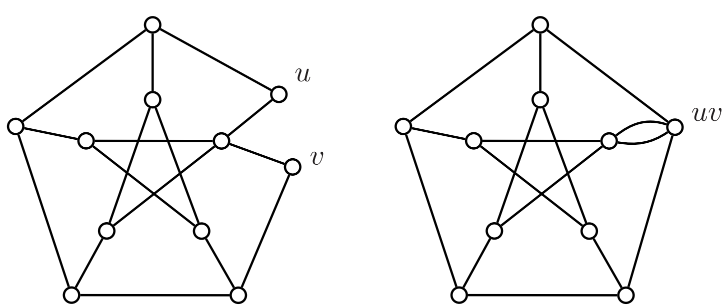
Conjecture 2.2 (4-Flow Conjecture) *Every 2-edge-connected graph with no Petersen minors admits a 4-flow.*

Conjecture 2.3 (3-Flow Conjecture) *Every 4-edge-connected graph admits a 3-flow.*

Conjectures 2.1 and 2.2 were proposed in 1954 [7], while Conjecture 2.3 was proposed much later in the early 80's [6]. All three conjectures are theorems for planar graphs. Robertson, Seymour and Thomas [4] have proved the 4-Flow Conjecture for cubic graphs. The 3-Flow Conjecture has been proved for planar graphs with up to three 3-cuts and for projective planar graphs with at most one 3-cut [6]. There are many more interesting results concerning these conjectures that are not mentioned here. Refer to Diestel [2], Seymour [5] or Zhang [9] for a more thorough review on related results. All three conjectures are still open.

Let G be a minimum counterexample for the 5-Flow Conjecture and u and v any pair of distinct vertices of G. Since the contraction of u and v does not create any new cuts, G_{uv} is not a counterexample and has a 5-flow. This argument is valid for any pair of vertices u and v. We thus conclude that every minimal counterexample for the 5-Flow Conjecture is vertex-5-critical. Using a similar argument we can prove that every minimal counterexample for the 3-Flow Conjecture must be vertex-3-critical.

We can also prove that every minimal counterexample for the 4-Flow Conjecture must be edge-4-critical. Let G be a minimum counterexample for the 4-Flow Conjecture and e an edge of G. Since G does not have a Petersen minor, G/e does not have one either. Thus, G is edge-4-critical. We do not know, however, whether G is vertex-4-critical. It may be the case that for some pair of distinct vertices u and v of G, graph G_{uv} has a Petersen minor, as illustrated by Figure 3.



(a) Has no Petersen minor (b) Petersen minor was created

Figure 3: Contraction of vertices may generate a Petersen minor

Therefore, the search for good characterizations of edge and vertex-k-critical graphs for $k = 3, 4, 5$ stands up as a new interesting approach towards proving these conjectures.

3 Characterizations and Properties of k-Critical Graphs

In this section we show characterizations of edge-k-critical and vertex-k-critical graphs, as well as some interesting properties satisfied by such graphs.

3.1 Characterization of Edge-k-Critical Graphs

Lemma 3.1 *Let D be a digraph with no mod-k-flow and $e = uv$ an edge of D. Then, the following properties are equivalent:*

(i) *Graph D/e has a mod-k-flow.*

(ii) *Graph $D - e$ has a mod-k-flow.*

(iii) *Graph D has a near-mod-k-flow that misses u and v.*

Proof:

(i) $\Rightarrow$ (ii): Let f be a mod-k-flow of D/e. Either f is a mod-k-flow of $D - e$ or f is a near-mod-k-flow of $D - e$ that misses the ends of e. But the latter cannot be the case, otherwise, by Lemma 1.1, D has a mod-k-flow, a contradiction. Thus, f is a mod-k-flow of $D - e$.

(ii) $\Rightarrow$ (iii): Let f' be a mod-k-flow of $D - e$. Extend f' to a mapping f of D by assigning to e a weight $f(e)$ that is not a multiple of k. That makes u and v unbalanced modulo k with respect to f while all other vertices are still balanced modulo k. Thus, f is a near-mod-k-flow of D that misses u and v.

(iii) $\Rightarrow$ (i): Let f be a near-mod-k-flow of D that misses u and v and f' be the restriction of f to D/e. All vertices, except perhaps the vertex of contraction w, are balanced modulo k with respect to f. Thus, so is vertex w and f' is a mod-k-flow of D/e. □

The next result is a characterization of edge-k-critical graphs and follows immediately from Lemma 3.1.

Corollary 3.2 *Let D be a digraph with no mod-k-flow. Then, the following properties are equivalent:*

(i) *Graph D is edge-k-critical.*

(ii) *For every edge e of D, graph $D - e$ has a mod-k-flow.*

(iii) *For every edge $e = uv$ of D, graph D has a near-mod-k-flow that misses u and v.*

3.2 Characterization of Vertex-k-Critical Graphs

The next result is an analog of Lemma 3.1 for any two vertices, not necessarily adjacent and is followed by a corollary that characterizes vertex-k-critical digraphs.

Lemma 3.3 *Let D be a digraph with no mod-k-flow and u and v an arbitrary pair of distinct vertices of D. Then, the following properties are equivalent:*

(i) *Graph D_{uv} has a mod-k-flow.*

(ii) *Graph $D + uv$ has a mod-k-flow.*

(iii) *Graph D has a near-mod-k-flow that misses u and v.*

Proof:

(i) $\Rightarrow$ (ii): If u and v are adjacent in D then, by Corollary 3.1(iii), there is a near-mod-k-flow that misses u and v. Then, by Lemma 1.1, $D + uv$ has a mod-k-flow. We may thus assume that u and v are not adjacent. By hypothesis, D_{uv} has a mod k-flow f. Let f' be an extension of f to D. Since D has no mod-k-flow, f' is a near-mod-k-flow of D that misses vertices u and v. Again, by Lemma 1.1, $D + uv$ has a mod-k-flow.

(ii) $\Rightarrow$ (iii) : By hypothesis, $D+uv$ has a mod-k-flow f'. Let e be edge uv in $D+uv$. Since $f'(e)$ is not a multiple of k, the restriction f of f' to D must be a near-mod-k-flow of D that misses u and v.

(iii) $\Rightarrow$ (i): By hypothesis, D has a near-mod-k-flow f that misses u and v. The restriction f' of f to D_{uv} is a mod-k-flow of D. □

Corollary 3.4 *Let D be a digraph with no k-flow. Then, the following properties are equivalent:*

(i) *Graph D is vertex-k-critical.*

(ii) *For every pair $\{u, v\}$ of two distinct vertices of D, the graph $D+uv$ has a mod-k-flow.*

(iii) *For every pair $\{u, v\}$ of two distinct vertices of D, there is a near-mod-k-flow of D that misses u and v.*

3.3 Other Properties

We now present some properties of edge-k-critical graphs. They are proved for edge-k-critical graphs, but since every loopless vertex-k-critical graph is edge-k-critical, they are valid for loopless vertex-k-critical graphs as well.

Proposition 3.5 *Every edge-k-critical digraph D has girth at least k.*

Proof: Suppose D has a circuit C of length $k-1$ or less. If C is not an oriented circuit in D, change the orientation of some of its edges so as to make it oriented. Let D' be the resulting orientation. Let e be an edge of C. By Corollary 3.2(ii), $D'-e$ has a mod-k-flow f'. Since $C-e$ has less than $k-1$ edges, there is some integer x not multiple of k such that $f'(e_c) \not\equiv x \pmod{k}$ for every edge $e_c \in C-e$. Obtain f from f' by adding x to the weight of every edge e_c of $C-e$ and assign $f(e)$ equal to x. Then, f is a mod-k-flow of D', which can be converted into a mod-k-flow of D, contradicting the hypothesis. Thus, D has girth at least k. □

Corollary 3.6 *Let D be an edge-k-critical digraph and u and v two vertices of D such that D_{uv} does not have a mod-k-flow. Then, the distance between u and v is at least k.*

Proof: Since D is edge-k-critical and D_{uv} does not have a mod-k-flow, u and v are not adjacent in D. Let e be an edge of D. By Corollary 3.2(ii), $D-e$ has a mod-k-flow, so $(D-e)_{uv}$ also has a mod-k-flow. But $(D-e)_{uv} = D_{uv} - e$. Thus, $D_{uv} - e$ has a mod-k-flow. This conclusion holds for every edge e, therefore, D_{uv} is edge-k-critical. By Proposition 3.5, D_{uv} has girth at least k. We conclude that the distance between u and v in D must be at least k. □

Corollary 3.7 *Let D be an edge-k-critical digraph with diameter less than k. Then, D is vertex-k-critical.*

Proposition 3.8 *Every connected edge-k-critical digraph D with $|V(D)| \geq 3$ is 2-connected.*

Proof: Suppose D has a cut-vertex v. Since $|V(D)| \geq 3$, D is the union of two non-trivial digraphs D_1 and D_2 that have only vertex v in common. Let e be an edge of D not in D_1. Since D is edge-k-critical, $D-e$ has a mod-k-flow f. In the restriction f_1 of f to D_1, balance is achieved a every vertex of $V(D_1)-v$; that implies v must be balanced as well and thus, f_1 is a mod-k-flow of D_1. Likewise, D_2 has a mod-k-flow f_2. Let f be the union of f_1 and f_2. Since both f_1 and f_2 are mod-k-flows, f is a mod-k-flow of D, contradicting the hypothesis. Thus, D has no cut-vertex. □

Proposition 3.9 *Every connected edge-k-critical digraph D with $|V(D)| \geq 3$ is 3-edge-connected.*

Proof: By Proposition 3.8, D is 2-connected and, consequently, 2-edge-connected. Suppose D has a 2-edge-cut $\{e_1, e_2\}$. By Corollary 3.2(ii) $D-e_1$ has a mod-k-flow. But e_2 is a cut-edge of $D-e_1$, so $D-e_1$ cannot have a mod-k-flow, a contradiction. Thus, D must be 3-edge-connected. □

Proposition 3.10 *Every edge-k-critical graph G admits a $(k+1)$-flow.*

Proof: Let $e = uv$ be an edge of G. By Corollary 3.2(ii) $G - e$ has a mod-k-flow. Thus, by Theorem 1.2, there is a k-flow f for some orientation D_e of $G - e$.

No cut of D_e is directed since every cut must be balanced with respect to f. Thus, there is a directed path P from v to u in D_e, and $P + e$ is a directed circuit of $D_e + e$. Obtain f' from f by adding 1 to the weight of every edge e_p of P and assign $f'(e)$ equal to 1. Then, f' is a $(k+1)$-flow of $D_e + e$. Indeed, G has a $(k+1)$-flow. □

The following Lemma is very helpful in finding examples of k-critical graphs.

Lemma 3.11 *Let D be a graph that does not admit a mod-k-flow. Then, there is a subset E' of the edges of D such that D/E' is edge-k-critical.*

Proof: The proof follows by induction on the number of edges of D. If D is edge-k-critical already, then we simply take $E' := \emptyset$. We may thus assume D is not edge-k-critical. Therefore, there is an edge e of D such that $D' := G/e$ does not admit a mod k-flow. By induction hypothesis, there is a subset E'' of the edges of D' such that D'/E'' is edge-k-critical. Take $E' := E'' + e$. Clearly, D'/E'' is also D/E', therefore, D/E' is edge-k-critical. □

Lemma 3.11 implies, for instance, that for every snark G there is at least one edge-4-critical graph that can be obtained from G by contracting a (possibly empty) set of edges. For instance, the flower-snark on Figure 4 is not edge-4-critical, but after contracting edges $\{ab, ac, bc\}$ we obtain the Petersen graph that is edge-4-critical.

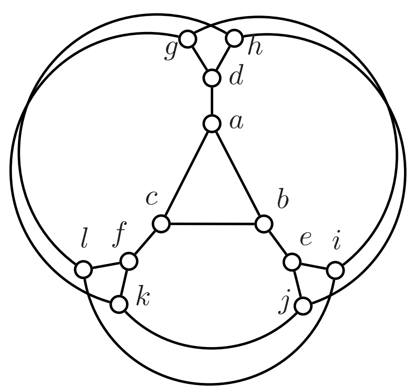
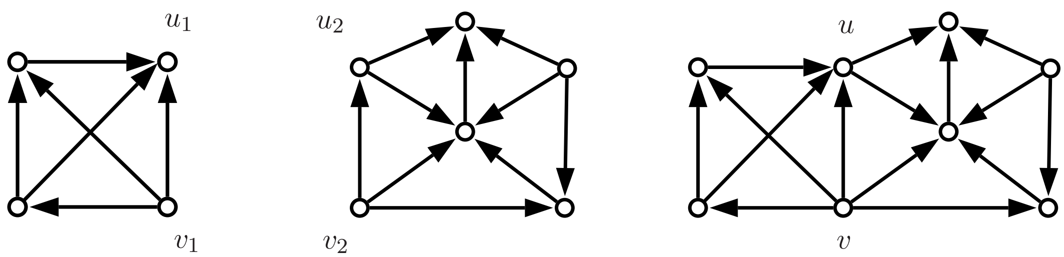


Figure 4: Flower snark J_3

4 Composition and Decomposition of k-Critical Graphs

Let D_1 and D_2 be two digraphs, $e = u_1v_1$ an edge of D_1 and u_2 and v_2 be two vertices of D_2. The *composition of* D_1 *and* D_2 *by* (u_1, v_1) *and* (u_2, v_2) is the graph obtained from $D_1 - e$ and D_2 by the identification of vertices u_1, u_2 into a vertex u and v_1, v_2 into a vertex v. We say a digraph D is a *composition of* D_1 *and* D_2 if for some edge $e = u_1v_1$ of D_1 and for two specified vertices u_2 and v_2 of D_2, D is a composition of D_1 and D_2 by (u_1, v_1) and (u_2, v_2). Figure 5 shows an example of a digraph obtained from the composition of K_4 and W_5.



(a) K_4 (b) W_5 (c) Composition

Figure 5: A composition of K_4 and W_5.

Kochol [3, Lemma 1] proves the following result:

Proposition 4.1 *Let D_1 and D_2 be two digraphs such that neither D_1 nor D_2 has a mod-k-flow. Then, no composition of D_1 and D_2 admits a mod-k-flow.*

We thus state the following conjecture:

Conjecture 4.2 *Let D_1 and D_2 be two vertex-k-critical graphs. Then any composition of D_1 and D_2 is vertex-k-critical.*

We present below some evidence in support of the validity of Conjecture 4.2.

Theorem 4.3 *Let k be a prime, D_1 an edge-k-critical digraph and D_2 a loopless vertex-k-critical digraph. Then any composition of D_1 and D_2 is edge-k-critical.*

Proof: Let $e = u_1v_1$ be an edge of D_1 and u_2 and v_2 two vertices of D_2 such that D is a composition of D_1 and D_2 by (u_1, v_1) and (u_2, v_2). Let e_2 be an edge of D_2. By hypothesis D_2 is loopless and vertex-k-critical. Thus, it is edge-k-critical. By Corollary 3.2(ii), $D_2 - e_2$ has a mod-k-flow f_2 and $D_1 - e$ has a mod-k-flow f_1. The union of f_1 and f_2 is a mod-k-flow of $D - e_2$.

Now let e_1 be an edge of $D_1 - e$. By Corollary 3.2(ii), $D_1 - e_1$ has a mod-k-flow f_1. Thus, $D_1 - e_1 - e$ has a near mod-k-flow f_1' that misses u_1 and v_1 by $f_1(e) = x$. By Corollary 3.4(iii), D_2 has a near mod-k-flow f_2 that misses u_2 and v_2 by, say, y. Since k is prime, the inverse $z \equiv y^{-1} \pmod{k}$ exists. Moreover, since $f_2(e)$ is not a multiple of k for every edge e, nor is $-xzf_2(e)$. Thus, $f_2' = -xzf_2$ is a near-mod-k-flow of D_2 that misses u_2 and v_2 by $-x \bmod k$. The union of f_1' and f_2' is a mod-k-flow of $D - e_1$. Every edge of D is either an edge of D_2 or of $D_1 - e$, thus, D is edge-k-critical. □

A composition of two digraphs D_1 and D_2 is *symmetric* if for some edge $e_1 = u_1v_1$ of D_1 and for some edge $e_2 = u_2v_2$ of D_2, D is a composition of D_1 and D_2 by (u_1, v_1) and (u_2, v_2).

Clearly, a symmetric composition D of D_1 and D_2 is also a (symmetric) composition of D_2 and D_1.

Theorem 4.4 *Let D_1 and D_2 be two edge-k-critical digraphs, k not necessarily prime. Then any symmetric composition of D_1 and D_2 is edge-k-critical.*

Proof: Let $e_1 = u_1v_1$ be an edge of D_1 and $e_2 = u_2v_2$ be an edge of D_2 such that D is a composition of D_1 and D_2 by (u_1, v_1) and (u_2, v_2). Let e' be an edge of D_2. By hypothesis D_2 is edge-k-critical. By Corollary 3.2(ii), $D_2 - e'$ has a mod-k-flow f_2 and $D_1 - e_1$ has a mod-k-flow f_1. The union of f_1 and f_2 is a mod-k-flow of $D - e'$. Thus, $D - e'$ has a mod-k-flow for every edge e' of D_2.

Since D is a symmetric composition, the roles of D_1 and D_2 can be switched in the argument above. We then conclude that $D - e'$ has a mod-k-flow for every edge e' of D_1. Every edge of D is either an edge of D_1 or of D_2, thus D is also edge-k-critical. □

Theorem 4.5 *Let k be a prime,*[1] *D an edge-k-critical digraph, $\{u, v\}$ a 2-vertex-cut of D and D_1 and D_2 subgraphs of D such that $V(D_1) \cap V(D_2) = \{u, v\}$ and $D = D_1 \cup D_2$. Then, either*

(i) *D_1 and $D_2 + uv$ are both edge-k-critical, or*

(ii) *$D_1 + uv$ and D_2 are both edge-k-critical.*

Proof: We first prove the following assertion:

$$\text{At most one of } D_1 \text{ and } D_2 \text{ has a near-mod-}k\text{-flow that misses } u \text{ and } v. \qquad (1)$$

Suppose (1) is not true. Let f_1 and f_2 be near-mod-k-flows of D_1 and D_2 that miss u and v by x and y respectively. Since k is prime, the inverse $z \equiv y^{-1} \pmod{k}$ exists. Moreover, since $f_2(e)$ is not a multiple of k for every edge e, nor is $-xzf_2(e)$. Thus, $f_2' = -xzf_2$ is a near-mod-k-flow of D_2 that misses u and v by $-x \bmod k$. The union of f_1 and f_2' is a mod-k-flow of D, a contradiction. Note that the restriction to k prime is only needed in the proof of assertion (1).

Adjust notation if necessary so that

$$D_2 \text{ does not have a near-mod-}k\text{-flow that misses } u \text{ and } v. \qquad (2)$$

We now prove that D_1 and $D_2 + uv$ are both edge-k-critical. Clearly,

$$D_2 + uv \text{ does not have a mod-}k\text{-flow.} \qquad (3)$$

Otherwise, the restriction of a mod-k-flow of $D_2 + uv$ to D_2 would be a near-mod-k-flow that misses u and v, contradicting (2).

[1] We recently proved that this Theorem holds for $k = 4$ also.

Let e_1 be an edge of D_1. By Corollary 3.2(ii), $D - e_1$ has a mod-k-flow f. The restriction f_1 of f to $D_1 - e_1$ is either a mod-k-flow of $D_1 - e_1$ or a near-mod-k-flow of $D_1 - e_1$ that misses u and v. If f_1 is a near-mod-k-flow of $D_1 - e_1$ that misses u and v, then the restriction f_2 of f to D_2 is a near-mod-k-flow of D_2 that misses u and v, contradicting (2). Thus, f_1 must be a mod-k-flow of $D_1 - e_1$ and, consequently, f_2 is a mod-k-flow of D_2. We conclude that

$$D_2 \text{ has a mod-}k\text{-flow} \tag{4}$$

and

$$D_1 - e_1 \text{ has a mod-}k\text{-flow.} \tag{5}$$

Assertion (5) holds for every edge e_1 of D_1. Moreover,

$$D_1 \text{ does not have a mod-}k\text{-flow.} \tag{6}$$

Otherwise, since D_2 has a mod-k-flow as stated in (4), a union of mod-k-flows of D_1 and D_2 would be a mod-k-flow of D, contradicting the hypothesis. We therefore conclude from (5) and (6) that D_1 is edge-k-critical.

Now let e_2 be an edge of D_2. By Corollary 3.2(ii), $D - e_2$ has a mod-k-flow f. The restriction f_2 of f to $D_2 - e_2$ is either a mod-k-flow of $D_2 - e_2$ or a near-mod-k-flow of $D_2 - e_2$ that misses u and v. If f_2 is a mod-k-flow of $D_2 - e_2$, then the restriction f_1 of f to D_1 is a mod-k-flow of D_1, contradicting (6). Thus, f_2 must be a near-mod-k-flow of $D_2 - e_2$ that misses u and v. By Lemma 1.1,

$$D_2 + uv - e_2 \text{ has a mod-}k\text{-flow.} \tag{7}$$

Assertion (7) holds for every edge e_2 of D_2. We conclude from (3), (4) and (7) that $D_2 + uv$ is edge-k-critical. Therefore, both D_1 and $D_2 + uv$ are edge-k-critical.

Note that if it is actually D_1 that does not have a near-mod-k-flow that misses u and v, an analogous argument implies that both $D_1 + uv$ and D_2 are edge-k-critical. □

Given a digraph D, a *decomposition of* D is a pair of graphs D_1 and D_2 such that D is a composition of D_1 and D_2. Clearly, a decomposition of D exists if and only if D has a 2-vertex-cut. Essentially, Theorem 4.5 states that, for k prime, we can decompose an edge-k-critical graph that has a 2-vertex-cut into two edge-k-critical graphs.

Curiously, we can also prove that a vertex-k-critical graph that is not 3-connected can be decomposed into two vertex-k-critical graphs if k is prime. The proof, presented below, is analogous to the proof of Theorem 4.5.

Theorem 4.6 *Let k be a prime, D a vertex-k-critical digraph, $\{u, v\}$ a 2-vertex-cut of D and D_1 and D_2 subgraphs of D such that $V(D_1) \cap V(D_2) = \{u, v\}$ and $D = D_1 \cup D_2$. Then, either*

(i) *D_1 and $D_2 + uv$ are both vertex-k-critical, or*

(ii) *$D_1 + uv$ and D_2 are both vertex-k-critical.*

Proof: We first prove the following assertion:

$$\text{At most one of } D_1 \text{ and } D_2 \text{ has a near-mod-}k\text{-flow that misses } u \text{ and } v. \tag{8}$$

Suppose (8) is not true. Let f_1 and f_2 be near-mod-k-flows of D_1 and D_2 that miss u and v by x and y respectively. Since k is prime, the inverse $z \equiv y^{-1} \pmod{k}$ exists. Moreover, since $f_2(e)$ is not a multiple of k for every edge e, nor is $-xzf_2(e)$. Thus, $f_2' = -xzf_2$ is a near-mod-k-flow of D_2 that misses u and v by $-x \bmod k$. The union of f_1 and f_2' is a mod-k-flow of D, a contradiction. Note that the restriction to k prime is only needed in the proof of assertion (8).

Adjust notation if necessary so that

$$D_2 \text{ does not have a near-mod-}k\text{-flow that misses } u \text{ and } v. \tag{9}$$

We now prove that D_1 and $D_2 + uv$ are both vertex-k-critical. Clearly,

$$D_2 + uv \text{ does not have a mod-}k\text{-flow.} \tag{10}$$

Otherwise, the restriction of a mod-k-flow of $D_2 + uv$ to D_2 would be a near-mod-k-flow that misses u and v, contradicting (9).

Let u_1 and v_1 be two vertices of of D_1. By Corollary 3.4(ii), $D + u_1v_1$ has a mod-k-flow f. The restriction f_1 of f to $D_1 + u_1v_1$ is either a mod-k-flow of $D_1 + u_1v_1$ or a near-mod-k-flow of $D_1 + u_1v_1$ that misses u and v. If f_1 is a near-mod-k-flow of $D_1 + u_1v_1$ that misses u and v, then the restriction f_2 of f to D_2 is a near-mod-k-flow of D_2 that misses u and v, contradicting (9). Thus, f_1 must be a mod-k-flow of $D_1 + u_1v_1$ and, consequently, f_2 is a mod-k-flow of D_2. We conclude that

$$D_2 \text{ has a mod-}k\text{-flow} \tag{11}$$

and

$$D_1 + u_1v_1 \text{ has a mod-}k\text{-flow.} \tag{12}$$

Assertion (12) holds for any two vertices u_1 and v_1 of D_1. Moreover,

$$D_1 \text{ does not have a mod-}k\text{-flow.} \tag{13}$$

Otherwise, since D_2 has a mod-k-flow as stated in (11), a union of mod-k-flows of D_1 and D_2 would be a mod-k-flow of D, contradicting the hypothesis. We therefore conclude from (12) and (13) that D_1 is vertex-k-critical.

Now let u_2 and v_2 be two vertices of D_2. By Corollary 3.4(ii), $D + u_2v_2$ has a mod-k-flow f. The restriction f_2 of f to $D_2 + u_2v_2$ is either a mod-k-flow of $D_2 + u_2v_2$ or a near-mod-k-flow of $D_2 + u_2v_2$ that misses u and v. If f_2 is a mod-k-flow of $D_2 + u_2v_2$, then the restriction f_1 of f to D_1 is a mod-k-flow of D_1, contradicting (13). Thus, f_2 must be a near-mod-k-flow of $D_2 + u_2v_2$ that misses u and v. By Lemma 1.1,

$$D_2 + uv + u_2v_2 \text{ has a mod-}k\text{-flow.} \tag{14}$$

Assertion (14) holds for any two vertices u_2 and v_2 of D_2. We conclude from (10) and (14) that $D_2 + uv$ is vertex-k-critical. Therefore, both D_1 and $D_2 + uv$ are vertex-k-critical.

Note that if it is actually D_1 that does not have a near-mod-k-flow that misses u and v, an analogous argument implies that both $D_1 + uv$ and D_2 are vertex-k-critical. □

5 Examples and Properties of 3-critical graphs

A *mod-3-orientation* of a graph G is an orientation D of G such that the mapping $f(e) = 1$ for every edge e is a mod-3-flow of D. It is not hard to see that a graph has a mod-3-flow if and only if it has a mod-3-orientation.

Given a graph G, we denote by $V_3(G)$ the set of vertices of G having degree 3. In any mod-3-orientation D of G, a vertex $v \in V_3(G)$ must have all its incident edges oriented in the same direction. We say v is a *source* in mod-3-orientation D if its incident edges have tail in v, otherwise v is a *sink*.

Proposition 5.1 *Let G be a graph such that $G[V_3(G)]$ is not bipartite. Then, G does not admit a 3-flow.*

Proof: Let $C = (v_0, v_1, \ldots, v_n = v_0)$ be an odd circuit of $G[V_3(G)]$. Suppose G has a mod-3-orientation D. Adjust labels, if necessary do that v_0 is a source in D. Therefore, for every even i in the range $1 \leq i \leq n$ vertex v_i is a source and for every odd i in this same range v_i is a sink. Then, $v_n = v_0$ is a sink, a contradiction. □

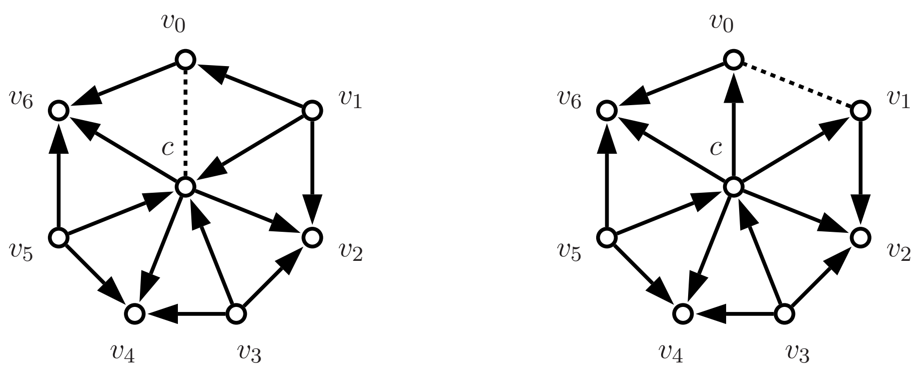
Lemma 5.2 *Every odd wheel is edge-3-critical.*

Proof: Let W_n be an odd wheel such that $v_0, v_1, \ldots, v_n = v_0$ are the rim vertices and c is the central vertex. By Proposition 5.1, W_n does not admit a 3-flow. Given the characterization of Corollary 3.2, we must show that for every edge e of W_n, $W_n - e$ has a mod 3-orientation. However, given the many symmetries present in odd wheels, it suffices to consider only one spoke and one rim edge. For the spoke edge $e_s = v_0c$, the graph $W_n - e_s$ certainly has a mod 3-orientation. Simply orient the edges so that v_i is either a source or a sink depending on whether i is odd or even, for all $1 \leq i \leq n - 1$. As illustrated by Figure 6(a), all edges have been oriented and vertex v_0 is balanced. Thus, all vertices are balanced modulo 3. Now consider the rim edge $e_r = v_0v_1$. Orient the edges of $W_n - e_r$ so that so that v_i is either a source or a sink depending on whether i is odd or even, for all $2 \leq i \leq n-1$. Then orient the remaining edges so as to balance vertices v_0 and v_1, as illustrated by Figure 6(b). This orientation is a mod 3-orientation for $W_n - e_r$. □

Corollary 5.3 *Every odd wheel is vertex-3-critical.*

Proof: Odd wheels have diameter 2 and are edge-3-critical. By Corollary 3.7, they are vertex-3-critical as well. □

Lemma 5.4 *Let G be an edge-3-critical graph. If $G[V_3(G)]$ has an odd circuit C, then every edge of G is incident with a vertex of C and C is chordless.*



(a) Graph $W_n - e_s$ (b) Graph $W_n - e_r$

Figure 6: Graphs $W_n - e_s$ and $W_n - e_r$ have mod 3-orientations

Proof: We first prove that every edge of G is incident with a vertex of C. Assume the contrary and let e be an edge of G that is not incident with any vertex of C. Then, $G' := G - e$ does not have a 3-flow since C is an odd circuit of $G'[V_3(G')]$. Thus, by Theorem 3.2, G is not edge-3-critical, a contradiction.

Now, if C is an odd circuit of $G[V_3]$ and e is a chord of C, then e splits C into two segments of different parity. Each segment, together with chord e, forms a circuit. The two circuits have different parities. Let us say C' is the circuit with odd length and C'' is the circuit with even length. By Proposition 3.5, D is simple and so C'' must have length at least four. Thus, one of its edges is not incident with odd circuit C', a contradiction. □

Corollary 5.5 *Let G be a loopless vertex-3-critical graph. If $G[V_3(G)]$ has an odd circuit C, then G is an odd wheel and C is its rim.*[2]

Proof: Since G is edge-3-critical as well, by Lemma 5.4, every edge of G is incident with C. Thus, $V(G)-V(C)$ is an independent set. If $V(G)-V(C)$ is non trivial, let u and v be two distinct vertices of $V(G)-V(C)$. By hypothesis, graph G_{uv} has a mod-3-flow. But C still is an odd circuit of G_{uv} induced by its vertices of degree 3. Thus, by Proposition 5.1, G_{uv} does not have a mod-3-flow, a contradiction. We conclude $V(G)-V(C)$ must be trivial. Moreover, C is chordless. Therefore G is an odd wheel with rim C. □

Lemma 5.6 *Let G be an edge-3-critical graph distinct from K_4. Then, $G[V_3(G)]$ has no even circuit.*

Proof: Assume by contradiction that $G[V_3(G)]$ has an even circuit. Let C be an even circuit of $G[V_3(G)]$ of minimum length. If C has a chord e, then e splits C into two segments of same parity. Each segment together with e form a circuit. Therefore, by the minimality of C, the two circuits must have odd length. By Lemma 5.4, each segment must have precisely

[2]We proved recently an analog of this assertion for G edge-3-critical.

length 2, otherwise the segment has an edge that is not incident with the odd circuit formed by the other segment and the chord e. Since all vertices in C have degree three and every edge of G must be incident with both odd circuits, G has one more edge only that joins the internal vertices of both segments. But then G is K_4, a contradiction.

We may thus assume that C has no chords. Let v be an arbitrary vertex of C with neighbours v_1, v_2 and v_3. Take v_1 and v_2 to be the neighbours of v in C and v_3 the one not in C. Let e be the edge vv_3. In any mod 3-orientation of graph $G' := G - e$, vertices v_1 and v_2 must be both sources or sinks, since v_1, v_2 are the ends of a path of odd length in $G'[V_3(G')]$. This implies that no mod 3-orientation G' will ever balance vertex v that has degree two in G'. So we conclude G is not edge-3-critical, a contradiction. □

By Lemma 5.6 no biwheel and no prism are edge-3-critical. Also, no Möbius ladder other than K_4 is edge-3-critical. All such graphs have an even circuit induced by the vertices of degree three.

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