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Technical Report - IC-07-015 - Relatório Técnico

May - 2007 - Maio

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Abstract

A *matching covered* graph is a nontrivial connected graph in which every edge is in some perfect matching. A matching covered graph G is *near-bipartite* if it is non-bipartite and there is a removable double ear R such that $G-R$ is matching covered and bipartite. In 2000, C. Little and I. Fischer characterized Pfaffian near-bipartite graphs in terms of forbidden subgraphs [3]. However, their characterization does not imply a polynomial time algorithm to determine whether a near-bipartite graph is Pfaffian. In this report, we give such an algorithm. Our algorithm is based in a Inclusion-Exclusion principle and uses as a subroutine an algorithm by McCuaig [5] and by Robertson, Seymour and Thomas [6] that determines whether a bipartite graph is Pfaffian.

1 Introduction

Let $A := (A_{ij})$ be an $n \times n$ skew-symmetric matrix. When n is even, there is a polynomial $P := P(A)$ in the a_{ij} called *Pfaffian* of A. This polynomial is defined as follows:

$$P := \sum \operatorname{sgn}(M) a_{i_1 j_1}\, a_{i_2 j_2}\, \ldots\, a_{i_k j_k},$$

where the sum is taken over the set of all partitions $M := (i_1 j_1, i_2 j_2, \ldots, i_k j_k)$ of $\{1, 2, \ldots, n\}$ into k unordered pairs, and $\operatorname{sgn}(M)$ is the sign of the permutation:

$$\pi(M) := \begin{pmatrix} 1 & 2 & 3 & 4 & . & . & . & 2k-1 & 2k \\ i_1 & j_1 & i_2 & j_2 & . & . & . & i_k & j_k \end{pmatrix}.$$

It can be seen that the definition of Pfaffian of A given above is independent of the order in which the constituent pairs in a partition M are listed, as also of the order in which the elements in a pair are listed. Since A is skew-symmetric, for each pair (i, j) of indices, either a_{ij} or a_{ji} is nonnegative.

*Institute of Computing, University of Campinas, 13084-971 Campinas, SP. Research supported by Fapesp – Fundação de Amparo à Pesquisa do Estado de São Paulo, proc. #05/04426-6

†Institute of Computing, University of Campinas, 13084-971 Campinas, SP. Research supported in part by CNPq — Conselho Nacional de Desenvolvimento Científico e Tecnológico, grant #306088/2005-3

Now suppose that G is a graph whose set of vertices is $\{1, 2, \ldots, n\}$. Let D be an orientation of G such that A is the adjacency matrix of D. Then each nonzero term in the expansion of the Pfaffian of A corresponds to a perfect matching M of G. Thus, if D is such that all $\mathrm{sgn}(M)$, for M perfect matching of G, are the same, then $|P|$ is the number of perfect matchings of G.

An orientation D of G is a *Pfaffian orientation* of G if all perfect matchings of G have the same sign. An undirected graph G is *Pfaffian* if it admits a Pfaffian orientation.

An edge e of a graph G is *admissible* if G has a perfect matching containing e. An edge $e = uw$ of G is admissible if and only if $G - u - w$ has a perfect matching. Thus, one can determine the set of admissible edges of G in polynomial time. The definition of Pfaffian orientation implies the following result:

Proposition 1.1
Let G be a graph and H the graph obtained from G by removing every non-admissible edge of G. An orientation D of G is Pfaffian if and only if the restriction of D to H is Pfaffian.

1.1 Removable Ears

A *single ear* of a connected graph G is a path $P := (v_0, e_1, v_1, e_2, v_2, \ldots, v_{2k-2}, e_{2k-1}, v_{2k-1})$ of odd length in G whose internal vertices $v_1, v_2, \ldots, v_{2k-2}$, if any, have degree two in G. The *order* of an edge e_i of P is *even* or *odd*, according to the parity of its index i. (Note that the order af an edge is preserved if one replaces P by its reverse, because the length of P is odd.) If P is a single ear of G then we denote by $G - P$ the graph obtained from G by deleting the edges and internal vertices of P. A *double ear* of G is a pair (R_1, R_2), where R_1 and R_2 are two vertex-disjoint single ears of G. An *ear* of G is either a single ear or a double ear of G. If R is an ear of G then we denote by $G - R$ the graph obtained from G by deleting the edges and internal vertices of the constituent paths of R.

A single ear R of a matching covered graph G is *removable* if the graph $G - R$ is matching covered. A removable single ear of length one is a *removable edge*. A double ear $R = (R_1, R_2)$ of G is removable if $G - R$ is matching covered and neither $G - R_1$ nor $G - R_2$ are matching covered. A *removable ear* of G is either a single or a double ear which is removable. A *partial ear decomposition* of a matching covered graph G that results in a graph H is a sequence $H = G_1 \subset G_2 \subset \ldots \subset G_r = G$ of subgraphs of G such that, for $2 \leq i \leq r$, $G_{i-1} = G_i - R_i$, where R_i is a removable ear of G_i. An *ear decomposition* of a matching covered graph G is a partial ear decomposition where $G_1 = K_2$.

Proposition 1.2 ([1] Proposition 4.1)
Let R be a removable ear and M a perfect matching of a matching covered graph G. Then, either M contains all the edges of even order of R or M contains all the edges of odd order of R.

Let G be a graph and H a subgraph of G. The graph H is *conformal* in G if $G - V(H)$ has a perfect matching.

PROPOSITION 1.3
Let G be a graph, H a conformal subgraph of G, and J a conformal subgraph of H. Then, J is a conformal subgraph of G.

1.2 Parities of Circuits

The *parity* of a circuit C of even length in a directed graph D is the parity of the number of its edges that are directed in agreement with a specified sense of orientation of C. As C has an even number of edges, the parity is the same in both senses and thus is well defined. If the parity of C is odd we say C is *oddly oriented* in D. For any two sets X and Y, we denote by $X \oplus Y$ the symmetric difference of X and Y.

THEOREM 1.4 ([4] LEMMA 8.3.1)
Let D be an arbitrary orientation of an undirected graph G. Let M_1 and M_2 be any two perfect matchings of G and let k denote the number of even parity circuits of $G[M_1 \oplus M_2]$. Then, M_1 and M_2 have the same sign if and only if k is even.

THEOREM 1.5 ([4] THEOREM 8.3.2)
Let G be a graph, M a perfect matching of G and D an orientation of G. Then the following are equivalent:

- *D is a Pfaffian orientation of G;*
- *Every M-alternating circuit of G is oddly oriented in D;*
- *Every conformal circuit of G is oddly oriented in D.*

COROLLARY 1.6
Let G be a graph, D a Pfaffian orientation of G, and H a conformal subgraph of G. Then, the restriction $D(H)$ of D to H is a Pfaffian orientation.

2 Inclusion-Exclusion Theorem

PROPOSITION 2.1
Let M and N be perfect matchings of a graph G. Let S be a subset of edges of G, such that $S \subseteq (M \cap N)$. Let Q be an M, N-alternating circuit. Then, Q is a conformal circuit of $H := G - V(S)$, where $G - V(S)$ is the graph obtained from G by removing every vertex incident with an edge of S.

Proof: Let $M' := M - S$ and $N' := N - S$. The matchings M' and N' are perfect matchings of H. Moreover, Q has no vertex in $V(S)$. Therefore, Q is an M', N'-alternating circuit. Thus, Q is a conformal circuit of H. □

Next, we present an important theorem for the rest of the paper. Let R be an ear of a graph G. Let $G - V(R)$ be the graph obtained from G by removing every vertex of R.

Theorem 2.2 (Inclusion-Exclusion Theorem)
Let D be an orientation of a matching covered graph G, R a removable ear of G, and Q a conformal circuit of G that contains some edge of R. Then, D is Pfaffian if and only if each of the following three properties holds:

1. *$D - R$ is Pfaffian;*

2. *$D - V(R)$ is Pfaffian;*

3. *Q is oddly oriented in D.*

Proof: (only if part) The graphs $D - R$, $D - V(R)$ and Q are conformal subgraphs of D. Therefore, if D is Pfaffian then each of the three directed graphs are also Pfaffian.

(if part) To prove the converse, assume that the three properties hold. Let $\mathcal{M}$ be the set of perfect matchings of G. According to Proposition 1.2, a perfect matching of G either contains all the edges of even order of R or all the edge of odd order of R. Therefore, $\mathcal{M}$ can be partitioned in two sets $\mathcal{M}_{even}$ and $\mathcal{M}_{odd}$, the set of perfect matchings of G that contain all the even order edges of R and the set of those that contain all the odd order edges of R, respectively. By Proposition 2.1, Property 1 implies that every perfect matching of $\mathcal{M}_{even}$ has the same sign s in D. Similarly, Property 2 implies that every perfect matching of $\mathcal{M}_{odd}$ has the same sign t in D. Circuit Q is conformal in G. So, let M be the union of a perfect matching of $G - V(Q)$ and a perfect matching of Q. Note that M is a perfect matching of G. Let $M' := M \oplus Q$. As Q contains some edge of R, one of M and M' is in $\mathcal{M}_{even}$ and the other is in $\mathcal{M}_{odd}$. On the other hand, Property 3 implies that $\mathrm{sgn}(M) = \mathrm{sgn}(M')$ in D. Therefore, $s = t$. Thus, every perfect matching of G has the same sign in D. We deduce that D is a Pfaffian orientation of G. □

3 Near-Bipartite Pfaffian Algorithm

Theorem 3.1 ([2] Theorem 3.9)
There exists a polynomial time algorithm that, given a matching covered graph G, determines an orientation D of G such that G is Pfaffian if and only if D is a Pfaffian orientation of G.

The following result was first proved by Vazirani and Yanakakis [7].

Theorem 3.2 ([2] Corollary 3.11)
The problem of determining whether or not a given orientation D of a matching covered graph G is Pfaffian is polynomially reducible to the problem of deciding whether or not G is Pfaffian.

The following algorithm is due to McCuaig and due to Robertson, Seymour and Thomas. We shall call it MRST algorithm.

Theorem 3.3 ([5, 6])
There exists a polynomial time algorithm that, given a matching covered bipartite graph G, determine whether G is Pfaffian.

From the above theorem and from Theorem 3.2, we have:

Corollary 3.4
There exists a polynomial time algorithm that, given an orientation D of a matching covered bipartite graph G, decides whether D is a Pfaffian orientation.

Proposition 1.1 tells us that a non-admissible edge of an orientation D does not influence whether D is Pfaffian or not. Morever, such edges can be detected in polynomial time. Therefore, we can derive the following corollary.

Corollary 3.5
There exists a polynomial time algorithm that, given an orientation D of a bipartite (possibly non matching covered) graph G, determines whether D is a Pfaffian orientation.

Finally, we are ready to prove that there exists a polynomial time algorithm to determine whether a near-bipartite graph G is Pfaffian. For that purpose, one first uses Theorem 3.1 to obtain an orientation D such that D is Pfaffian if and only if G is Pfaffian, then uses the following theorem.

Theorem 3.6 (Main Theorem)
There exists a polynomial time algorithm that, given an orientation D of a near-bipartite graph G and a removable double ear R of G such that $G - R$ is bipartite, decides whether D is Pfaffian.

Proof: First decide in polynomial time whether both $D - R$ and $D - V(R)$ are Pfaffian or not, using Corollary 3.5. If one of them is not Pfaffian, then D is not Pfaffian, because these graphs are conformal subgraphs of D. Let M be a perfect matching of G containing the odd order edges of R, and N a perfect matching of G containing the even order edges of R. Then, there is an M, N-alternating circuit Q containing some edge of R. If Q is not oddly oriented in D, then D is not Pfaffian. So, if none of the above conditions tells us that D is not Pfaffian, then by Theorem 2.2, D is a Pfaffian orientation. □

The previous algorithm uses an inclusion-exclusion technique based on the algorithm described in Corollary 3.5. There is another more intuitive method to decide whether a bipartite orientation is Pfaffian. Let G be a graph and D_1 and D_2 orientations of G. We say D_1 and D_2 are *similar* if these orientations differ precisely in a cut of G.

Theorem 3.7 ([2] Corollary 3.5)
Every Pfaffian bipartite matching covered graph G has precisely one dissimilar Pfaffian orientations.

In view of the previous theorem, we have that any two Pfaffian orientations of a bipartite matching covered graph are similar. Let D be an orientation of a Pfaffian bipartite matching covered graph G. Then, D is Pfaffian if and only if D is similar to an orientation to G obtained by Theorem 3.1.

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