A Polynomial Time Algorithm for Recognizing Near-Bipartite Pfaffian Graphs

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Abstract

A matching covered graph is a nontrivial connected graph in which every edge is in some perfect matching. A matching covered graph $G$ is near-bipartite if it is non-bipartite and there is a removable double ear $R$ such that $G - R$ is matching covered and bipartite. In 2000, C. Little and I. Fischer characterized Pfaffian near-bipartite graphs in terms of forbidden subgraphs [3]. However, their characterization does not imply a polynomial time algorithm to determine whether a near-bipartite graph is Pfaffian. In this report, we give such an algorithm. Our algorithm is based in an Inclusion-Exclusion principle and uses as a subroutine an algorithm by McCuaig [5] and by Robertson, Seymour and Thomas [6] that determines whether a bipartite graph is Pfaffian.

1 Introduction

Let $A := (A_{ij})$ be an $n \times n$ skew-symmetric matrix. When $n$ is even, there is a polynomial $P := P(A)$ in the $a_{ij}$ called Pfaffian of $A$. This polynomial is defined as follows:

$$P := \sum \text{sgn}(M) a_{i_1 j_1} a_{i_2 j_2} \ldots a_{i_k j_k},$$

where the sum is taken over the set of all partitions $M := (i_1 j_1, i_2 j_2, \ldots, i_k j_k)$ of $\{1, 2, \ldots, n\}$ into $k$ unordered pairs, and sgn$(M)$ is the sign of the permutation:

$$\pi(M) := \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ i_1 & j_1 & i_2 & j_2 & \ldots & i_k & j_k \end{pmatrix}.$$ 

It can be seen that the definition of Pfaffian of $A$ given above is independent of the order in which the constituent pairs in a partition $M$ are listed, as also of the order in which the elements in a pair are listed. Since $A$ is skew-symmetric, for each pair $(i, j)$ of indices, either $a_{ij}$ or $a_{ji}$ is nonnegative.

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Now suppose that $G$ is a graph whose set of vertices is $\{1, 2, \ldots, n\}$. Let $D$ be an orientation of $G$ such that $A$ is the adjacency matrix of $D$. Then each nonzero term in the expansion of the Pfaffian of $A$ corresponds to a perfect matching $M$ of $G$. Thus, if $D$ is such that all $\text{sgn}(M)$, for $M$ perfect matching of $G$, are the same, then $|P|$ is the number of perfect matchings of $G$.

An orientation $D$ of $G$ is a Pfaffian orientation of $G$ if all perfect matchings of $G$ have the same sign. An undirected graph $G$ is Pfaffian if it admits a Pfaffian orientation.

An edge $e$ of a graph $G$ is admissible if $G$ has a perfect matching containing $e$. An edge $e = uv$ of $G$ is admissible if and only if $G - u - v$ has a perfect matching. Thus, one can determine the set of admissible edges of $G$ in polynomial time. The definition of Pfaffian orientation implies the following result:

**Proposition 1.1**
Let $G$ be a graph and $H$ the graph obtained from $G$ by removing every non-admissible edge of $G$. An orientation $D$ of $G$ is Pfaffian if and only if the restriction of $D$ to $H$ is Pfaffian.

### 1.1 Removable Ears

A single ear of a connected graph $G$ is a path $P := (v_0, e_1, v_1, e_2, v_2, \ldots, v_{2k-2}, e_{2k-1}, v_{2k-1})$ of odd length in $G$ whose internal vertices $v_1, v_2, \ldots, v_{2k-2}$, if any, have degree two in $G$. The order of an edge $e_i$ of $P$ is even or odd, according to the parity of its index $i$. (Note that the order of an edge is preserved if one replaces $P$ by its reverse, because the length of $P$ is odd.) If $P$ is a single ear of $G$ then we denote by $G - P$ the graph obtained from $G$ by deleting the edges and internal vertices of $P$. A double ear of $G$ is a pair $(R_1, R_2)$, where $R_1$ and $R_2$ are two vertex-disjoint single ears of $G$. An ear of $G$ is either a single ear or a double ear of $G$. If $R$ is an ear of $G$ then we denote by $G - R$ the graph obtained from $G$ by deleting the edges and internal vertices of the constituent paths of $R$.

A single ear $R$ of a matching covered graph $G$ is removable if the graph $G - R$ is matching covered. A removable single ear of length one is a removable edge. A double ear $R = (R_1, R_2)$ of $G$ is removable if $G - R$ is matching covered and neither $G - R_1$ nor $G - R_2$ are matching covered. A removable ear of $G$ is either a single or a double ear which is removable. A partial ear decomposition of a matching covered graph $G$ that results in a graph $H$ is a sequence $H = G_1 \subset G_2 \subset \ldots \subset G_r = G$ of subgraphs of $G$ such that, for $2 \leq i \leq r$, $G_{i-1} = G_i - R_i$, where $R_i$ is a removable ear of $G_i$. An ear decomposition of a matching covered graph $G$ is a partial ear decomposition where $G_1 = K_2$.

**Proposition 1.2** ([1] Proposition 4.1)
Let $R$ be a removable ear and $M$ a perfect matching of a matching covered graph $G$. Then, either $M$ contains all the edges of even order of $R$ or $M$ contains all the edges of odd order of $R$.

Let $G$ be a graph and $H$ a subgraph of $G$. The graph $H$ is conformal in $G$ if $G - V(H)$ has a perfect matching.
Proposition 1.3
Let $G$ be a graph, $H$ a conformal subgraph of $G$, and $J$ a conformal subgraph of $H$. Then, $J$ is a conformal subgraph of $G$.

1.2 Parities of Circuits

The parity of a circuit $C$ of even length in a directed graph $D$ is the parity of the number of its edges that are directed in agreement with a specified sense of orientation of $C$. As $C$ has an even number of edges, the parity is the same in both senses and thus is well defined. If the parity of $C$ is odd we say $C$ is oddly oriented in $D$. For any two sets $X$ and $Y$, we denote by $X \oplus Y$ the symmetric difference of $X$ and $Y$.

Theorem 1.4 ([4] Lemma 8.3.1)
Let $D$ be an arbitrary orientation of an undirected graph $G$. Let $M_1$ and $M_2$ be any two perfect matchings of $G$ and let $k$ denote the number of even parity circuits of $G[M_1 \oplus M_2]$. Then, $M_1$ and $M_2$ have the same sign if and only if $k$ is even.

Theorem 1.5 ([4] Theorem 8.3.2)
Let $G$ be a graph, $M$ a perfect matching of $G$ and $D$ an orientation of $G$. Then the following are equivalent:

- $D$ is a Pfaffian orientation of $G$;
- Every $M$-alternating circuit of $G$ is oddly oriented in $D$;
- Every conformal circuit of $G$ is oddly oriented in $D$.

Corollary 1.6
Let $G$ be a graph, $D$ a Pfaffian orientation of $G$, and $H$ a conformal subgraph of $G$. Then, the restriction $D(H)$ of $D$ to $H$ is a Pfaffian orientation.

2 Inclusion-Exclusion Theorem

Proposition 2.1
Let $M$ and $N$ be perfect matchings of a graph $G$. Let $S$ be a subset of edges of $G$, such that $S \subseteq (M \cap N)$. Let $Q$ be an $M,N$-alternating circuit. Then, $Q$ is a conformal circuit of $H := G - V(S)$, where $G - V(S)$ is the graph obtained from $G$ by removing every vertex incident with an edge of $S$.

Proof: Let $M' := M - S$ and $N' := N - S$. The matchings $M'$ and $N'$ are perfect matchings of $H$. Moreover, $Q$ has no vertex in $V(S)$. Therefore, $Q$ is an $M',N'$-alternating circuit. Thus, $Q$ is a conformal circuit of $H$. □

Next, we present an important theorem for the rest of the paper. Let $R$ be an ear of a graph $G$. Let $G - V(R)$ be the graph obtained from $G$ by removing every vertex of $R$. 
Theorem 2.2 (Inclusion-Exclusion Theorem)
Let $D$ be an orientation of a matching covered graph $G$, $R$ a removable ear of $G$, and $Q$ a conformal circuit of $G$ that contains some edge of $R$. Then, $D$ is Pfaffian if and only if each of the following three properties holds:

1. $D - R$ is Pfaffian;
2. $D - V(R)$ is Pfaffian;
3. $Q$ is oddly oriented in $D$.

Proof: (only if part) The graphs $D - R$, $D - V(R)$ and $Q$ are conformal subgraphs of $D$. Therefore, if $D$ is Pfaffian then each of the three directed graphs are also Pfaffian.

(if part) To prove the converse, assume that the three properties hold. Let $M$ be the set of perfect matchings of $G$. According to Proposition 1.2, a perfect matching of $G$ either contains all the edges of even order of $R$ or all the edge of odd order of $R$. Therefore, $M$ can be partitioned in two sets $M_{\text{even}}$ and $M_{\text{odd}}$, the set of perfect matchings of $G$ that contain all the even order edges of $R$ and the set of those that contain all the odd order edges of $R$, respectively. By Proposition 2.1, Property 1 implies that every perfect matching of $M_{\text{even}}$ has the same sign $s$ in $D$. Similarly, Property 2 implies that every perfect matching of $M_{\text{odd}}$ has the same sign $t$ in $D$. Circuit $Q$ is conformal in $G$. So, let $M$ be the union of a perfect matching of $G - V(Q)$ and a perfect matching of $Q$. Note that $M$ is a perfect matching of $G$. Let $M' := M \oplus Q$. As $Q$ contains some edge of $R$, one of $M$ and $M'$ is in $M_{\text{even}}$ and the other is in $M_{\text{odd}}$. On the other hand, Property 3 implies that $\text{sgn}(M) = \text{sgn}(M')$ in $D$. Therefore, $s = t$. Thus, every perfect matching of $G$ has the same sign in $D$. We deduce that $D$ is a Pfaffian orientation of $G$. \qed

3 Near-Bipartite Pfaffian Algorithm

Theorem 3.1 ([2] Theorem 3.9)
There exists a polynomial time algorithm that, given a matching covered graph $G$, determines an orientation $D$ of $G$ such that $G$ is Pfaffian if and only if $D$ is a Pfaffian orientation of $G$.

The following result was first proved by Vazirani and Yanakakis [7].

Theorem 3.2 ([2] Corollary 3.11)
The problem of determining whether or not a given orientation $D$ of a matching covered graph $G$ is Pfaffian is polynomially reducible to the problem of deciding whether or not $G$ is Pfaffian.

The following algorithm is due to McCuaig and due to Robertson, Seymour and Thomas. We shall call it MRST algorithm.

Theorem 3.3 ([5, 6])
There exists a polynomial time algorithm that, given a matching covered bipartite graph $G$, determine whether $G$ is Pfaffian.
From the above theorem and from Theorem 3.2, we have:

**Corollary 3.4**

There exists a polynomial time algorithm that, given an orientation \( D \) of a matching covered bipartite graph \( G \), decides whether \( D \) is a Pfaffian orientation.

Proposition 1.1 tells us that a non-admissible edge of an orientation \( D \) does not influence whether \( D \) is Pfaffian or not. Moreover, such edges can be detected in polynomial time. Therefore, we can derive the following corollary.

**Corollary 3.5**

There exists a polynomial time algorithm that, given an orientation \( D \) of a bipartite (possibly non matching covered) graph \( G \), determines whether \( D \) is a Pfaffian orientation.

Finally, we are ready to prove that there exists a polynomial time algorithm to determine whether a near-bipartite graph \( G \) is Pfaffian. For that purpose, one first uses Theorem 3.1 to obtain an orientation \( D \) such that \( D \) is Pfaffian if and only if \( G \) is Pfaffian, then uses the following theorem.

**Theorem 3.6 (Main Theorem)**

There exists a polynomial time algorithm that, given an orientation \( D \) of a near-bipartite graph \( G \) and a removable double ear \( R \) of \( G \) such that \( G - R \) is bipartite, decides whether \( D \) is Pfaffian.

**Proof:** First decide in polynomial time whether both \( D - R \) and \( D - V(R) \) are Pfaffian or not, using Corollary 3.5. If one of them is not Pfaffian, then \( D \) is not Pfaffian, because these graphs are conformal subgraphs of \( D \). Let \( M \) be a perfect matching of \( G \) containing the odd order edges of \( R \), and \( N \) a perfect matching of \( G \) containing the even order edges of \( R \). Then, there is an \( M, N \)-alternating circuit \( Q \) containing some edge of \( R \). If \( Q \) is not oddly oriented in \( D \), then \( D \) is not Pfaffian. So, if none of the above conditions tells us that \( D \) is not Pfaffian, then by Theorem 2.2, \( D \) is a Pfaffian orientation. \( \square \)

The previous algorithm uses an inclusion-exclusion technique based on the algorithm described in Corollary 3.5. There is another more intuitive method to decide whether a bipartite orientation is Pfaffian. Let \( G \) be a graph and \( D_1 \) and \( D_2 \) orientations of \( G \). We say \( D_1 \) and \( D_2 \) are similar if these orientations differ precisely in a cut of \( G \).

**Theorem 3.7 ([2] Corollary 3.5)**

Every Pfaffian bipartite matching covered graph \( G \) has precisely one dissimilar Pfaffian orientations.

In view of the previous theorem, we have that any two Pfaffian orientations of a bipartite matching covered graph are similar. Let \( D \) be an orientation of a Pfaffian bipartite matching covered graph \( G \). Then, \( D \) is Pfaffian if and only if \( D \) is similar to an orientation to \( G \) obtained by Theorem 3.1.
References


