Edge-Colouring of Join Graphs

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A Comparative Study of Brazilian Beers

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Abstract
We discuss the following conjecture:
If \( G \) is a graph with \( n \) vertices and maximum degree \( \Delta > n/3 \), then \( G \) is 1-factorizable.

1 Introduction

The graphs in this paper are simple, that is they have no loops or multiple edges. Let \( G = (V, E) \) be a graph; the degree of a vertex \( v \), denoted by \( d_G(v) \), is the number of edges incident to \( v \); the maximum degree of \( G \), denoted by \( \Delta(G) \), is the maximum vertex degree in \( G \); \( G \) is regular if the degree of every vertex is the same.

An edge-colouring of a graph \( G = (V, E) \) is an assignment of colours to its edges so that no two edges incident with the same vertex receive the same colour. An edge-colouring of \( G \) using \( k \) colours (\( k \) edge-colouring) is then a partition of the edge set \( E \) into \( k \) disjoint matchings.

The chromatic index of \( G \), denoted by \( \chi'(G) \), is the least \( k \) for which \( G \) has a \( k \) edge-colouring. In [10] it was shown that every graph \( G \) with \( m \) edges and \( \chi'(G) \leq k \) has an equalized \( k \) edge-colouring \( \mathcal{C} \): each colour \( f_i \) in \( \mathcal{C} \) appears on exactly either \( \lfloor \frac{m}{k} \rfloor \) edges or \( \lceil \frac{m}{k} \rceil \) edges.

A celebrated theorem of Vizing [17] states that

\[
\chi'(G) = \Delta(G) \quad \text{or} \quad \chi'(G) = \Delta(G) + 1.
\]

Graphs with \( \chi'(G) = \Delta(G) \) are said to be Class 1; graphs with \( \chi'(G) = \Delta(G) + 1 \) are said to be Class 2. The graphs that are Class 1 are also known as 1-factorizable graphs. Fournier [6] gave a polynomial time algorithm that finds a \( \Delta(G) + 1 \) edge-colouring of a graph \( G \).

Since it is NP-complete to determine if a cubic graph has chromatic index three [9], it follows that deciding whether a graph is Class 1 or Class 2 is NP-hard. The problem remains open for several classes of graphs, including the class of graphs that are \( P_4 \)-free (cographs) [1].

A graph \( G \) is overfull if

\[\frac{m}{\Delta} \geq n \geq \frac{2m}{\Delta(G) + 1} \]

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An easy counting argument shows that if $G$ is overfull then $|V(G)|$ must be odd and $G$ is Class 2 (in every edge-colouring at most $1/2(|V(G)| - 1)$ edges of $G$ can have the same colour). If $G$ is not overfull but it contains an overfull subgraph $H$ with $\Delta(H) = \Delta(G)$, then $G$ is Class 2.

Not every Class 2 graph necessarily contains an overfull subgraph with the same maximum degree. Examples of such graphs are very rare. The smallest one is $P^*$, the graph obtained from the Petersen graph by removing an arbitrary vertex. For all known of these graphs, the maximum degree is relatively small compared with the number of vertices ($\Delta(P^*) = |V(P^*)|/3$).

In 1985, Hilton proposed the following conjecture, known as Hilton’s Overfull Subgraph Conjecture [3]:

Conjecture 1 (Hilton) If $G$ is a graph with $\Delta(G) > |V(G)|/3$ and $G$ contains no overfull subgraph $H$ with $\Delta(H) = \Delta(G)$, then $G$ is Class 1.

Conjecture 1 was proved to be true for many special cases: when $G$ is a multipartite graph [8]; when $\Delta(G) \geq |V(G)| - 3$ [4, 15, 16]; and when the number of the vertices of maximum degree is “relatively small” and some other conditions on the maximum degree or the minimum degree hold [3, 5, 12].

If Conjecture 1 were true, then the problem of deciding whether a graph $G$ with $\Delta(G) > |V(G)|/3$ is Class 1 would be polynomially solvable [11, 13]. One more consequence of the validity of Conjecture 1 is that an old conjecture on regular graphs would be true:

Conjecture 2 Let $G$ be a $k$-regular graph with an even number of vertices. If $k \geq |V(G)|/2$ then $G$ is Class 1.

(Here, $k$-regular means that the degree of every vertex is equal to $k$.) Conjecture 2 appeared in [2] but may go back to G.A. Dirac in the early 1950s. To see that Conjecture 1 implies Conjecture 2, it is sufficient to observe that no $k$-regular graph $G$ with an even number of vertices, such that $k \geq |V(G)|/2$, contains an overfull subgraph $H$ with $\Delta(H) = k$ [3, 7].

Conjecture 2 was proved to be true when $k \geq 1/2(\sqrt{7} - 1)|V(G)|$ [4], and for large graphs when $|V(G)| < (2 - \epsilon)\Delta(G)$ [14].

The goal of this paper is to prove Conjecture 1 or Conjecture 2 for the class of join graphs.

2 The join graphs

Let $G = (V, E)$ be a graph with $n$ vertices. We say that $G$ is a join graph if $G$ is the complete union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. In other words, $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. If $G$ is the join graph of $G_1$ and $G_2$, we shall write $G = G_1 + G_2$. Note that the class of join graphs strictly contains the class of connected $P_4$-free graphs.
Write \( n_1 = |V_1|, n_2 = |V_2|, \Delta_1 = \Delta(G_1), \) and \( \Delta_2 = \Delta(G_2) \). Clearly, \( n = n_1 + n_2 \) and \( \Delta(G) = \max\{n_1 + \Delta_2, n_2 + \Delta_1\} \). Fig. 1 shows a join graph \( G = G_1 + G_2 \) with \( n = 5 \) and \( \Delta(G) = 3 \).

Without loss of generality we shall assume that \( |V_1| \leq |V_2| \). Note that a join graph \( G \) with \( n \) vertices satisfies \( \Delta(G) \geq n/2 \). Hence, if Conjecture 1 were true, then every join graph \( G \) that contains no overfull subgraph \( H \) with \( \Delta(H) = \Delta(G) \), would be Class 1; moreover if Conjecture 2 were true, then every regular join graph would be Class 1.

We shall show that:

- if \( \Delta_1 > \Delta_2 \), then Conjecture 1 holds true;
- if \( \Delta_1 = \Delta_2 \), then Conjecture 2 holds true;
- if \( \Delta_1 = \Delta_2 \), then Conjecture 1 holds true under some hypothesis;
- if \( \Delta_1 < \Delta_2 \) and \( n_1 = n_2 \), then Conjecture 1 holds true.

To every join graph \( G = G_1 + G_2 \) we shall associate the complete bipartite graph \( B_G \) obtained from \( G \) by removing all edges of \( G_1 \) and \( G_2 \). For every maximum matching \( M \) in \( B_G \), let \( G_M \) denote the subgraph of \( G \) obtained by removing all edges of \( B_G \) but the edges in \( M \). Fig. 2 shows two \( G_M \) for the graph \( G \) in Fig. 1.

Our results are based on the following key observation:

**Observation 1** Let \( G = G_1 + G_2 \) be a join graph with \( n_1 \leq n_2 \) such that \( \Delta_1 \geq \Delta_2 \) or \( \Delta_1 < \Delta_2 \) and \( n_1 = n_2 \). If there exists a maximum matching \( M \) in \( B_G \) such that the corresponding graph \( G_M \) is Class 1, then \( G \) is Class 1.
**Theorem 1**

In view of Observation 1, it is natural to ask when there exist a maximum matching of $G$. Assume that the theorem is not true. Then there exists a maximum matching of $G$ such that the corresponding graph $G_M$ is Class 1. The following result shows that such a matching always exists and that, in fact, every maximum matching of $B_G$ has the desired property.

**Theorem 1** Let $G = G_1 + G_2$ be a join graph with $n_1 \leq n_2$. If $\Delta_1 > \Delta_2$ then for every maximum matching $M$ of $B_G$, the corresponding graph $G_M$ is Class 1.

**Proof** Assume that the theorem is not true. Then there exists a maximum matching $M$ of $B_G$ such that the corresponding graph $G_M$ is Class 2, and so $\chi'(G_M) > \Delta(G_M) = \Delta_1 + 1$.

We shall find a contradiction. For this purpose, colour $G_1$ and $G_2$ with $\Delta_1 + 1$ colours $a_0, a_1, \ldots, a_{\Delta_1}$ (this can be done because $\Delta_1 > \Delta_2$). Now, extend this colouring to as many edges in $M$ as possible. By assumption, not every edge in $M$ has been coloured; in particular, some edge $uv$ in $M$ with $u \in V_2$ and $v \in V_1$ is not coloured. We shall show how we can extend our $(\Delta_1 + 1)$ edge-colouring in the graph $G_M$ so to colour also edge $uv$, getting then a contradiction. Until the end of the proof, we shall consider only the graph $G_M$.

For this purpose, first note that every neighbor of $u$, but vertex $v$, has degree less than or equal to $\Delta_2 + 1$ (every such a neighbor of $u$ is a vertex of $G_2$). Since $\Delta_2 < \Delta_1$, and since we used $\Delta_1 + 1$ colours, it follows that every neighbor of $u$, but vertex $v$, misses at least one colour $a_i$. Moreover, since $uv$ is not coloured, both $u$ and $v$ miss at least one colour. Let $a_0$ be a colour missing at $u$, and let $a_1$ be a colour missing at $v$; clearly, colour $a_1$ must appear at $u$, or we could use $a_1$ on $uv$. Suppose $v_1$ is the neighbor of $u$ along the edge coloured $a_1$. At $v_1$ some colour $a_2$ is missing; clearly, $a_2$ must appear at $u$, or we could recolour $uv_1$ from $a_1$ to $a_2$ and then use $a_1$ on $uv$ to extend the colouring.

For $i \geq 2$ we continue this process. Having selected a new colour $a_i$ that appears at $u$, let $v_i$ be the neighbor of $u$ along the edge coloured $a_i$. Let $a_{i+1}$ be a colour missing at $v_i$. If $a_{i+1}$ is missing at $u$, we recolour $uv_i$ from $a_i$ to $a_{i+1}$ and then we recolour every $uv_j$ with $j < i$ from $a_j$ to $a_{j+1}$ and then use $a_1$ on $uv$ to extend the colouring. Hence, we may assume that each colour missing at every neighbor of $u$ in $G_M$ is present at $u$. Since the neighbors of $u$, but $v$, are at most $\Delta_2$, the iterative selection of $a_{i+1}$ eventually repeats a colour. Let $l$ be the index of the vertex at which the first repetition occurs. In other words,
$a_i \neq a_j$ for every $i \neq j$ with $i, j \leq l$, and $a_{l+1} = a_k$ with $k < l + 1$. Colour $a_{l+1}$ is missing at $v_l$; colour $a_k$ is missing at $v_{k-1}$ and appears on $uv_k$. If $a_0$ is missing at $v_l$, then we could recolour $uv_l$ from $a_l$ to $a_0$ and then recolour $uv_i$ from $a_i$ to $a_{i+1}$ for every $i < l$ and finally use $a_1$ to colour $uv$.

Hence, we may assume that $a_0$ appears at $v_l$ and that $a_k(= a_{l+1})$ does not. Let $P$ be the unique maximal alternating path of edges coloured $a_0$ and $a_k$ that begins at $v_l$.

If $P$ reaches $v_k$, then it must reach $v_k$ along an edge coloured $a_0$, it continues along edge $v_ku$ coloured $a_k$ and finally it stops at $u$ (because $a_0$ is missing at $u$). But in this case, we could interchange colours $a_0$ and $a_k$ along $P$, recolour $uv_j$ from $a_j$ to $a_{j+1}$ for every $j < k$, and then use colour $a_1$ on $uv$. Similarly, if $P$ reaches $v_{k-1}$, then it must reach $v_{k-1}$ along an edge coloured $a_0$, and it stops there (because $a_k$ does not appear at $v_{k-1}$). But in this case, we could interchange colours $a_0$ and $a_k$ along $P$, recolour $uv_{k-1}$ from $a_{k-1}$ to $a_0$, recolour $uv_j$ from $a_j$ to $a_{j+1}$ for every $j < k - 1$, and then use colour $a_1$ on $uv$.

Hence, we may assume that $P$ does not reach $v_k$ and it does not reach $v_{k-1}$, and so $P$ ends in a vertex outside $\{u, v_l, v_k, v_{k-1}\}$. But then, we could interchange colours $a_0$ and $a_k$ along $P$, recolour $uv_l$ from $a_l$ to $a_0$, recolour $uv_j$ from $a_j$ to $a_{j+1}$ for every $j < l$, and then use colour $a_1$ on $uv$.

\begin{corollary}
Let $G = G_1 + G_2$ be a join graph with $n_1 \leq n_2$. If $\Delta_1 > \Delta_2$ then $G$ is Class 1 and Conjecture 1 holds true.
\end{corollary}

Note that the proof of Theorem 1 yields a polynomial-time algorithm to colour the edges of a join graph $G = G_1 + G_2$ with $\Delta(G)$ colours, whenever $n_1 \leq n_2$ and $\Delta_1 > \Delta_2$.

4 \hspace{1cm} \Delta_1 = \Delta_2

Let $G = G_1 + G_2$ be a join graph with $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $n_1 \leq n_2$. Assume that $\Delta_1 = \Delta_2$. In view of Observation 1, it is natural to ask whether a result similar to Theorem 1 is still valid. Unfortunately, this is not the case. For instance, consider the join graph $G = G_1 + G_2$ in Fig. 3: it is easy to see that, for every maximum matching $M$ of $B_G$, the corresponding graph $G_M$ is Class 2 ($G_M$ is overfull). On the other hand, consider the case of the join graph $G = C_5 + C_5$ (where $C_5$ denotes the chordless cycle with five vertices); if $M$ is chosen so that $G_M$ is the Petersen graph then $G_M$ is Class 2; on the other hand it is easy to see that there exist maximum matchings $M$ such that the corresponding $G_M$ are Class 1.
Fig. 3

It follows that we can still make use of Observation 1 by finding sufficient conditions under which join graphs $G = G_1 + G_2$ with $\Delta_1 = \Delta_2$ have the property of the existence of a maximum matching $M$ of $B_G$ such that $G_M$ is Class 1.

**Theorem 2** Let $G = G_1 + G_2$ be a join graph with $\Delta_1 = \Delta_2$. If one of the following three conditions holds

(i) both $G_1$ and $G_2$ are Class 1;
(ii) $G_1$ is a subgraph of $G_2$;
(iii) both $G_1$ and $G_2$ are disjoint unions of cliques;

then there exists a maximum matching $M$ of $B_G$ such that the corresponding graph $G_M$ is Class 1.

**Proof** Without loss of generality, we can assume that $n_1 \leq n_2$.

First, assume that (i) holds. Let $M$ be an arbitrary maximum matching of $B_G$. Since $G_1$ and $G_2$ are Class 1, and since $\Delta_2 = \Delta_1$, it follows that we can colour the edges of $G_1$ and $G_2$ with $\Delta_1$ colours. If we use an extra colour to colour all edges in $M$, then $\chi'(G_M) \leq \Delta_1 + 1 = \Delta(G_M)$, and so $G_M$ is Class 1.

Secondly, assume that (ii) holds. Let $V_1 = \{u_1, \ldots, u_{n_1}\}$ and $V_2 = \{v_1, \ldots, v_{n_2}\}$. Colour the edges of $G_2$ with $\Delta_2 + 1$ colours $f_1, \ldots, f_{\Delta_2+1}$. Since $G_1$ is a subgraph of $G_2$, it follows that for every edge $u_iu_j$ of $G_1$ $v_iv_j$ is an edge of $G_2$; let $f_k$ be the colour of $v_iv_j$. Colour $u_iu_j$ with colour $f_k$. Hence, we can extend the $\Delta_2 + 1$ edge-colouring of $G_2$ to all the edges of $G_1$. Now, let $u_i$ be an arbitrary vertex of $G_1$ and let $v_i$ be the corresponding vertex of $G_2$ (such a vertex exists because $G_1$ is a subgraph of $G_2$). Let $f_k$ be a colour missing at $v_i$ (such a colour exists because $d_{G_2}(v_i) \leq \Delta_2$). By construction, colour $f_k$ is missing also at $u_i$ and so we can colour $u_iv_i$ with colour $f_k$. Since we can repeat this operation for every vertex of $G_1$, it follows that for the matching $M = \{u_iv_i, i = 1, \ldots, n_1\}$ the graph $G_M$ is $\Delta_2 + 1$ edge-colourable, and so $G_M$ is Class 1.

Finally, assume that (iii) holds. Order the vertices of $G_1$, $u_1, \ldots, u_{n_1}$, so that all the vertices in a same connected component of $G_1$ are consecutive, and such that if $u_i$ belongs
to a clique $K_i$ and $u_j$ belongs to a clique $K_s$ with $t > s$ then $i > j$. Similarly, we can order the vertices of $G_2$, $v_1, \ldots, v_{n_2}$, so that all the vertices in a same connected component of $G_2$ are consecutive, and such that if $v_i$ belongs to a clique $K_t$ and $v_j$ belongs to a clique $K_s$ with $t > s$ then $i > j$.

Let $C = \{f_0, \ldots, f_{\Delta_1}\}$ be the $\Delta_1 + 1$ edge-colouring of $G_1$ obtained in the following way: to every edge $u_i u_j$ assign colour $f_h$ with $h = (i + j) \mod (\Delta_1 + 1)$. To show that this colouring is admissible, we only need verify that any two arbitrary adjacent edges of $G_1$ have different colours. For this purpose, assume that the edges $u_i u_j$ and $u_j u_k$ (with $i \neq k$) have been assigned the same colour $f_h$. Then, by construction, $h = (i + j) \mod (\Delta_1 + 1)$ and $h = (j + k) \mod (\Delta_1 + 1)$. It follows that $h = i + j - t_1(\Delta_1 + 1)$ (for some nonnegative integer $t_1$) and $h = j + k - t_2(\Delta_1 + 1)$ (for some nonnegative integer $t_2$), with $t_2 \neq t_1$ (because $k \neq i$). But then we can write $(\Delta_1 + 1)(t_2 - t_1) - (k - i) = 0$, and so $|k - i| = |t_2 - t_1|(\Delta_1 + 1)$, which implies that $|k - i| \geq \Delta_1 + 1$. On the other hand, the chosen ordering of the vertices of $G_1$ implies that $|k - i| \leq \Delta_1$ (because $u_i$ and $u_j$ belong to a same clique whose size is at most $\Delta_1 + 1$), a contradiction.

Note that, by construction, for every $i = 1, \ldots, n_1$, vertex $u_i$ misses colour $f_{(2i) \mod (\Delta_1 + 1)}$. Since $\Delta_2 = \Delta_1$, we can colour the edges of $G_2$ in a similar way using the same colours in $C$: to every edge $v_i v_j$ of $G_2$, assign colour $f_{h}$ with $h = (i + j) \mod (\Delta_1 + 1)$. By construction, for every $i = 1, \ldots, n_2$, vertex $v_i$ misses colour $f_{(2i) \mod (\Delta_1 + 1)}$.

Now we are ready to choose the desired maximum matching $M$ of $BG$: $M = \{u_i v_i, i = 1, \ldots, n_1\}$. Indeed, for every $i = 1, \ldots, n_1$, we can assign to edge $u_i v_i$ the colour $f_{(2i) \mod (\Delta_1 + 1)}$. Thus, $G_M$ is Class 1 and the theorem follows.

**Corollary 2** Let $G = G_1 + G_2$ be a join graph with $\Delta_1 = \Delta_2$. If one of the following three conditions holds
(i) both $G_1$ and $G_2$ are Class 1
(ii) $G_1$ is a subgraph of $G_2$
(iii) both $G_1$ and $G_2$ are disjoint unions of cliques,
then $G$ is Class 1.

Note that the proof of Theorem 2 gives an algorithm to colour the edges of a join graph $G = G_1 + G_2$ with $\Delta(G)$ colours, whenever $\Delta_1 = \Delta_2$, and $G_1$ is a subgraph of $G_2$ or both $G_1$ and $G_2$ are disjoint unions of cliques.

**Theorem 3** Every regular join graph $G = G_1 + G_2$ with $\Delta_1 = \Delta_2$ is Class 1.

**Proof** Let $m_i$ denote the number of edges of $G_i$, $i = 1, 2$. Since $G$ is regular and that $\Delta_1 = \Delta_2$, it follows that $n_1 = n_2$ and $m_1 = m_2$. Let $C_1 = \{f_1, \ldots, f_{\Delta_1 + 1}\}$ be an equalized edge-colouring of $G_1$; and let $C_2 = \{g_1, \ldots, g_{\Delta_2 + 1}\}$ be an equalized edge-colouring of $G_2$.

Since $C_1$ is equalized, each colour $f_i$ ($i = 1, \ldots, \Delta_1 + 1$) is missed by exactly $n_1 - 2\left\lceil \frac{m_1}{\Delta_1 + 1} \right\rceil$ or $n_1 - 2\left\lceil \frac{m_1}{\Delta_1 + 1} \right\rceil$ vertices of $G_1$; similarly, each colour $g_i$ ($i = 1, \ldots, \Delta_2 + 1$) is missed by exactly $n_2 - 2\left\lceil \frac{m_2}{\Delta_2 + 1} \right\rceil$ or $n_2 - 2\left\lceil \frac{m_2}{\Delta_2 + 1} \right\rceil$ vertices of $G_2$. Without loss of generality, we can assume that colours $f_1, \ldots, f_p$ are missed by exactly $n_1 - 2\left\lceil \frac{m_1}{\Delta_1 + 1} \right\rceil$ vertices of $G_1$, that colours $f_{p+1}, \ldots, f_{\Delta_1 + 1}$ are missed by exactly $n_1 - 2\left\lceil \frac{m_1}{\Delta_1 + 1} \right\rceil$ vertices of $G_1$, that colours
$g_1, \ldots, g_q$ are missed by exactly $n_2 - 2\left\lceil \frac{m_2}{\Delta_2 + 1} \right\rceil$ vertices of $G_2$, that colours $g_{q+1}, \ldots, g_{\Delta_2+1}$ are missed by exactly $n_2 - 2\left\lceil \frac{m_2}{\Delta_2 + 1} \right\rceil$ vertices of $G_2$.

Since $G$ is regular, it follows that $G_1$ is $\Delta_1$-regular and that $G_2$ is $\Delta_2$-regular, and so each vertex $u_i$ of $G_1$ misses exactly one colour $f_j$ and each vertex $v_i$ of $G_2$ misses exactly one colour $g_h$. Thus we can write

\[ n_1 = p \left( n_1 - 2 \left\lceil \frac{m_1}{\Delta_1 + 1} \right\rceil \right) + (\Delta_1 + 1 - p) \left( n_1 - 2 \left\lceil \frac{m_1}{\Delta_1 + 1} \right\rceil \right) \]

\[ n_2 = q \left( n_2 - 2 \left\lceil \frac{m_2}{\Delta_2 + 1} \right\rceil \right) + (\Delta_2 + 1 - q) \left( n_2 - 2 \left\lceil \frac{m_2}{\Delta_2 + 1} \right\rceil \right). \]

Since $n_1 = n_2$ and $m_1 = m_2$, we can write

\[ (p - q) \left( n_1 - 2 \left\lceil \frac{m_1}{\Delta_1 + 1} \right\rceil \right) = (p - q) \left( n_1 - 2 \left\lceil \frac{m_1}{\Delta_1 + 1} \right\rceil \right). \]

But then,

\[ p = q \quad \text{or} \quad \left\lceil \frac{m_1}{\Delta_1 + 1} \right\rceil = \left\lceil \frac{m_1}{\Delta_1 + 1} \right\rceil. \]

Note that in the latter case, we must have $p = \Delta_1 + 1$ and $q = \Delta_2 + 1$. Hence, $p = q$, and so we can assume that $g_i = f_i$ for every $i = 1, \ldots, \Delta_1 + 1$.

Now, let $M = \{u_i v_i : i = 1, \ldots, n_1\}$. For every $i = 1, \ldots, \Delta_1 + 1$, since both $u_i$ and $v_i$ miss the same colour, say $f_k$, we can assign to edge $u_i v_i$ the colour $f_k$. But then we get a $\Delta_1 + 1$ edge-colouring of $G_M$, and so $G_M$ is Class 1.

Note that the proof of Theorem 3 gives an algorithm to colour the edges of a regular join graph $G = G_1 + G_2$ with $\Delta(G)$ colours, whenever $\Delta_1 = \Delta_2$.

**Corollary 3** Conjecture 2 holds true for every regular join graph $G = G_1 + G_2$ with $\Delta_1 = \Delta_2$.

5 $\Delta_1 < \Delta_2$

Let $G = G_1 + G_2$ be a join graph with $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $n_1 \leq n_2$.

If $n_1 < n_2$, then Observation 1 does not help. Indeed, there are join graphs $G = G_1 + G_2$ with $\Delta_1 < \Delta_2$ and $n_1 < n_2$ such that $G$ is Class 2 even though $G_M$ is Class 1 for every maximum matching $M$. This is, for instance, the case of the graph in Fig. 4.
Moreover, when $\Delta_1 < \Delta_2$ and $n_1 < n_2$ there are graphs $G = G_1 + G_2$ that satisfy some of the three conditions in Theorem 2 and are Class 2. For instance, every complete graph $G$ with an odd number of vertices satisfies conditions $(ii)$ and $(iii)$; the graph in Fig. 4 satisfies conditions $(i)$ and $(ii)$.

However, if we assume that $n_1 = n_2$, then we can apply Observation 1. In fact, we can get a strong result similar to Theorem 1:

**Theorem 4** Let $G = G_1 + G_2$ be a join graph with $n_1 = n_2$. If $\Delta_1 < \Delta_2$ then for every maximum matching $M$ of $B_G$, the corresponding graph $G_M$ is Class 1.

**Proof** Interchange the roles of $G_1$ and $G_2$ and apply Theorem 1.  

**Corollary 4** Let $G = G_1 + G_2$ be a join graph with $n_1 = n_2$. If $\Delta_1 < \Delta_2$ then $G$ is Class 1 and Conjecture 1 holds true.

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