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Edge-Colouring of Join Graphs

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A Comparative Study of Brazilian Beers

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Abstract

We discuss the following conjecture:
If G is a graph with n vertices and maximum degree $\Delta > n/3$, then G is 1-factorizable.

1 Introduction

The graphs in this paper are simple, that is they have no loops or multiple edges. Let $G = (V, E)$ be a graph; the *degree* of a vertex v, denoted by $d_G(v)$, is the number of edges incident to v; the *maximum degree* of G, denoted by $\Delta(G)$, is the maximum vertex degree in G; G is *regular* if the degree of every vertex is the same.

An *edge-colouring* of a graph $G = (V, E)$ is an assignment of colours to its edges so that no two edges incident with the same vertex receive the same colour. An edge-colouring of G using k colours (k *edge-colouring*) is then a partition of the edge set E into k disjoint matchings.

The *chromatic index* of G, denoted by $\chi'(G)$, is the least k for which G has a k edge-colouring. In [10] it was shown that every graph G with m edges and $\chi'(G) \leq k$ has an *equalized* k edge-colouring $\mathcal{C}$: each colour f_i in $\mathcal{C}$ appears on exactly either $\lfloor \frac{m}{k} \rfloor$ edges or $\lceil \frac{m}{k} \rceil$ edges.

A celebrated theorem of Vizing [17] states that

$$\chi'(G) = \Delta(G) \quad \text{or} \quad \chi'(G) = \Delta(G) + 1.$$

Graphs with $\chi'(G) = \Delta(G)$ are said to be *Class 1*; graphs with $\chi'(G) = \Delta(G)+1$ are said to be *Class 2*. The graphs that are Class 1 are also known as 1-factorizable graphs. Fournier [6] gave a polynomial time algorithm that finds a $\Delta(G) + 1$ edge-colouring of a graph G.

Since it is NP-complete to determine if a cubic graph has chromatic index three [9], it follows that deciding whether a graph is Class 1 or Class 2 is NP-hard. The problem remains open for several classes of graphs, including the class of graphs that are P_4-free (cographs) [1].

A graph G is *overfull* if

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$$|E(G)| > \Delta(G)\frac{|V(G)|-1}{2}. \qquad (1)$$

An easy counting argument shows that if G is overfull then $|V(G)|$ must be odd and G is Class 2 (in every edge-colouring at most $1/2(|V(G)|-1)$ edges of G can have the same colour). If G is not overfull but it contains an overfull subgraph H with $\Delta(H) = \Delta(G)$, then G is Class 2.

Not every Class 2 graph necessarily contains an overfull subgraph with the same maximum degree. Examples of such graphs are very rare. The smallest one is P^*, the graph obtained from the Petersen graph by removing an arbitrary vertex. For all known of these graphs, the maximum degree is relatively small compared with the number of vertices ($\Delta(P^*) = |V(P^*)|/3$).

In 1985, Hilton proposed the following conjecture, known as *Hilton's Overfull Subgraph Conjecture* [3]:

Conjecture 1 *(Hilton) If G is a graph with $\Delta(G) > |V(G)|/3$ and G contains no overfull subgraph H with $\Delta(H) = \Delta(G)$, then G is Class 1.*

Conjecture 1 was proved to be true for many special cases: when G is a multipartite graph [8]; when $\Delta(G) \geq |V(G)| - 3$ [4, 15, 16]; and when the number of the vertices of maximum degree is "relatively small" and some other conditions on the maximum degree or the minimum degree hold [3, 5, 12].

If Conjecture 1 were true, then the problem of deciding whether a graph G with $\Delta(G) > |V(G)|/3$ is Class 1 would be polynomially solvable [11, 13]. One more consequence of the validity of Conjecture 1 is that an old conjecture on regular graphs would be true:

Conjecture 2 *Let G be a k-regular graph with an even number of vertices. If $k \geq |V(G)|/2$ then G is Class 1.*

(Here, k-regular means that the degree of every vertex is equal to k.) Conjecture 2 appeared in [2] but may go back to G.A. Dirac in the early 1950s. To see that Conjecture 1 implies Conjecture 2, it is sufficient to observe that no k-regular graph G with an even number of vertices, such that $k \geq |V(G)|/2$, contains an overfull subgraph H with $\Delta(H) = k$ [3, 7]. Conjecture 2 was proved to be true when $k \geq 1/2(\sqrt{7}-1)|V(G)|$ [4], and for large graphs when $|V(G)| < (2-\epsilon)\Delta(G)$ [14].

The goal of this paper is to prove Conjecture 1 or Conjecture 2 for the class of join graphs.

2 The join graphs

Let $G = (V, E)$ be a graph with n vertices. We say that G is a *join* graph if G is the complete union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. In other words, $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. If G is the join graph of G_1 and G_2, we shall write $G = G_1 + G_2$. Note that the class of join graphs strictly contains the class of connected P_4-free graphs.

Write $n_1 = |V_1|$, $n_2 = |V_2|$, $\Delta_1 = \Delta(G_1)$, and $\Delta_2 = \Delta(G_2)$. Clearly, $n = n_1 + n_2$ and $\Delta(G) = \max\{n_1 + \Delta_2, n_2 + \Delta_1\}$. Fig. 1 shows a join graph $G = G_1 + G_2$ with $n = 5$ and $\Delta(G) = 3$.

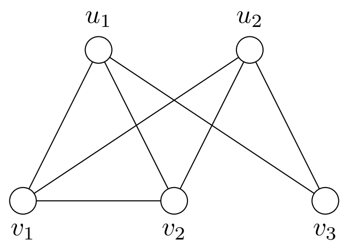

Fig. 1

Without loss of generality we shall assume that $|V_1| \leq |V_2|$. Note that a join graph G with n vertices satisfies $\Delta(G) \geq n/2$. Hence, if Conjecture 1 were true, then every join graph G that contains no overfull subgraph H with $\Delta(H) = \Delta(G)$, would be Class 1; moreover if Conjecture 2 were true, then every regular join graph would be Class 1.

We shall show that:

- if $\Delta_1 > \Delta_2$, then Conjecture 1 holds true;
- if $\Delta_1 = \Delta_2$, then Conjecture 2 holds true;
- if $\Delta_1 = \Delta_2$, then Conjecture 1 holds true under some hypothesys;
- if $\Delta_1 < \Delta_2$ and $n_1 = n_2$, then Conjecture 1 holds true.

To every join graph $G = G_1 + G_2$ we shall associate the complete bipartite graph B_G obtained from G by removing all edges of G_1 and G_2. For every maximum matching M in B_G, let G_M denote the subgraph of G obtained by removing all edges of B_G but the edges in M. Fig. 2 shows two G_M for the graph G in Fig. 1.

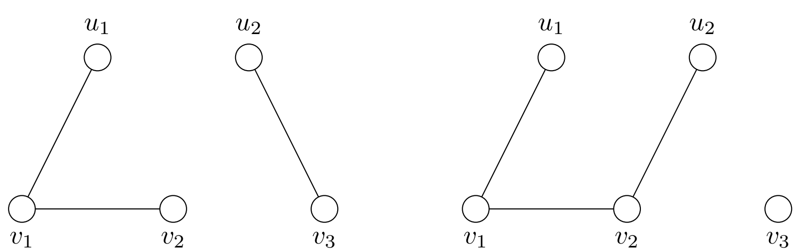

Fig. 2

Our results are based on the following key observation:

Observation 1 *Let $G = G_1 + G_2$ be a join graph with $n_1 \leq n_2$ such that $\Delta_1 \geq \Delta_2$ or $\Delta_1 < \Delta_2$ and $n_1 = n_2$. If there exists a maximum matching M in B_G such that the corresponding graph G_M is Class 1, then G is Class 1.*

Proof Let M be a maximum matching of B_G such that $\chi'(G_M) = \Delta(G_M)$; and let B' be the bipartite graph obtained from B_G by removing all edges in M. Note that $\chi'(G) \leq \chi'(G_M) + \chi'(B')$, and that $\chi'(B') = \Delta(B') = n_2 - 1$ (because $n_1 \leq n_2$). If $\Delta_1 \geq \Delta_2$, then $\Delta(G) = \Delta_1 + n_2$ and $\Delta(G_M) = \Delta_1 + 1$, and so $\chi'(G) \leq \Delta(G)$. If $\Delta_1 < \Delta_2$ and $n_1 = n_2$, then $\Delta(G) = \Delta_2 + n_2$ and $\Delta(G_M) = \Delta_2 + 1$, and so $\chi'(G) \leq \Delta(G)$. ■

In section 3 we shall study join graphs $G = G_1 + G_2$ with $n_1 \leq n_2$ and $\Delta_1 > \Delta_2$; in section 4 we shall study join graphs $G = G_1 + G_2$ with $n_1 \leq n_2$ and $\Delta_1 = \Delta_2$; in section 5 we shall study join graphs $G = G_1 + G_2$ with $n_1 \leq n_2$ and $\Delta_1 < \Delta_2$.

3 $\Delta_1 > \Delta_2$

Let $G = G_1 + G_2$ be a join graph with $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $n_1 \leq n_2$. In view of Observation 1, it is natural to ask when there exists a maximum matching M of B_G such that the corresponding graph G_M is Class 1. The following result shows that such a matching always exists and that, in fact, every maximum matching of B_G has the desired property.

Theorem 1 *Let $G = G_1 + G_2$ be a join graph with $n_1 \leq n_2$. If $\Delta_1 > \Delta_2$ then for every maximum matching M of B_G, the corresponding graph G_M is Class 1.*

Proof Assume that the theorem is not true. Then there exists a maximum matching M of B_G such that the corresponding graph G_M is Class 2, and so $\chi'(G_M) > \Delta(G_M) = \Delta_1 + 1$.

We shall find a contradiction. For this purpose, colour G_1 and G_2 with $\Delta_1 + 1$ colours $a_0, a_1, \ldots, a_{\Delta_1}$ (this can be done because $\Delta_1 > \Delta_2$). Now, extend this colouring to as many edges in M as possible. By assumption, not every edge in M has been coloured; in particular, some edge uv in M with $u \in V_2$ and $v \in V_1$ is not coloured. We shall show how we can extend our $(\Delta_1 + 1)$ edge-colouring in the graph G_M so to colour also edge uv, getting then a contradiction. Until the end of the proof, we shall consider only the graph G_M.

For this purpose, first note that every neighbor of u, but vertex v, has degree less than or equal to $\Delta_2 + 1$ (every such a neighbor of u is a vertex of G_2). Since $\Delta_2 < \Delta_1$, and since we used $\Delta_1 + 1$ colours, it follows that every neighbor of u, but vertex v, misses at least one colour a_i. Moreover, since uv is not coloured, both u and v miss at least one colour. Let a_0 be a colour missing at u, and let a_1 be a colour missing at v; clearly, colour a_1 must appear at u, or we could use a_1 on uv. Suppose v_1 is the neighbor of u along the edge coloured a_1. At v_1 some colour a_2 is missing; clearly, a_2 must appear at u, or we could recolour uv_1 from a_1 to a_2 and then use a_1 on uv to extend the colouring.

For $i \geq 2$ we continue this process. Having selected a new colour a_i that appears at u, let v_i be the neighbor of u along the edge coloured a_i. Let a_{i+1} be a colour missing at v_i. If a_{i+1} is missing at u, we recolour uv_i from a_i to a_{i+1} and then we recolour every uv_j with $j < i$ from a_j to a_{j+1} and then use a_1 on uv to extend the colouring. Hence, we may assume that each colour missing at every neighbor of u in G_M is present at u. Since the neighbors of u, but v, are at most Δ_2, the iterative selection of a_{i+1} eventually repeats a colour. Let l be the index of the vertex at which the first repetition occurs. In other words,

$a_i \neq a_j$ for every $i \neq j$ with $i, j \leq l$, and $a_{l+1} = a_k$ with $k < l + 1$. Colour a_{l+1} is missing at v_l; colour a_k is missing at v_{k-1} and appears on uv_k. If a_0 is missing at v_l, then we could recolour uv_l from a_l to a_0 and then recolour uv_i from a_i to a_{i+1} for every $i < l$ and finally use a_1 to colour uv.

Hence, we may assume that a_0 appears at v_l and that $a_k(= a_{l+1})$ does not. Let P be the unique maximal alternating path of edges coloured a_0 and a_k that begins at v_l.

If P reaches v_k, then it must reach v_k along an edge coloured a_0, it continues along edge $v_k u$ coloured a_k and finally it stops at u (because a_0 is missing at u). But in this case, we could interchange colours a_0 and a_k along P, recolour uv_j from a_j to a_{j+1} for every $j < k$, and then use colour a_1 on uv. Similarly, if P reaches v_{k-1}, then it must reach v_{k-1} along an edge coloured a_0, and it stops there (because a_k does not appear at v_{k-1}). But in this case, we could interchange colours a_0 and a_k along P, recolour uv_{k-1} from a_{k-1} to a_0, recolour uv_j from a_j to a_{j+1} for every $j < k - 1$, and then use colour a_1 on uv.

Hence, we may assume that P does not reach v_k and it does not reach v_{k-1}, and so P ends in a vertex outside $\{u, v_l, v_k, v_{k-1}\}$. But then, we could interchange colours a_0 and a_k along P, recolour uv_l from a_l to a_0, recolour uv_j from a_j to a_{j+1} for every $j < l$, and then use colour a_1 on uv. ■

Corollary 1 *Let $G = G_1 + G_2$ be a join graph with $n_1 \leq n_2$. If $\Delta_1 > \Delta_2$ then G is Class 1 and Conjecture 1 holds true.*

Note that the proof of Theorem 1 yields a polynomial-time algorithm to colour the edges of a join graph $G = G_1 + G_2$ with $\Delta(G)$ colours, whenever $n_1 \leq n_2$ and $\Delta_1 > \Delta_2$.

4 $\Delta_1 = \Delta_2$

Let $G = G_1 + G_2$ be a join graph with $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $n_1 \leq n_2$. Assume that $\Delta_1 = \Delta_2$. In view of Observation 1, it is natural to ask whether a result similar to Theorem 1 is still valid. Unfortunately, this is not the case. For instance, consider the join graph $G = G_1 + G_2$ in Fig. 3: it is easy to see that, for every maximum matching M of B_G, the corresponding graph G_M is Class 2 (G_M is overfull). On the other hand, consider the case of the join graph $G = C_5 + C_5$ (where C_5 denotes the chordless cycle with five vertices); if M is chosen so that G_M is the Petersen graph then G_M is Class 2; on the other hand it is easy to see that there exist maximum matchings M such that the corresponding G_M are Class 1.

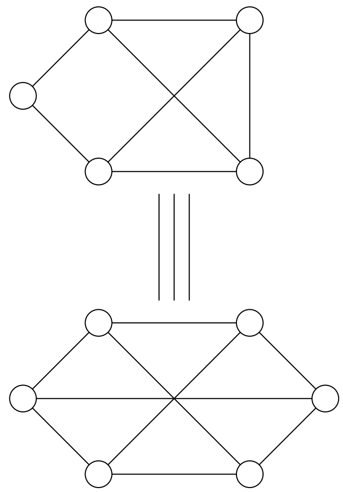

Fig. 3

It follows that we can still make use of Observation 1 by finding sufficient conditions under which join graphs $G = G_1 + G_2$ with $\Delta_1 = \Delta_2$ have the property of the existence of a maximum matching M of B_G such that G_M is Class 1.

Theorem 2 *Let $G = G_1 + G_2$ be a join graph with $\Delta_1 = \Delta_2$. If one of the following three conditions holds*
(i) both G_1 and G_2 are Class 1;
(ii) G_1 is a subgraph of G_2;
(iii) both G_1 and G_2 are disjoint unions of cliques;
then there exists a maximum matching M of B_G such that the corresponding graph G_M is Class 1.

Proof Without loss of generality, we can assume that $n_1 \leq n_2$.

First, assume that (i) holds. Let M be an arbitrary maximum matching of B_G. Since G_1 and G_2 are Class 1, and since $\Delta_2 = \Delta_1$, it follows that we can colour the edges of G_1 and G_2 with Δ_1 colours. If we use an extra colour to colour all edges in M, then $\chi'(G_M) \leq \Delta_1 + 1 = \Delta(G_M)$, and so G_M is Class 1.

Secondly, assume that (ii) holds. Let $V_1 = \{u_1, \ldots, u_{n_1}\}$ and $V_2 = \{v_1, \ldots, v_{n_2}\}$. Colour the edges of G_2 with $\Delta_2 + 1$ colours $f_1, \ldots, f_{\Delta_2+1}$. Since G_1 is a subgraph of G_2, it follows that for every edge u_iu_j of G_1 v_iv_j is an edge of G_2; let f_k be the colour of v_iv_j. Colour u_iu_j with colour f_k. Hence, we can extend the $\Delta_2 + 1$ edge-colouring of G_2 to all the edges of G_1. Now, let u_i be an arbitrary vertex of G_1 and let v_i be the corresponding vertex of G_2 (such a vertex exists because G_1 is a subgraph of G_2). Let f_k be a colour missing at v_i (such a colour exists because $d_{G_2}(v_i) \leq \Delta_2$). By construction, colour f_k is missing also at u_i and so we can colour u_iv_i with colour f_k. Since we can repeat this operation for every vertex of G_1, it follows that for the matching $M = \{u_iv_i, i = 1, \ldots, n_1\}$ the graph G_M is $\Delta_2 + 1$ edge-colourable, and so G_M is Class 1.

Finally, assume that (iii) holds. Order the vertices of G_1, $u_1, \ldots, u_{n_1}$, so that all the vertices in a same connected component of G_1 are consecutive, and such that if u_i belongs to a clique K_t and u_j belongs to a clique K_s with $t > s$ then $i > j$. Similarly, we can order the vertices of G_2, $v_1, \ldots, v_{n_2}$, so that all the vertices in a same connected component of G_2 are consecutive, and such that if v_i belongs to a clique K_t and v_j belongs to a clique K_s with $t > s$ then $i > j$.

Let $\mathcal{C} = \{f_0, \ldots, f_{\Delta_1}\}$ be the $\Delta_1 + 1$ edge-colouring of G_1 obtained in the following way: to every edge $u_i u_j$ assign colour f_h with $h = (i + j)\ mod(\Delta_1 + 1)$. To show that this colouring is admissible, we only need verify that any two arbitrary adjacent edges of G_1 have different colours. For this purpose, assume that the edges $u_i u_j$ and $u_j u_k$ (with $i \neq k$) have been assigned the same colour f_h. Then, by construction, $h = (i+j)\ mod(\Delta_1+1)$ and $h = (j+k)\ mod(\Delta_1+1)$. It follows that $h = i+j-t_1(\Delta_1+1)$ (for some nonnegative integer t_1) and $h = j + k - t_2(\Delta_1 + 1)$ (for some nonnegative integer t_2), with $t_2 \neq t_1$ (because $k \neq i$). But then we can write $(\Delta_1+1)(t_2-t_1)-(k-i) = 0$, and so $|k-i| = |t_2-t_1|(\Delta_1+1)$, which implies that $|k - i| \geq \Delta_1 + 1$. On the other hand, the chosen ordering of the vertices of G_1 implies that $|k - i| \leq \Delta_1$ (because u_i and u_j belong to a same clique whose size is at most $\Delta_1 + 1$), a contradiction.

Note that, by construction, for every $i = 1, \ldots, n_1$, vertex u_i misses colour $f_{(2i) mod(\Delta_1+1)}$. Since $\Delta_2 = \Delta_1$, we can colour the edges of G_2 in a similar way using the same colours in $\mathcal{C}$: to every edge $v_i v_j$ of G_2, assign colour f_h with $h = (i + j)\ mod(\Delta_1 + 1)$. By construction, for every $i = 1, \ldots, n_2$, vertex v_i misses colour $f_{(2i) mod(\Delta_1+1)}$.

Now we are ready to choose the desired maximum matching M of B_G: $M = \{u_i v_i, i = 1, \ldots, n_1\}$. Indeed, for every $i = 1, \ldots, n_1$, we can assign to edge $u_i v_i$ the colour $f_{(2i) mod(\Delta_1+1)}$. Thus, G_M is Class 1 and the theorem follows. ■

Corollary 2 *Let $G = G_1 + G_2$ be a join graph with $\Delta_1 = \Delta_2$. If one of the following three conditions holds*
(i) both G_1 and G_2 are Class 1
(ii) G_1 is a subgraph of G_2
(iii) both G_1 and G_2 are disjoint unions of cliques,
then G is Class 1.

Note that the proof of Theorem 2 gives an algorithm to colour the edges of a join graph $G = G_1 + G_2$ with $\Delta(G)$ colours, whenever $\Delta_1 = \Delta_2$, and G_1 is a subgraph of G_2 or both G_1 and G_2 are disjoint unions of cliques.

Theorem 3 *Every regular join graph $G = G_1 + G_2$ with $\Delta_1 = \Delta_2$ is Class 1.*

Proof Let m_i denote the number of edges of G_i, $i = 1, 2$. Since G is regular and that $\Delta_1 = \Delta_2$, it follows that $n_1 = n_2$ and $m_1 = m_2$. Let $\mathcal{C}_1 = \{f_1, \ldots, f_{\Delta_1+1}\}$ be an equalized edge-colouring of G_1; and let $\mathcal{C}_2 = \{g_1, \ldots, g_{\Delta_2+1}\}$ be an equalized edge-colouring of G_2.

Since $\mathcal{C}_1$ is equalized, each colour f_i ($i = 1, \ldots, \Delta_1+1$) is missed by exactly $n_1 - 2\lfloor \frac{m_1}{\Delta_1+1} \rfloor$ or $n_1 - 2\lceil \frac{m_1}{\Delta_1+1} \rceil$ vertices of G_1; similarly, each colour g_i ($i = 1, \ldots, \Delta_2 + 1$) is missed by exactly $n_2 - 2\lfloor \frac{m_2}{\Delta_2+1} \rfloor$ or $n_2 - 2\lceil \frac{m_2}{\Delta_2+1} \rceil$ vertices of G_2. Without loss of generality, we can assume that colours $f_1, \ldots, f_p$ are missed by exactly $n_1 - 2\lfloor \frac{m_1}{\Delta_1+1} \rfloor$ vertices of G_1, that colours $f_{p+1}, \ldots, f_{\Delta_1+1}$ are missed by exactly $n_1 - 2\lceil \frac{m_1}{\Delta_1+1} \rceil$ vertices of G_1, that colours

$g_1, \ldots, g_q$ are missed by exactly $n_2 - 2\lfloor \frac{m_2}{\Delta_2+1} \rfloor$ vertices of G_2, that colours $g_{q+1}, \ldots, g_{\Delta_2+1}$ are missed by exactly $n_2 - 2\lceil \frac{m_2}{\Delta_2+1} \rceil$ vertices of G_2.

Since G is regular, it follows that G_1 is Δ_1-regular and that G_2 is Δ_2-regular, and so each vertex u_i of G_1 misses exactly one colour f_j and each vertex v_i of G_2 misses exactly one colour g_h. Thus we can write

$$\begin{aligned} n_1 &= p\left(n_1 - 2\left\lfloor \tfrac{m_1}{\Delta_1+1} \right\rfloor\right) + (\Delta_1 + 1 - p)\left(n_1 - 2\left\lceil \tfrac{m_1}{\Delta_1+1} \right\rceil\right) \\ n_2 &= q\left(n_2 - 2\left\lfloor \tfrac{m_2}{\Delta_2+1} \right\rfloor\right) + (\Delta_2 + 1 - q)\left(n_2 - 2\left\lceil \tfrac{m_2}{\Delta_2+1} \right\rceil\right). \end{aligned}$$

Since $n_1 = n_2$ and $m_1 = m_2$, we can write

$$(p-q)\left(n_1 - 2\left\lfloor \frac{m_1}{\Delta_1+1} \right\rfloor\right) = (p-q)\left(n_1 - 2\left\lceil \frac{m_1}{\Delta_1+1} \right\rceil\right).$$

But then,

$$p = q \quad \text{or} \quad \left\lfloor \frac{m_1}{\Delta_1+1} \right\rfloor = \left\lceil \frac{m_1}{\Delta_1+1} \right\rceil.$$

Note that in the latter case, we must have $p = \Delta_1 + 1$ and $q = \Delta_1 + 1$. Hence, $p = q$, and so we can assume that $g_i = f_i$ for every $i = 1, \ldots, \Delta_1 + 1$.

Now, let $M = \{u_i v_i : i = 1, \ldots, n_1\}$. For every $i = 1, \ldots, \Delta_1 + 1$, since both u_i and v_i miss the same colour, say f_k, we can assign to edge $u_i v_i$ the colour f_k. But then we get a $\Delta_1 + 1$ edge-colouring of G_M, and so G_M is Class 1. ■

Note that the proof of Theorem 3 gives an algorithm to colour the edges of a regular join graph $G = G_1 + G_2$ with $\Delta(G)$ colours, whenever $\Delta_1 = \Delta_2$.

Corollary 3 *Conjecture 2 holds true for every regular join graph $G = G_1 + G_2$ with $\Delta_1 = \Delta_2$.*

5 $\Delta_1 < \Delta_2$

Let $G = G_1 + G_2$ be a join graph with $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $n_1 \leq n_2$.

If $n_1 < n_2$, then Observation 1 does not help. Indeed, there are join graphs $G = G_1 + G_2$ with $\Delta_1 < \Delta_2$ and $n_1 < n_2$ such that G is Class 2 even though G_M is Class 1 for every maximum matching M. This is, for instance, the case of the graph in Fig. 4.

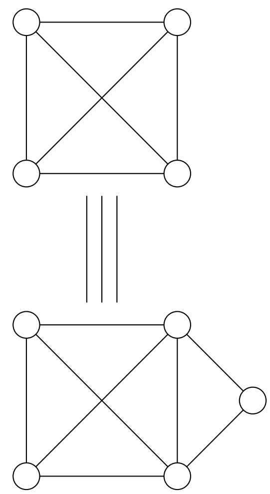

Fig. 4

Moreover, when $\Delta_1 < \Delta_2$ and $n_1 < n_2$ there are graphs $G = G_1 + G_2$ that satisfy some of the three conditions in Theorem 2 and are Class 2. For instance, every complete graph G with an odd number of vertices satisfies conditions (ii) and (iii); the graph in Fig. 4 satisfies conditions (i) and (ii).

However, if we assume that $n_1 = n_2$, then we can apply Observation 1. In fact, we can get a strong result similar to Theorem 1:

Theorem 4 *Let $G = G_1 + G_2$ be a join graph with $n_1 = n_2$. If $\Delta_1 < \Delta_2$ then for every maximum matching M of B_G, the corresponding graph G_M is Class 1.*

Proof Interchange the roles of G_1 and G_2 and apply Theorem 1. ■

Corollary 4 *Let $G = G_1 + G_2$ be a join graph with $n_1 = n_2$. If $\Delta_1 < \Delta_2$ then G is Class 1 and Conjecture 1 holds true.*

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