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The Vertex Separator Problem: A Polyhedral Investigation

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Abstract

The vertex separator (VS) problem in a graph $G = (V, E)$ asks for a partition of V into nonempty subsets A, B, C such that there is no edge between A and B, and $|C|$ is minimized subject to a bound on $\max\{|A|, |B|\}$. We give a mixed integer programming formulation of the problem and investigate the vertex separator polytope (VSP), the convex hull of incidence vectors of vertex separators. Necessary and sufficient conditions are given for the VSP to be full dimensional. Central to our investigation is the relationship between separators and dominators. Several classes of valid inequalities are investigated, along with the conditions under which they are facet defining for the VSP. Some of our proofs combine in new ways projection with lifting.

In a companion paper we develop a branch-and-cut algorithm for the (VS) problem based on the inequalities discussed here, and report on computational experience with a wide variety of (VS) problems drawn from the literature and inspired by various applications.

1 Introduction

A *vertex separator* in an undirected graph is a subset of the vertices, whose removal disconnects the graph. Formally, the vertex separator problem VSP can be stated as follows:
INSTANCE: A connected undirected graph $G = (V, E)$, with $|V| = n$, an integer $b(n) \leq n$ and a cost c_i associated with each vertex $i \in V$.
PROBLEM: Find a partition of V into disjoint sets A, B, C, with A and B non empty, such that
(i) E contains no edge (i, j) with $i \in A$, $j \in B$,
(ii) $\max\{|A|, |B|\} \leq b(n)$
(iii) $\sum(c_j : j \in C)$ is minimized subject to (i), (ii).

A and B are called the *shores* of the separator C. A separator C that satisfies (i) but violates (ii) is termed *infeasible*; one that satisfies (i) and (ii) is *feasible*; and a separator

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that satisfies (i), (ii), (iii) is optimal. When we use the term separator we mean feasible separator, unless otherwise specified.

To the best of our knowledge, this is the first *polyhedral* study of the VSP, which otherwise has received considerable attention in the literature, due to its widespread applicability to all kinds of practical connectivity problems (see, for instance, [2, 3, 5]). One particularly important area of application is linear algebra, namely to minimize the work involved in solving systems of equations [4, 6]. Another one is finite element and finite difference problems [7].

The VSP is $\mathcal{NP}$-hard. For the case when G is planar and $b(n) = 2n/3$, a celebrated result of Lipton and Tarjan [5] states that a separator of size bounded by $2\sqrt{2}\sqrt{n}$ can be found in $O(n)$ time; but the question whether the VSP on planar graphs can be solved in polynomial time is still open. On the other hand, the VSP defined on an arbitrary graph becomes polynomially solvable if $b(n)$ in (ii) is replaced by $n - k$ for some positive constant k. To see this, construct a bipartite graph $G^* = (V^*, E^*)$, with bipartition $V^* = (V_1^*, V_2^*)$, as follows: (a) for every $i \in V$, let $i_1 \in V_1^*$, $i_2 \in V_2^*$, and $(i_1, i_2) \in E^*$. (b) for every $(i, j) \in E$, let $\{(i_1, j_2), (j_1, i_2)\} \subset E^*$. Then the (VSP) with the modified condition (ii) is equivalent to the problem of finding a maximum-weight stable set S in G^* (with weights $c_j, j \in V^*$), such that $\max\{|S \cap V_1^*|, |S \cap V_2^*|\} \leq n - k$. Clearly, this problem is solvable in $O(n^3 \cdot n^k)$ time.

Before we proceed, we introduce some notation and recall a few basic concepts from graph theory, to be used throughout this paper.

Consider a simple undirected graph $G = (V, E)$. We denote $x(S) = \sum(x_j : j \in S)$ for any $S \subset V$. For $i \in V$, we write $\deg(i)$ for the degree of i. $G - i$ denotes the graph obtained from G by removing vertex i. For $S \subseteq V$, $G[S]$ is the subgraph of G induced by S. For $S, S' \subseteq V$, $(S, S') := \{(i,j) \in E : i \in S, j \in S'\}$. For $S \subset V$, $\delta(S) := \{(i,j) \in E : |\{i,j\} \cap S| = 1\}$, and $E(S) = \{(i,j)\} \in E : i,j \in S\}$. When $S = \{i\}$, we write $\delta(i)$ instead of $\delta(\{i\})$. Also for $S \subset V$, $\mathrm{Adj}(S) := \{i \in V \setminus S : (i,j) \in E$ for some $j \in S\}$, and when $S = \{i\}$, we write $\mathrm{Adj}(i)$ for $\mathrm{Adj}(\{i\})$. For $F \subseteq E$, $V(F)$ denotes the set of endpoints of the edges in F. A bipartite graph with vertex bipartition V_1, V_2 is denoted $G = (V_1, V_2; E)$.

A set $S \subseteq V$ such that $E(S) = \emptyset$ is called *stable* or *independent*. $S \subseteq V$ such that $(S, E(S))$ is a complete graph, is called a *clique*. $S \subseteq V$ such that $V \subseteq (S \cup \mathrm{Adj}(S))$ is called a *dominating* set for G or for V. A dominating set S for G is *minimal* if no proper subset of S is a dominating set for G.

A vertex $i \in V$ is *universal* if it is adjacent to every $j \in V \setminus \{i\}$.

For the sake of brevity, for the rest of this paper a vertex separator, and a dominating set, will be simply referred to as a *separator* and a *dominator*, respectively.

Next we outline the structure of the paper. Section 2 states the mixed integer programming formulation used throughout the paper. Section 3 establishes necessary and sufficient conditions for the VS polytope to be full dimensional. Section 4 deals with the connection between vertex separators and vertex dominators. The remaining five sections, which constitute the bulk of the paper, describe various classes of valid inequalities and investigate the conditions under which they are facet defining. Section 5 introduces a class of symmetric inequalities associated with minimal connected dominators, and shows that under mild and easily verifiable conditions they define facets of the VS polytope in all but a few exceptional situations. For those exceptional cases an alternative inequality is derived that is facet defining. Section 6 introduces a class of asymmetric inequalities associated with minimal dominators (not necessarily connected), and states the (rather restrictive) conditions under which they are facet defining. When those restrictive conditions are not present, this asymmetric class of inequalities can be lifted or otherwise generalized to yield facet defining inequalities, and this is the object of the last three sections. In Sections 7 and 8 two classes of lifted inequalities are derived. The novel feature of this derivation is that the inequality to be lifted, when restricted to the subspace of its support, is invalid. A combination of projection, restriction and sequential lifting is used to overcome this difficulty. Finally, Section 9 generalizes the inequality of Section 6 in a different direction.

2 A Mixed Integer Programming Formulation

Let c_i be the cost of assigning vertex i to the separator, and let

$$u_{i1} = \begin{cases} 1 & \text{if vertex } i \text{ is assigned to shore } A \\ 0 & \text{else} \end{cases}$$

$$u_{i2} = \begin{cases} 1 & \text{if vertex } i \text{ is assigned to shore } B \\ 0 & \text{else.} \end{cases}$$

For any $S \subset V$ and for $k = 1, 2$, we write $u_k(S) = \sum_{i \in S} u_{ik}$, and $u(S) = \sum_{k=1}^{2} \sum_{i \in S} u_{ik}$. Then (VSP) can be formulated as

$$\max \sum_{i \in V} c_i(u_{i1} + u_{i2}) \tag{2.1}$$

$$u_{i1} + u_{i2} \leq 1, \quad i \in V \tag{2.2}$$

$$u_{i1} + u_{j2} \leq 1, \quad (i,j) \in E \tag{2.3}$$

$$u_{j1} + u_{i2} \leq 1, \quad (i,j) \in E \tag{2.4}$$

$$u_1(V) \leq b \tag{2.5}$$

$$u_2(V) \leq b \tag{2.6}$$

$$u_1(V) \geq 1 \tag{2.7}$$

$$u_2(V) \geq 1 \tag{2.8}$$

$$u_{i1}, u_{i2} \geq 0, \quad i \in V \tag{2.9}$$

$$u_{i1} \text{ integer}, \quad i \in V \tag{2.10}$$

It is not hard to see that for any set of $u_{i1} \in \{0, 1\}$, $i \in V$, the variables u_{i2}, $i \in V$, will take on 0-1 values in any basic solution to the resulting linear program. Indeed, if the coefficient matrix of the system (2.1)–(2.7) is written as (A_1, A_2), where for $k = 1, 2$, A_k represents the columns corresponding to u_{ik}, $i \in V$, we see that A_2 is totally unimodular. Hence substituting any 0-1 values for u_{i1}, $i \in V$, we get 0-1 values for u_{i2}, $i \in V$.

With this information, the above system has the following interpretation. Condition (2.1) states that vertex i cannot be assigned to both A and B, but it leaves open the possibility that it is assigned to none, in which case it belongs to C, the separator. Constraints (2.2) and (2.3) prevent the endpoints of any edge to be assigned one to A, the other to B. Inequalities (2.4) and (2.5) restrict the size of each of A and B to b, while (2.6) and (2.7) impose the conditions $A \neq \emptyset \neq B$. It is easy to see that the above formulation is correct. In the following sections we study the VS polytope, defined as

$$P(G, b) := \text{conv}\{u \in \mathbb{B}^{2n} : u \text{ satisfies } (2.1) - (2.10)\}.$$

Sometimes we will write $P(G)$ for $P(G, b)$.

3 The dimension of $P(G, b)$

Clearly, if G is complete, (VSP) is trivial. On the other hand, if $b = 1$, then inequalities (2.5)-(2.8) hold as equations and (VSP) is again trivial; whereas if $b \geq n-1$, then constraints (2.5),(2.6) are redundant and (VSP) is polynomially solvable as shown in section 1. Thus from now on we assume that G is incomplete and connected, $|V| \geq 3$ and that $2 \leq b \leq n-2$.

A vertex i is called *regular*, if there exists a separator $C \subset V \setminus \{i\}$ such that $C \cup \{i\}$ is also a separator. Thus i is regular if and only if there exists a separator C with shores A, B such that $i \in A$ and $|A| \geq 2$. A vertex that is not regular is called *irregular*.

We now give a sufficient condition for $P(G, b)$ to be full dimensional.

Lemma 3.1. *If every $i \in V$ is regular, then $P(G, b)$ is full dimensional.*

Proof. Suppose every $i \in V$ is regular. Then any equation $\alpha u = \alpha_0$ satisfied by all $u \in P(G, b)$ must have coefficients $\alpha_j = 0$ for $j = 0, 1, \ldots, 2n$. Indeed, let $C \subset V \setminus \{i\}$ and $C' = C \cup \{i\}$ be two separators with shores A, B and A', B', respectively, such that $i \in A$, $A' = A \setminus \{i\}$, and $B' = B$. Further, let $u, u' \in P(G, b)$ be the two solutions associated with C and C', respectively. Then

$$\begin{aligned}
\alpha u &= \alpha_0 = \alpha_{i1} + \alpha_1(A \setminus \{i\}) + \alpha_2(B) \\
\alpha u' &= \alpha_0 = \alpha_1(A \setminus \{i\}) + \alpha_2(B)
\end{aligned}$$

and $\alpha u - \alpha u' = \alpha_{i1} = 0$.

Since this argument applies to all regular vertices, and since the roles of A and B are interchangeable, it follows that $\alpha_{i1} = \alpha_{i2} = 0$ for all $i \in V$, hence $\alpha_0 = 0$. □

Note that checking the regularity of a vertex is an $O(|E|)$ operation.

Next we characterize irregular vertices. First a definition: if G has two nonadjacent vertices i and k, such that $\text{Adj}(i) = \text{Adj}(k) = V \setminus \{i, k\}$, then both i and k are irregular, and we say that they form a *polar* pair of irregular vertices. In such a case every separator of G either contains both i and k or none of them, i.e. $u_{i1} + u_{i2} = u_{k1} + u_{k2}$ and $P(G, b)$ is not full dimensional. A graph that has a universal vertex, or a polar pair of irregular vertices, will be called *degenerate*.

Lemma 3.2. *Let $i \in V$ be irregular. Then*

(a) For every separator C with a shore $A = \{i\}$, we have that $\text{Adj}(i) \subseteq C$ and every $j \in C$ is adjacent to every k in B. Furthermore, if B is a singleton, G is degenerate; and if $|B| \geq 2$, $G[B]$ is a clique whose vertices are all regular.

(b) If $G[\text{Adj}\,(i)]$ is a clique or a clique short of an edge, G is degenerate.

Proof. (a) Let C be a separator with a shore $A = \{i\}$. This clearly implies $\text{Adj}(i) \subseteq C$. If there exists $j \in C$ that is not adjacent to some $k \in B$, then $C' := (C \setminus \{j\}) \cup (B \setminus \{k\})$ is a separator with shores $A' = \{i, j\}$, $B' = \{k\}$, contrary to i being irregular. Thus every $j \in C$ is adjacent to every $k \in B$. Further, if B is a singleton, say k, then i and k form a polar pair of irregular vertices and G is degenerate. Finally, assume $|B| \geq 2$. Then every vertex in B is regular, since removing it from B and adding it to C yields a valid solution. Also, $G[B]$ must be a clique; for otherwise, if $k, \ell \in B$ and $(k, \ell) \notin E$, then $C' := C \cup (B \setminus \{k, \ell\})$ is a separator with shores $A' = \{i, k\}$, $B' = \{\ell\}$, and $C'' := C' \cup \{i\}$ is a separator with shores $A'' = A' \setminus \{i\}$, $B'' = B'$, contrary to i being irregular.

(b) If $G[\text{Adj}(i)]$ is a clique, then every vertex in $\text{Adj}(i)$ is universal (from (a)). If, on the other hand, $G[\text{Adj}(i)]$ is a complete graph minus an edge, say the one between vertices k and ℓ, then $\text{Adj}(k) = \text{Adj}(\ell) = V \setminus \{k, \ell\}$ i.e. k and ℓ are polar irregular vertices, and G is degenerate. □

Lemma 3.3. *If i and k are irregular vertices not adjacent to each other, then they are polar.*

Proof. If $V \setminus (\{i, k\} \cup \text{Adj}(k))$ is nonempty, then i is regular. Similarly, if $V \setminus (\{i, k\} \cup \text{Adj}(i))$ is nonempty, then k is regular. Hence $\text{Adj}(i) = \text{Adj}(k) = V \setminus \{i, k\}$, i.e. i and k are polar. □

Lemma 3.4. *If G is nondegenerate, then all irregular vertices of G are adjacent to each other.*

Proof. If i is irregular, from Lemma 3.2(a) all vertices in $V \setminus (\{i\} \cup \text{Adj}(i))$ are regular. Hence all irregular vertices other than i belong to $\text{Adj}(i)$. Applying this reasoning to all irregular vertices we conclude that they induce a clique in G. □

Lemma 3.5. *Let G be nondegenerate, and let $S \subset V$ be the set of irregular vertices of G, with $|S| \geq 2$. Then*

(a) Every $i \in V$ either is in S or is adjacent to some $k \in S$.

(b) Every $i \in \text{Adj}(S)$ is adjacent to all but possibly one of the vertices in S.

(c) Every $i \in \bigcap_{k \in S} \text{Adj}(k)$ is adjacent to every $j \in V \setminus \bigcap_{k \in S} \text{Adj}(k)$

(d) If there exists a nonadjacent pair $\{i,k\} \in \text{Adj}(S)$*, then both* i *and* k *are adjacent to all vertices in* S*.*

Proof. (a) If $V \setminus (S \cup \text{Adj}(S)) \neq \emptyset$, then $A = \{i,k\} \subseteq S$ and $B = \{\ell\} \subseteq V \setminus (S \cup \text{Adj}(S))$ are the shores of a separator, contrary to the assumed irregularity of the vertices in S.

(b) If $i \in \text{Adj}(S)$ is nonadjacent to $k \in S$ and $\ell \in S$, then $A = \{k,\ell\}$ and $B = \{i\}$ are the shores of a separator, again contrary to the assumed irregularity of k and ℓ.

(c) Let $i \in \bigcap_{k\in S} \text{Adj}(k)$ and $j \in V \setminus \bigcap_{k\in S} \text{Adj}(k)$ be nonadjacent. Then clearly $j \notin S$, and since $j \notin \bigcap_{k\in S} \text{Adj}(k)$, there exists some $\ell \in S$ such that $(j,\ell) \notin E$. But then $A = \{i,\ell\}$ and $B = \{j\}$ are the shores of a separator, contrary to the assumption that ℓ is irregular.

(d) Let $\{i,k\} \in \text{Adj}(S)$, $(i,k) \notin E$, and suppose i is not adjacent to some $\ell \in S$. Then $\{k,\ell\}$ and $\{i\}$ are the shores of a separator, contrary to the assumed irregularity of ℓ. $\square$

The conditions of Lemma 3.5 can be restated as

(a′) $V = S \cup \text{Adj}(S)$

(b′) $\text{Adj}(S) = (\bigcup_{i\in S} \bigcap_{k\in S\setminus\{i\}} \text{Adj}(k)) \cup (\bigcap_{k\in S} \text{Adj}(k))$

(c′) Every $i \in \bigcap_{k\in S} \text{Adj}(k)$ is adjacent to every $j \in \bigcap_{k\in S\setminus\{\ell\}} \text{Adj}(k)$, for all $\ell \in S$.

(d′) $G[\bigcup_{\ell\in S} \bigcap_{k\in S\setminus\{\ell\}} \text{Adj}(k)]$ is a clique.

Corollary 3.6. *Let* S *be as defined in Lemma 3.5. Then every pair* $\{i,\ell\} \subseteq S$ *is a dominator.*

Proof. $i \in S$ dominates each of S, $\bigcap_{j\in S} \text{Adj}(j)$, and $\bigcap_{k\in S\setminus\{j\}} \text{Adj}(k)$ for all $j \neq i$. Further, any $\ell \neq i$ dominates $\bigcap_{k\in S\setminus\{i\}} \text{Adj}(k)$. Thus i and ℓ together dominate $S \cup \text{Adj}(S)$. $\square$

Theorem 3.7. *Let* G *be nondegenerate, and let* S *be the set of irregular vertices of* G*. Then* $P(G,b)$ *is full dimensional if and only if* $G[\bigcap_{i\in S} \text{Adj}(i)]$ *is not a clique.*

Proof. Necessity. If $G[\bigcap_{i\in S} \text{Adj}(i)]$ is a clique, then from Lemma 3.5(c) every $k \in \bigcap_{i\in S} \text{Adj}(i)$ is a universal vertex in G, hence $P(G,b)$ is not full dimensional.

Sufficiency. Let us write $G^* := G[\bigcap_{i\in S} \text{Adj}(i)]$, and assume G^* is not a clique. If the complement $\bar{G}^*$ of G^* has exactly one edge, say $(k,\ell) \notin E$, then k and ℓ are polar irregular vertices of G, a case ruled out by the assumption that G is nondegenerate. If $\bar{G}^*$ has two adjacent edges, say (k,ℓ) and (ℓ,j), and no others, then ℓ is easily seen to be an irregular vertex of G, a case ruled out by the assumption that S is the set of irregular vertices. Thus $\bar{G}^*$ either has at least two disjoint, i.e. nonadjacent, edges, or it has three edges that form a triangle. In either case G has a separator that contains S.

Now let $\alpha u = \alpha_0$ be any equation satisfied by all $u \in P(G,b)$. As shown in the proof of Lemma 3.1, if $j \in V$ is regular, then $\alpha_{j1} = \alpha_{j2} = 0$. Hence this holds for all $j \in V \setminus S$. Now let $j \in S$, and consider the separator C with shores $A = \{j\}$ and $B \subseteq (\bigcap_{k \in S \setminus \{j\}} \mathrm{Adj}(k)) \setminus \mathrm{Adj}(j)$, $C \supseteq \mathrm{Adj}(j)$, as well as a separator C' whose shores A', B' are both contained in $\bigcap_{k \in S} \mathrm{Adj}(k)$. The existence of C' was pointed out at the end of the preceding paragraph. Let $u, u' \in P(G,b)$ correspond to C and C', respectively. Then

$$\begin{aligned} u = \alpha_0 &= \alpha_{j1} \quad\;\; + \alpha_2(B), \\ \alpha u' = \alpha_0 &= \alpha_1(A') + \alpha_2(B'). \end{aligned}$$

and $\alpha u - \alpha' u = 0 = \alpha_{j1}$, since $(B \cup A' \cup B') \subseteq V \setminus S$, hence all the coefficients indexed by these sets are 0 as the corresponding vertices are regular. Since $j \in S$ was chosen arbitrarily, $\alpha_{j1} = 0$ for all $j \in S$. Reversing the roles of A, B then yields $\alpha_{j2} = 0$ for all $j \in S$, hence $\alpha_0 = 0$. Thus $\alpha_j = 0$ for $j = 0, 1, \ldots, 2n$, which proves that $P(G,b)$ is full dimensional. □

4 Separators and Dominators

In Section 1 we defined a dominator of V as a set $S \subseteq V$ such that $V \subseteq (S \cup \mathrm{Adj}(S))$, and a minimal dominator as one that does not contain any dominator as a proper subset. Now we call a dominator S *connected*, if $G[S]$ is connected; and we define a *minimal connected dominator* (CD), as a CD that does not contain any CD as a proper subset; i.e. $S \subseteq V$ is a minimal CD if for every $i \in S$, $S \setminus \{i\}$ is either disconnected, or is not a dominator of G (or both). Thus a minimal CD may or may not be a minimal dominator, but it always contains one.

Separators and connected dominators are in a fundamental relationship similar to that between spanning trees and cutsets:

Proposition 4.1. *In a connected graph, any separator and any connected dominator have at least one vertex in common.*

Proof. Let C be a separator with shores A and B, and let S be a connected dominator. If $C \cap S = \emptyset$, then $S \subseteq (A \cup B)$; but since S is connected, this implies either $S \subseteq A$, $S \cap B = \emptyset$, or vice versa, which contradicts the fact that S is a dominator. Hence $C \cap S \neq \emptyset$. □

For $S \subset V$ and $k \in V \setminus S$, we denote $\mathrm{Adj}_S(k) := \{i \in S : (i,k) \in E\}$.

Definition 1. *Let $S \subset V$ be a dominator of V. For $i \in S$,*

$$P(i) := \{k \in V \setminus S : \mathrm{Adj}_S(k) = \{i\}\}$$

is the set of pendent *vertices of i.*

Notice that if the dominator S is minimal and $P(i) = \emptyset$ for some $i \in S$, then the presence of i in S is needed only to dominate i itself. We call such a vertex a *self-dominator*.

Proposition 4.2. *If S is a minimal dominator, then for every $i \in S$, either i is a self-dominator or $P(i) \neq \emptyset$.*

Proof. Follows from the definitions and the minimality of S. □

The next proposition characterizes the structure of minimal connected dominators.

Proposition 4.3. *Let S be a minimal connected dominator, and let $S_D := \{i \in S : P(i) \neq \emptyset\}$, $S_Q := S \setminus S_D$. Then*

(a) If $S_Q \neq \emptyset$, every $i \in S_Q$ is an articulation point of $G[S]$

(b) S_D contains no self-dominating vertices.

(c) S_D is the unique minimal dominator of $V \setminus S$ contained in S.

Proof. (a) Let $i \in S_Q$. Then $P(i) = \emptyset$, hence $S \setminus \{i\}$ is a dominator, and the only possible reason for the presence of i in S is to make $G[S]$ connected. On the other hand, if $G[S \setminus \{i\}]$ is also connected, then S is not a minimal connected dominator. Hence $G[S \setminus \{i\}]$ is disconnected, i.e. i is an articulation point of $G[S]$.

(b) Suppose $i \in S_D$ is a self-dominator. Then i is an isolated vertex of S_D, and $\mathrm{Adj}_S(i) \subseteq S_Q$. But then $S \setminus \{i\}$ is a minimal connected dominating set, since i is adjacent to one or more $j \in S_Q$, a contradiction.

(c) Since S is a dominator of V, hence of $V \setminus S$, and $P(j) = \emptyset$ for all $j \in S_Q$, $S_D = S \setminus S_Q$ is itself a dominator of $V \setminus S$. Further, since $P(i) \neq \emptyset$ for $i \in S_D$, S_D is a minimal dominator of $V \setminus S$. The uniqueness of S_D follows from the fact that it is the set of precisely those vertices in S that cover some vertex in $V \setminus S$ not covered by any other vertex in S. □

The next two sections of our paper examine valid inequalities for $P(G, b)$ and the conditions under which they are facet defining. From now on we will assume that $P(G, b)$ is full dimensional.

5 A Class of Symmetric Facets of $P(G, b)$

A valid inequality for $P(G, b)$ is one that is satisfied by every $u \in P(G, b)$.We call such an inequality *symmetric* if for all $j \in V$, the coefficients of u_{j1} and u_{j2} are equal. A valid inequality $\alpha u \leq \alpha_0$ is *maximal* if there exists no valid inequality $\alpha' u \leq \alpha_0$ with $\alpha' \geq \alpha$ and $\alpha'_j > \alpha_j$ for some j. For any polyhedron in $\mathbb{R}^n_+$, all essential (i.e. facet defining) inequalities are maximal, but the converse is of course not true.

Proposition 5.1. *Let S be a minimal connected dominator of V. Then*

$$u(S) \leq |S| - 1 \tag{5.1}$$

is a valid inequality for $P(G, b)$.

Proof. It follows directly from Proposition 4.1: as S is a dominator, it must have at least one vertex in any separator. □

There is no easy, simple necessary and sufficient condition for the inequality (5.1) to be facet defining. Maximality is somewhat easier to establish. To put it simply, (5.1) is maximal whenever G does not have a certain kind of vertices.

Given a minimal connected dominator S of V and a vertex $v \in V \setminus S$, we will say that S is v-*decomposable* if $G[S \cup \{v\}]$ has an articulation point i such that $G[S \cup \{v\} \setminus \{i\}]$ either has two components neither of which is the singleton v, or has at least three components. We call the vertex $v \in V \setminus S$ *forbidden* if it has each of the following three properties:

(i) S is not v-decomposable

(ii) v is adjacent to every $j \in \bigcup_{i \in S} P(i)$

(iii) v is adjacent to at least two $j \in S$.

Proposition 5.2. *The inequality (5.1) is maximal if and only if G has no forbidden vertices.*

Proof. Necessity. Suppose G has a forbidden vertex $v \in V \setminus S$. Then the inequality $u(S) + u_{v2} \leq |S| - 1$ can be shown to be valid for $P(G, b)$, hence (5.1) is not maximal. Indeed, let $u_{v2} = 1$, and call B the separator shore containing v. We claim that the shores of such a separator can contain at most $|S| - 2$ vertices of S. For if $S \cap B = \emptyset$, then $(S \cap A) \subseteq (S \setminus \mathrm{Adj}(v))$, and from property (iii) of v, $|S \cap A| \leq |S| - 2$. If, on the other hand, $S \cap B \neq \emptyset$, there are two cases: (a) $S \cap A = \emptyset$, and (b) $S \cap A \neq \emptyset$. In case (a), from property (ii) of v, A cannot contain any vertex of $\bigcup_{i \in S} P(i)$, which implies that each vertex in $A \subseteq V \setminus S$ is adjacent to at least two vertices in S; hence $|S \cap B| \leq |S| - 2$. In case (b), since $S \cap A$ can have no vertex adjacent to $S \cap B$, and from (i) it requires the removal of at least two vertices from S to disconnect $G[S \cup \{v\}]$ without creating a singleton component consisting of v (while the creation of such a component is excluded by (iii)), it follows that $|S \cap A| + |S \cap B| \leq |S| - 2$. Thus $u(S) + u_{v2} \leq |S| - 1$ is valid for $P(G, b)$, i.e. (5.1) is not maximal.

Sufficiency. Suppose (5.1) is not maximal. Then $\alpha u \leq |S| - 1$ is valid for some α such that $\alpha_{jk} \geq 1$ for all $j \in S$, $\alpha_{jk} \geq 0$ for all $j \in V \setminus S$, $k = 1, 2$, and at least one of the inequalities holds strictly. If $\alpha_{jk} > 1$, for some $j \in S$, then any $u \in P(G, b)$ corresponding to a separator with shore $A := S \setminus \{i\}$ for some $i \neq j$ violates $\alpha u \leq |S| - 1$. Thus $\alpha_{jk} = 1$ for all $j \in S$, $k = 1, 2$. Now let $\alpha_{v1} > 0$ for some $v \in V \setminus S$. Then v must satisfy (ii); for if there exists $\ell \in P(i)$ for some $i \in S$ such that $(v, \ell) \notin E$, then there is a separator with shores $A = \{\ell\}$ and $B = S \cup \{v\} \setminus \{i\}$ such that the corresponding solution $\bar{u}$ satisfies $\alpha \bar{u} = |S| - 1 + \alpha_{v1} > |S| - 1$. Also, v must have property (iii); for otherwise $v \in P(i)$ for some $i \in S$, and there is a separator with shores $A = \{v\}$ and $B = S \setminus \{i\}$, whose associated solution $\hat{u}$ satisfies $\alpha \hat{u} + \alpha_{v1} = |S| - 1 + \alpha_{v1} > |S| - 1$. Finally, v must also have property (i), for if S is v-decomposable with articulation point i, then there is a separator with shores $A = \{v\} \cup S'$, $B = S''$, where $S' \cup S'' = S \setminus \{i\}$, $S'' \cap \mathrm{Adj}_S(v) = \emptyset$, whose associated solution $\tilde{u}$ satisfies $\alpha \tilde{u} + \alpha_{v1} = |S' \cup S''| + \alpha_{v1} > |S| - 1$. □

Example. In the graph G shown in Figure 1, $S = \{1, \ldots, 4\}$ is a minimal connected dominator, and the inequality

$$u_{11} + u_{21} + u_{31} + u_{41} + u_{12} + u_{22} + u_{32} + u_{42} \leq 3$$

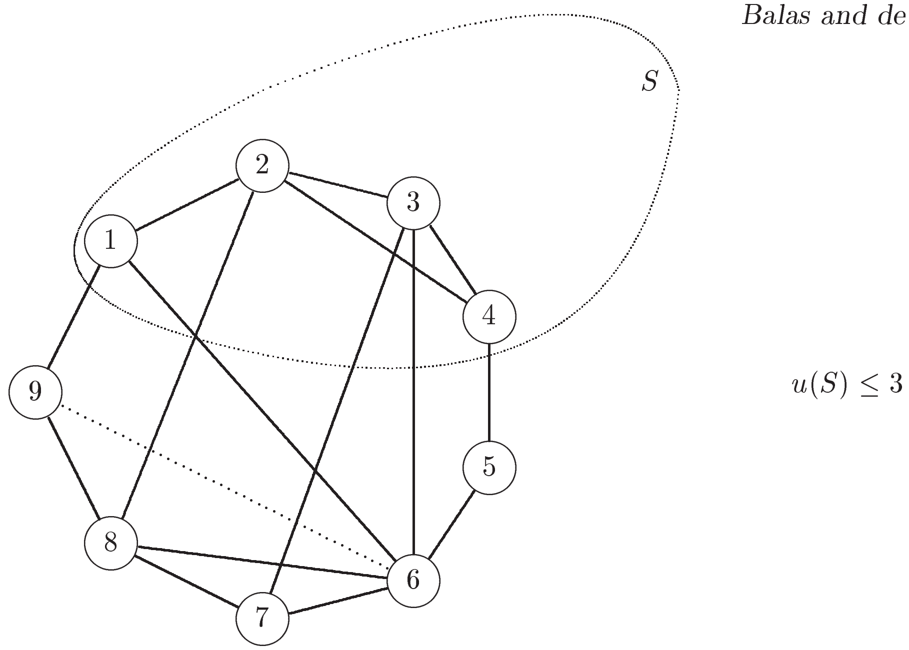


Figure 1: Maximal vs. non-maximal inequality

is valid for $P(G,b)$, where $4 \le b \le 6$. S is v-decomposable for $v = 5, 7$ and 8 (with articulation point $i = 2$ in each case), but not for $v = 6$, which has property (i). Vertex 6 also has property (iii), and is the only such vertex in $V \setminus S$. However, without edge $(6,9)$ shown as a dotted line, vertex 6 does not have property (ii), and so the above inequality is maximal. Upon insertion of edge $(6,9)$ vertex 6 acquires property (ii) and the inequality can be strengthened by changing the coefficient of u_{62} from 0 to 1.

We are now ready to address the issue of when (5.1) is facet defining. Certainly, the conditions on the vertices in $V \setminus S$ required for maximality are also required, i.e. necessary, for (5.1) to be facet defining. However, they are in general not sufficient. Furthermore, there are conditions concerning the vertices in S that are rather complex.

Let F be the face of $P(G,b)$ defined by the inequality (5.1), i.e. let

$$F := \{u \in P(G,b) : u(S) = |S| - 1\}.$$

F is a facet of $P(G,b)$ if and only if every equation $\alpha u = |S| - 1$ satisfied by every $u \in F$ has coefficients

$$\alpha_{j1} = \alpha_{j2} = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{if } j \in V \setminus S. \end{cases}$$

We will use the notation $\alpha u = |S| - 1$ for a generic equation satisfied by all $u \in F$, and will start by stating some sufficient conditions for a vertex $j \in V \setminus S$ to have coefficients $\alpha_{j1} = \alpha_{j2} = 0$ in this equation. We will say that a separator is in F if the corresponding point $u \in P(G,b)$ is in F.

Throughout this section we will repeatedly use the following argument.

Proposition 5.3. *If for some $v \in V \setminus S$ there exist two separators in F, C and C', such that $C = C' \cup \{v\}$, then $\alpha_{v1} = \alpha_{v2} = 0$.*

Proof. Let u and u' be the points in $P(G, b)$ corresponding to C and C', respectively. Then $u_{jk} = u'_{jk}$ for $j \in V \setminus \{v\}$ and $k = 1, 2$. Hence $\alpha u - \alpha u' = 0 = \alpha_{v1}(u_{v1} - u'_{v1}) + \alpha_{v2}(u_{v2} - u'_{v2})$. But since either $u_{v1} \neq u'_{v1}$ or $u_{v2} \neq u'_{v2}$, it follows that at least one of α_{v1} and α_{v2} is equal to 0. Since the shores are interchangeable, it then follows that $\alpha_{v1} = \alpha_{v2} = 0$. □

Proposition 5.4. *If S is v-decomposable for some $v \in V \setminus S$ and $|S| \leq b + 2$, then $\alpha_{v1} = \alpha_{v2} = 0$.*

Proof. Let S be v-decomposable with articulation point i, and let $G[S' \cup \{v\}]$ be a component of $G[S \cup \{v\} \setminus \{i\}]$, with $S' \neq \emptyset$. Then there exists a separator C with shores $A = S' \cup \{v\}$, $B = S \setminus (S' \cup \{i\})$, such that $|(A \cup B) \cap S| = |S| - 1$, i.e. C is in F. But $C' = C \cup \{v\}$ is also a separator in F, with shores $A' = A \setminus \{v\}$, $B' = B$; hence from Proposition 5.3, $\alpha_{v1} = \alpha_{v2} = 0$. □

Proposition 5.5. *If there exists $\ell \in P(i)$ for some $i \in S$ such that $(v, \ell) \notin E$ for some $v \in V \setminus S$ and $|S| \leq b$, then $\alpha_{v1} = \alpha_{v2} = 0$.*

Proof. Let ℓ be as assumed; then there exists a separator C in F with shores $A = S \cup \{v\} \setminus \{i\}$ and $B = \{\ell\}$; but $C' = C \cup \{v\}$ is also a separator in F, with shores $A' = A \setminus \{v\}$, $B' = B$. Hence from Proposition 5.3, $\alpha_{v1} = \alpha_{v2} = 0$. □

Proposition 5.6. *If $v \in P(i)$ for some $i \in S$ such that $|P(i)| \geq 2$ and $|S| \leq b + 1$, then $\alpha_{v1} = \alpha_{v2} = 0$.*

Proof. By assumption, there exists $\ell \neq v$, $\ell \in P(i)$. Therefore there exists a separator C in F with shores $A = S \setminus \{i\}$, $B = \{\ell, v\}$; but $C' = C \cup \{v\}$ is also a separator in F, with shores $A' = A$, $B' = B \setminus \{v\}$; hence $\alpha_{v1} = \alpha_{v2} = 0$. □

Notice that, while the conditions on v stated in Proposition 5.4 and 5.5 are the exact converse of the conditions (i) and (ii), the condition of Proposition 5.6 is stronger than the converse of (iii), which would only require that $v \in P(i)$ for some i. This is consonant with the fact that the maximality of an inequality does not imply that it is also facet defining. For this to be the case, additional properties are required.

Next we show that if G has a vertex for which none of the three conditions listed in Propositions 5.4-5.6 is satisfied, then (5.1) is not facet defining.

Proposition 5.7. *Suppose there exists $v \in V \setminus S$ with the properties*

(a) $G[S \cup \{v\}]$ and $G[S]$ have no common articulation point

(b) v is adjacent to every $j \in \bigcup_{k \in S} P(k)$

(c) $\{v\} = P(i)$ for some $i \in S$. Then the inequality (5.1) does not define a facet of $P(G, b)$.

Proof. We show that, under conditions (a), (b), (c), all $u \in F$ satisfy the equation $wu := u_{i1} + u_{i2} + u_{v1} + u_{v2} = 1$. Let C be the separator associated with u, and A, B its shores.

Assume first that both v and i belong to C, i.e. $wu = 0$. From (a), if $A \cap S \neq \emptyset \neq B \cap S$, then $|(A \cup B) \cap S| \leq |S| - 2$. Thus either $|A \cap S| = 0$ or $|B \cap S| = 0$. Assume $|A \cap S| \neq 0$. From (c), v is the only vertex not adjacent to A and which could thus belong to B. But by hypothesis, $v \in C$, a contradiction.

Assume now that $wu = 2$ and, w.l.o.g., that both v and i belong to B. From (b), for any $\ell \in S \setminus \{i\}$, $P(\ell) \cap A = \emptyset$. Also, from (a), $S \cap A = \emptyset$ (since $i \in S \cap B$). But then $A = \emptyset$, a contradiction. □

We are now ready to state necessary and sufficient conditions for a large class of inequalities of the form (5.1) to be facet defining for $P(G, b)$.

Let S be a minimal connected dominator, with $S = S_D \cup S_Q$, where $S_D = \{i \in S : P(i) \neq \emptyset\}$ is the unique minimal dominator contained in S, and $S_Q = S \setminus S_D$, where every $j \in S_Q$ is an articulation point of $G[S]$. We call the set S *orderly*, if either $S_Q = \emptyset$, or else S_D contains no articulation point of $G[S]$, and S_Q can be ordered into a sequence $i_1, \ldots, i_q$, with the property that for $r = 1, \ldots, q$, $G[S \setminus \{i_r\}]$ has exactly two components with vertex sets S', S'', such that $\{i_1, \ldots, i_{r-1}\} \subset S'$, $\{i_{r+1}, \ldots, i_q\} \subset S''$.

We need some notation. Let $s = |S|$, $d = |S_D|$, $q = |S_Q|$. For any separator C_i in F with shores A_i, B_i, let $a_i = |A_i \cap S_D|$, $b_i = |B_i \cap S_D|$. Since any separator C_i in F contains exactly one vertex $i \in S$, we will call C_i of type 1 if $S \setminus \{i\}$ is contained in a single shore, and of type 2 if $(S \setminus \{i\}) \subseteq A_i \cup B_i$, with $A_i \cap S \neq \emptyset \neq B_i \cap S$. Here we are concerned with separators of type 2, with $i \in S_Q$. Notice that for such a separator $a_i + b_i = d$. A collection $\mathcal{C}$ of type 2 separators will be called representative if it contains exactly one member C_i for each $i \in S_Q$. We order the members of such a collection according to the rule

$$a_i \geq a_{i+1} \ (b_i \leq b_{i+1}), \quad i = 1, \ldots, q-1,$$

and we denote

$$\begin{aligned} a_1^{2k+1} &= a_1 + a_3 + \cdots a_{2k+1} \\ a_2^{2k} &= a_2 + a_4 + \cdots a_{2k}, \end{aligned} \tag{5.2}$$

with b_1^{2k+1} and b_2^k defined in the same way.

With this notation, a minimal connected dominator S with an orderly S_Q is called *exceptional* if

(i) s is odd and

(ii) for any representative collection of type 2 separators

$$\begin{aligned} a_1^{q-1} - a_2^q &= (d-1)/2 \quad \text{if } q \text{ is even} \\ a_1^q - a_2^{q-1} &= d/2 \quad \text{if } q \text{ is odd.} \end{aligned}$$

We now state the main result of this section. From Proposition 5.7 we know that if none of the conditions of Propositions 5.4-5.6 are satisfied, (5.1) does not define a facet of $P(G, b)$. So we can assume the opposite.

Theorem 5.8. *Let S be a minimal connected dominator that is orderly, $|S| \leq b$, and assume that every $v \in V \setminus S$ satisfies at least one of the conditions stated in Propositions 5.4, 5.5 and 5.6. Then the inequality (5.1) defines a facet of $P(G,b)$ if and only if S is not exceptional.*

Proof. Since every $j \in V \setminus S$ satisfies at least one of the conditions stated in Propositions 5.4, 5.5 and 5.6, $\alpha_{j1} = \alpha_{j2} = 0$ for all $j \in V \setminus S$ for any α such that $\alpha u = |S| - 1$ for all $u \in F = \{u \in P(G,b) : u(S) = |S| - 1\}$. We seek necessary and sufficient conditions for having $\alpha_{j1} = \alpha_{j2} = 1$ for all $j \in S$, assuming that S is orderly.

If we set to 0 all α_{jk} for $j \in V \setminus S$, $k = 1, 2$, we are left with a system

$$\sum_{j \in S} (\alpha_{j1} u_{j1} + \alpha_{j2} u_{j2}) = |S| - 1 \tag{5.3}$$

in the unknowns α_{jk}, $j \in S$, $k = 1, 2$. If we denote by F_S the projection of F onto the subspace indexed by S, i.e. $F_S := \{u^S \in \mathbb{R}^{2s} : (u^S, u^{V \setminus S}) \in F \text{ for some } u^{V \setminus S} \in \mathbb{R}^{2(n-s)}\}$, then (5.3) must be satisfied by α for all $u \in F_S$. We will show that $\alpha_{jk} = 1$ for all $j \in S$, $k = 1, 2$, is the unique solution to (5.3) if and only if S is not exceptional, by exhibiting $2s$ points $u \in F_S$ that are affinely independent if and only if S is not exceptional.

We will use the two types of separators in F defined above. The first type, C, has shores $A = S \setminus \{i\}$ and $B \subseteq P(i)$ for some $i \in S_D$. Since by assumption $|S| \leq b + 1$, such a separator obviously exists for each $i \in S_D$, and its incidence vector u satisfies $u_1(S) = |S| - 1$, $u_2(S) = 0$, hence belongs to F_S. The second type, C', has shores A', B' such that $(A' \cup B') \supseteq (S \setminus \{i\})$ for some $i \in S_Q$. Again, at least one such separator in F exists for every $i \in S_Q$, since i is an articulation point of $G[S]$: assigning the vertex set S' of one component of $G[S \setminus \{i\}]$, to A', and the vertex set S'' of the second component to B' (since S is orderly, there are only two components), or vice versa, yields a separator whose incidence vector u satisfies $u_1(S') = |S'|$, $u_2(S') = 0$, $u_1(S'') = 0$, $u_2(S'') = |S''|$, with $|S'| + |S''| = |S| - 1$. Clearly, $u \in F_S$.

If we choose d vectors $u^i \in F_S$ corresponding to separators of the first type, one for each $i \in S_D$, and q vectors $u^i \in F_S$ corresponding to a representative collection of separators of the second type, that yields $d + q = s$ vectors $u^i \in F_S$. If we now choose for every $u^i \in F_S$ its symmetric counterpart obtained by interchanging the two shores of each separator, i.e. interchanging u_1^i and u_2^i for each $i \in S$, we obtain another s points in F_S. We claim that these $2s$ points are affinely independent if and only if S is not exceptional. We will show this by representing each point as the row of a square matrix M of order $2s$, and proving that M is non-singular if and only if our condition is satisfied.

We will denote the $m \times n$ matrix of all 1's by $J_{m \times n}$, the identity of order n by I_n, and we will write J_n for $J_{n \times n}$. Let $H = J_d - I_d$, i.e. H is the $d \times d$ matrix of all 1's, except for the diagonal, which has all 0's. Let D and $\bar{D}$ be $q \times d$ matrices of 0's and 1's, with $D + \bar{D} = J_{q \times d}$, and let Q and $\bar{Q}$ be $q \times q$ matrices of 0's and 1's, Q upper triangular and $\bar{Q}$ lower triangular, with 0's on the diagonal, and with $Q + \bar{Q} = J_q - I_q$. Then our matrix is of the form

$$M = \left(\begin{array}{cc|cc} H & J & 0 & 0 \\ D & Q & \bar{D} & \bar{Q} \\ \hline 0 & 0 & H & J \\ \bar{D} & \bar{Q} & D & Q \end{array} \right).$$

Figure 2 shows the components of an instance of M with $d = 4$ and $q = 3$.

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\bar{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Figure 2: Illustration of the components of M.

It is not hard to see that each row of $(H, J, 0, 0)$ is the incidence vector of a $u \in F_S$ corresponding to a separator C of the first type, with $A = S \setminus \{i\}$ for some $i \in S_D$, and $B \cap S = \emptyset$ ($B \subset P(i)$), i.e. $u^i_{j1} = 1$ for all $j \in S \setminus \{i\}$, $u^i_{i1} = 0$, and $u^i_{j2} = 0$ for all $j \in S$. The 0's on the diagonal of H represent the entry corresponding to i in $S \setminus \{i\}$. Similarly, each row of $(D, Q, \bar{D}, \bar{Q})$ is the incidence vector of a $u \in F_S$ corresponding to a separator C' of the second type, with $A' = S'$ represented by the 1's in the first half of the row, and $B' = S''$ represented by the 1's in the second half of the row, whereas the i in $S \setminus \{i\}$ is represented by the 0's on the diagonal of Q and $\bar{Q}$. This describes the upper half of M. The lower half is obtained by interchanging the roles of u_1 and u_2.

Since S is orderly, if $S_Q \neq \emptyset$ then the representative collection of type 2 separators C'_i corresponding to the rows of $(D, Q, \bar{D}, \bar{Q})$ can be ordered according to increasing or decreasing size of their shores A'_i. Here we choose to order them decreasingly, which corresponds to having Q upper triangular, $\bar{Q}$ lower triangular (with 0's on the diagonal), and the row sums of D and $\bar{D}$ satisfying $a_i \geq a_{i+1}$ and $b_i \leq b_{i+1}$, $i = 1, \ldots, q-1$, respectively.

Now let's first look at the case when $S_Q = \emptyset$ and so $S = S_D$, i.e. the minimal connected separator S is also minimal as a separator. Then our matrix M reduces to $\begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}$, which is obviously nonsingular.

Next assume $S_Q \neq \emptyset$, i.e. all the submatrices of M are present.

Using the standard procedure for inversion of partitioned matrices, it is straightforward to show that the $2s \times 2s$ matrix M is nonsingular if and only if the $2q \times 2q$ matrix

$$R = \left(\begin{array}{c|c} Q - \frac{1}{d-1}DJ & \bar{Q} - \frac{1}{d-1}\bar{D}J \\ \hline \bar{Q} - \frac{1}{d-1}\bar{D}J & Q - \frac{1}{d-1}DJ \end{array}\right) = \begin{pmatrix} Q_0 & \bar{Q}_0 \\ \bar{Q}_0 & Q_0 \end{pmatrix}$$

is nonsingular. Furthermore, the same techniques can be used to show that the $2q \times 2q$ matrix R is nonsingular, if and only if the $q \times q$ matrix $\hat{R} := Q_0 - \bar{Q}_0 Q_0^{-1} \bar{Q}_0$ is nonsingular.

For $\hat{R}$ to be well defined, Q_0^{-1} must exist, i.e. Q_0 must be nonsingular. This can be shown to be always the case. Indeed,

$$Q_0 = Q - \frac{1}{d-1} D\,J = \frac{1}{d-1} \begin{pmatrix} -a_1 & d-1-a_1 & \dots & d-1-a_1 \\ -a_2 & -a_2 & \dots & d-1-a_2 \\ \vdots & \vdots & & \\ -a_q & -a_q & \dots & -a_q \end{pmatrix}.$$

Subtracting column 1 of this matrix from every other column yields a matrix that is upper triangular except for its first column, and whose determinant, like that of Q_0, has absolute value $a_q/(d-1)$. Hence Q_0 is nonsingular, i.e. $\hat{R}$ is well defined.

To examine the conditions under which $\hat{R}$ is nonsingular, we start by computing its elements. We have

$$Q_0^{-1} = \begin{pmatrix} -1 & 0 & 0 & \cdots & \frac{d-1-a_1}{-a_q} \\ 1 & -1 & 0 & \cdots & \frac{a_1-a_2}{-a_q} \\ 0 & 1 & -1 & \cdots & \frac{a_2-a_3}{-a_q} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{a_{q-2}-a_{q-1}}{-a_q} \\ 0 & 0 & 0 & \cdots & \frac{a_{q-1}}{-a_q} \end{pmatrix}$$

$$\bar{Q} = \frac{1}{d-1} \begin{pmatrix} -b_1 & -b_1 & \cdots & -b_1 & -b_1 \\ d-1-b_2 & -b_2 & \cdots & -b_2 & -b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ d-1-b_{q-1} & d-1-b_{q-1} & \cdots & -b_{q-1} & -b_{q-1} \\ d-1-b_q & d-1-b_q & \cdots & d-1-b_q & -b_q \end{pmatrix}$$

and

$$\hat{R} = Q_0 - \bar{Q}_0 Q_0^{-1} \bar{Q}_0 = \frac{1}{d-1} \tilde{R},$$

where $\tilde{R}$ is the matrix given by

$$\begin{pmatrix} \frac{b_1}{a_q}-d & \frac{b_1}{a_q}-1 & \frac{b_1}{a_q}-1 & \cdots & \frac{b_1}{a_q}-1 & \frac{b_1}{a_q}b_q+b_1-1 \\ \frac{b_2-b_1+1}{a_q}-(d+1) & \frac{b_2-b_1+1}{a_q}-(d+1) & \frac{b_2-b_1+1}{a_q}-2 & \cdots & \frac{b_2-b_1+1}{a_q}-2 & \frac{b_2-b_1+1}{a_q}b_q+b_2-b_1-1 \\ \frac{b_3-b_2+1}{a_q}-2 & \frac{b_3-b_2+1}{a_q}-(d+1) & \frac{b_3-b_2+1}{a_q}-(d+1) & \cdots & \frac{b_3-b_2+1}{a_q}-2 & \frac{b_3-b_2+1}{a_q}b_q+b_3-b_2-1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{b_{q-1}-b_{q-2}+1}{a_q}-2 & \frac{b_{q-1}-b_{q-2}+1}{a_q}-2 & \frac{b_{q-1}-b_{q-2}+1}{a_q}-2 & \cdots & \frac{b_{q-1}-b_{q-2}+1}{a_q}-(d+1) & \frac{b_{q-1}-b_{q-2}+1}{a_q}b_q+b_{q-1}-b_{q-2}-1 \\ \frac{b_q-b_{q-1}+1}{a_q}-2 & \frac{b_q-b_{q-1}+1}{a_q}-2 & \frac{b_q-b_{q-1}+1}{a_q}-2 & \cdots & \frac{b_q-b_{q-1}+1}{a_q}-(d+1) & \frac{b_q-b_{q-1}+1}{a_q}b_q-a_q-b_{q-1} \end{pmatrix}$$

We claim that $\hat{R}$ is nonsingular if and only if S is not exceptional. To prove this, we will perform a linear transformation on $\hat{R}$ that does not affect the absolute value of its determinant. We will use the fact that many elements of $\hat{R}$ are equal to each other and some pairs of entries differ by the same constant. By subtracting column $i-1$ from column i for $i = q, q-1, \ldots, 2$, and dividing every entry by $d-1$, we obtain the following matrix whose determinant has the same absolute value as that of $\hat{R}$:

$$\hat{R}' = (r_{ij}) = \begin{pmatrix} \frac{b_1 - da_q}{a_q(d-1)} & 1 & 0 & \cdots & 0 & \frac{b_1}{a_q} \\ \frac{(b_2-b_1+1)-(d+1)a_q}{a_q(d-1)} & 0 & 1 & \cdots & 0 & \frac{b_2-b_1+1}{a_q} \\ \frac{(b_3-b_2+1)-2a_q}{a_q(d-1)} & -1 & 0 & \cdots & 0 & \frac{b_3-b_2+1}{a_q} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{(b_{q-1}-b_{q-2}+1)-2a_q}{a_q(d-1)} & 0 & 0 & \cdots & 0 & \frac{b_{q-1}-b_{q-2}+1+a_q}{a_q} \\ \frac{(b_q-b_{q-1}+1)-2a_q}{a_q(d-1)} & 0 & 0 & \cdots & -1 & \frac{b_q-b_{q-1}+1}{a_q} \end{pmatrix}$$

Next, letting r_j denote column j of $\hat{R}'$, we subtract from column 1

$$r_{11} \cdot r_2 + r_{21} \cdot r_3 + (r_{11} + r_{31}) \cdot r_4 + (r_{21} + r_{41}) r_5 + \cdots$$
$$\begin{cases} +(r_{11} + r_{31} + \cdots + r_{q-1,1}) r_q \text{ (if } q \text{ is even) or} \\ +(r_{21} + r_{41} + \cdots + r_{q-1,1}) r_q \text{ (if } q \text{ is odd),} \end{cases}$$

and we subtract from column q

$$r_{1q} \cdot r_2 + r_{2q} \cdot r_3 + (r_{1q} + r_{3q}) \cdot r_4 + (r_{2q} + r_{4q}) r_5 + \cdots$$
$$\begin{cases} +(r_{1q} + r_{3q} + \cdots + r_{q-1,q}) r_q \text{ (if } q \text{ is even) or} \\ +(r_{2q} + r_{4q} + \cdots + r_{q-1,q}) r_q \text{ (if } q \text{ is odd).} \end{cases}$$

The outcome is the matrix W, whose determinant also has the same absolute value as that of $\hat{R}$:

$$W = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ w_{q-1,1} & 0 & 0 & \cdots & -1 & 0 & w_{q-1,q} \\ w_{q,1} & 0 & 0 & \cdots & 0 & -1 & w_{q,q} \end{pmatrix}$$

It is not hard to see that $|\det(W)| = |w_{q-1,1} \cdot w_{q,q} - w_{q,1} \cdot w_{q-1,q}|$. Here

$$w_{q-1,1} = r_{q-1,1} + r_{q-3,1} + \cdots + \begin{cases} r_{11} & \text{if } q \text{ is even} \\ r_{21} & \text{if } q \text{ is odd} \end{cases}$$

$$w_{q,1} = r_{q,1} \quad + r_{q-2,1} + \cdots + \begin{cases} r_{21} & \text{if } q \text{ is even} \\ r_{11} & \text{if } q \text{ is odd} \end{cases}$$

and

$$w_{q-1,q} = r_{q-1,q} + r_{q-3,q} + \cdots + \begin{cases} r_{1q} & \text{if } q \text{ is even} \\ r_{2q} & \text{if } q \text{ is odd} \end{cases}$$

$$w_{q,q} = r_{q,q} \quad + r_{q-2,q} + \cdots + \begin{cases} r_{2q} & \text{if } q \text{ is even} \\ r_{1q} & \text{if } q \text{ is odd} \end{cases}$$

Substituting for each r_{ij} its value from $\hat{R}$ and using the notation (5.2), we obtain:

- for q even,

$$\begin{aligned} w_{q-1,1} &= \left((b_1^{q-1} - b_2^{q-2}) - (d+q-2)a_q + \tfrac{q}{2} - 1\right) / a_q(d-1) \\ w_{q,1} &= \left((b_2^{q} - b_1^{q-1}) - (d+q-1)a_q + \tfrac{q}{2}\right) / a_q(d-1) \\ w_{q-1,q} &= \left((b_1^{q-1} - b_2^{q-2}) + \tfrac{q}{2} - 1 + a_q\right) / a_q \\ w_{q,q} &= \left((b_2^{q} - b_1^{q-1}) + \tfrac{q}{2}\right) / a_q; \end{aligned}$$

- for q odd,

$$\begin{aligned} w_{q-1,1} &= \left((b_2^{q-1} - b_1^{q-2}) - (d+q-2)a_q + \tfrac{q-1}{2}\right) / a_q(d-1) \\ w_{q,1} &= \left((b_1^{q} - b_2^{q-1}) - (d+q-1)a_q + \tfrac{q-1}{2}\right) / a_q(d-1) \\ w_{q-1,q} &= \left((b_2^{q-1} - b_1^{q-2}) + \tfrac{q-1}{2} + a_q\right) / a_q \\ w_{q,q} &= \left((b_1^{q} - b_2^{q-1}) + \tfrac{q-1}{2}\right) / a_q \end{aligned}$$

Thus for q even, we have

$$\begin{aligned} & w_{q-1,1} \cdot w_{q,q} - w_{q,1} \cdot w_{q-1,q} = \\ & \qquad \Big[\left((b_1^{q-1} - b_2^{q-2}) - (d+q-2)a_q + \tfrac{q}{2} - 1\right) \cdot \left((b_2^{q} - b_1^{q-1}) + \tfrac{q}{2}\right) \\ & \qquad - \left((b_2^{q} - b_1^{q-1}) - (d+q-1)a_q + \tfrac{q}{2}\right) \cdot \left((b_1^{q-1} - b_2^{q-2}) + \tfrac{q}{2} - 1 + a_q\right)\Big] / a_q^2(d-1). \end{aligned}$$

Clearly, $\det(W) \neq 0$ if and only if the numerator is nonzero, so we may ignore the denominator. Multiplying through and collecting terms then yields for the numerator the expression

$$E = a_q(d+q-1)(2(b_1^{q-1} - b_2^q) + d - 1).$$

Similarly, for q odd we have

$$w_{q-1,1} \cdot w_{q,q} - w_{q,1} \cdot w_{q-1,q} = \Big[\Big((b_2^{q-1} - b_1^{q-2}) - (d+q-2)a_q + \tfrac{q-1}{2}\Big) \cdot \Big((b_1^q - b_2^{q-1}) + \tfrac{q-1}{2}\Big) - \Big((b_1^q - b_2^{q-1}) - (d+q-1)a_q + \tfrac{q-1}{2}\Big) \cdot \Big((b_2^{q-1} - b_1^{q-2}) + \tfrac{q-1}{2} + a_q\Big)\Big] / a_q^2(d-1),$$

and the expression we get for the numerator, after multiplying through and collecting terms, is

$$E' = -a_q(d+q-1)(2(b_1^q - b_2^{q-1}) - d).$$

Now we claim that for admissible values of d, q, a_j, b_j, $j = 1, \ldots, q$, the expression E, if q is even, or E', if q is odd, vanishes if and only if S is exceptional.

Assume first that q is even. Then $E = 0$ if and only if $b_1^{q-1} - b_2^q = (1-d)/2$. But since $b_1^{q-1} - b_2^q = \frac{q}{2}d - a_1^{q-1} - \frac{q}{2}d + a_2^q$, $E = 0$ if and only if $a_1^{q-1} - a_2^q = (d-1)/2$. This is precisely condition (ii) of the definition of S being exceptional when q is even. Furthermore, the last equation implies that d is odd, which in turn implies that s is odd (since q is even), thus condition (i) of that definition is also satisfied.

Assume now that q is odd. Then $E' = 0$ if and only if $b_1^q - b_2^{q-1} = d/2$. But $b_1^q - b_2^{q-1} = \frac{q+1}{2}d - a_1^q - \frac{q-1}{d} + a_2^{q-1}$, hence $E' = 0$ if and only if $a_1^q - a_2^{q-1} = d/2$, which is precisely condition (ii) of the exceptionality of S when q is odd. Furthermore, in this case d is obviously even, and since q is odd, s is also odd, which is condition (i).

Assume now that S is exceptional, i.e. conditions (i) and (ii) are satisfied. We claim that in that case there exists no set of $2s$ affinely independent points $u \in F_S$. Indeed, the only candidates are separators of type 1 with $i \in S_D$, or separators of type 2 with $i \in S_Q$, since the only other possible case, that of a type 2 separator with $i \in S_D$, is excluded by the fact that if $S_Q \neq \emptyset$, S_D contains no articulation point of $G[S]$. But the only separators of type 1 with $i \in S_D$ are those corresponding to the rows of H, and by assumption every representative set of separators of type 2 satisfies the conditions that make S exceptional, which proves our claim. □

Turning now to the case when S is exceptional, we start by examining the situation where $q = |S_Q| = 1$, say $S_Q = \{i\}$.

By the definition of exceptionality, d is even, $a_1 = d/2 = b_1$, and each of the two components of S_D is 2-connected. Let $p := d/2$.

Proposition 5.9. *Let S, with $S_q = \{i\}$, be exceptional. Then the inequality*

$$p\,u_1(S \setminus \{i\}) + (p-1)u_2(S \setminus \{i\}) + (2p-1)u_{i2} \leq p(2p-1) \tag{5.4}$$

is valid for $P(G, b)$. Furthermore, (5.4) is facet defining if and only if every $v \in V \setminus S$ satisfies at least one of the conditions of Propositions 5.4, 5.5 and 5.6.

Proof. Suppose that S, with $S_Q = \{i\}$, is exceptional. Then the maximum of the left hand side of (5.4), say $f(u)$, over all $u \in P(G,b)$, is $p(2p-1)$. Indeed, let $\bar{u} \in F$ be a point for which $f(u)$ attains its maximum. If i belongs to the separator C associated with $\bar{u}$, then the shores of C must be $A = S'$ and $B = S''$ or vice versa, where $G[S']$ and $G[S'']$ are the two components of $G[S \setminus \{i\}]$. Thus $\bar{u}_{j1} = 1$, $\bar{u}_{j2} = 0$ for all $j \in S'$, $\bar{u}_{j1} = 0$, $\bar{u}_{j2} = 1$ for all $j \in S''$, $\bar{u}_{jk} = 0$, $k = 1,2$, otherwise, and $f(\bar{u}) = p \cdot p + (p-1)p = p(2p-1)$. If the separator does not contain i, it must contain some $\ell \in S$, $\ell \neq i$, but since ℓ is not an articulation point, all of $S \setminus \{\ell\}$ must belong to the same shore. If this shore is A, with $B \subseteq P(\ell)$, then $\bar{u}_{j1} = 1$, $\bar{u}_{j2} = 0$ for all $j \in S \setminus \{i, \ell\}$, $\bar{u}_{jk} = 0$, $k = 1,2$ otherwise, and $f(\bar{u}) = p(2p-1) + 0 = p(2p-1)$, as claimed. If $(S \setminus \{\ell\}) \subseteq B$, with $A \subseteq P(\ell)$, then $\bar{u}_{j1} = 0$, $\bar{u}_{j2} = 1$ for all $j \in S \setminus \{\ell\}$ (including $j = i$), $u_{jk} = 0$, $k = 1,2$ otherwise, and again $f(\bar{u}) = 0 + (p-1)(2p-1) + (2p-1) = p(2p-1)$. This proves that (5.4) is valid.

Now let $F' := \{u \in P(G,b) : u \text{ satisfies (5.4) at equality}\}$. Then F' is a facet of $P(G,b)$ if and only if every α such that $\alpha u = p(2p-1)$ for all $u \in F'$ satisfies $\alpha_{jk} = 0$ for $j \in V \setminus S$, $k = 1,2$, and

$$\alpha_{j1} = \begin{cases} p & j \in S \setminus \{i\} \\ 0 & j = 1 \end{cases} \qquad \alpha_{j2} = \begin{cases} p-1 & j \in S \setminus \{i\} \\ 2p-1 & j = i \end{cases} \tag{5.5}$$

It is easy to see that Propositions 5.3-5.7, which were stated for the case of inequality (5.1), remain valid for the case of inequality (5.4). Therefore, $\alpha_{j1} = \alpha_{j2} = 0$ for all $j \in V \setminus S$ if and only if every $v \in V \setminus S$ satisfies at least one of the conditions of Proposition 5.4, 5.5 and 5.6, with the implication that, if there exists $v \in V \setminus S$ that violates all three of these conditions, then (5.4) does not define a facet of $P(G,b)$. Assuming now that $\alpha_{jk} = 0$ for all $j \in V \setminus S$, $k = 1,2$, we show that the coefficients α_{jk} for $j \in S$, $k = 1,2$, satisfy (5.5), by the procedure used in the proof of Theorem 5.8, i.e. by exhibiting $2s$ affinely independent points $u \in F'_S = \{u^S \in \mathbb{R}^{2s} : (u^S, u^{V \setminus S}) \in F' \text{ for some } u^{V \setminus S} \in \mathbb{R}^{2(n-s)}\}$. As in the case of that proof, we use the d points $u^j \in F'_S$ corresponding to the separators C_j of type 1 with shores $A_j = S \setminus \{j\}$, $j \in S_D$, $B_j \subseteq P(j)$, and the one point $u^i \in F'_S$ corresponding to the separator of type 2 C'_i with shores $A'_i = S'$, $B'_i = S''$, where $G[S']$ and $G[S'']$ are the two components of $G[S \setminus \{i\}]$. This is a collection of $d + 1 = s$ points in F'_S, and taking the symmetric counterparts of these points, obtained by interchanging u_1 and u_2, we obtain an additional s points. The matrix M whose rows are the incidence vectors of these points has the same structure as the corresponding matrix in the proof of Theorem 5.8, with the only difference that here $\bar{Q} = 1$. As in that proof, M is nonsingular if and only if $\hat{R} = Q_0 - \bar{Q}_0 Q_0^{-1} \bar{Q}_0$ is nonsingular, where

$$Q_0 = Q - \frac{1}{d-1} DJ = 0 - \frac{1}{d-1} \cdot a_1, \quad Q_0^{-1} = -\frac{d-1}{a_1}, \quad \bar{Q}_0 = 1 - \frac{1}{d-1} \bar{D} J = 1 - \frac{1}{d-1} b_1$$

and

$$\begin{aligned} \hat{R} &= -\frac{a_1}{d-1} - \frac{d-1-b_1}{d-1} \left(-\frac{d-1}{a_1} \right) \frac{d-1-b_1}{d-1} \\ &= \frac{(d-1-b_1)^2 - a_1^2}{(d-1)a_1} = \frac{1-2a_1}{(d-1)a_1} \neq 0 \end{aligned}$$

This proves that $\hat{R}$ is nonsingular, hence (5.4) is facet defining. □

Remark 5.10. *If S_D contains an articulation point $\ell \neq i$ of $G[S]$, then the inequality (5.4) is not valid.*

Proof. Suppose $G[S]$ has a second articulation point, say $\ell \neq i$. Let $G[S']$ be a component of $G[S \setminus \{\ell\}]$ such that $i \notin S'$, and let $S'' := S \setminus (S' \cup \{\ell\})$. We claim that in this case (5.4) is not valid. Indeed, consider the separator C whose shores are $A = S'$ and $B = S''$, with associated $\bar{u} \in P(G, b)$. Then

$$\begin{aligned} & p \cdot \bar{u}_1(S \setminus \{i\}) + (p-1)\bar{u}_2(S \setminus \{i\}) + (2p-1)\bar{u}_{i2} = \\ = \;& p \cdot |S'| + (p-1)(|S''| - 1) + (2p-1) \\ = \;& |S'| + (p-1)(|S'| + |S''| - 1) + (2p-1) \\ = \;& |S'| + (p-1)(2p-1) + (2p-1) > p(2p-1). \end{aligned}$$

□

When the minimal connected dominator S is not orderly, inequality (5.1) may or may not be facet defining. For any *specific* non-orderly S it is not hard to tell whether (5.1) is facet-defining, by applying the same analysis as in the proof of Theorem 5.8, with Q and $\bar{Q}$ modified to reflect the structure of S; but this analysis becomes unwieldy for a general S. However, when (5.1) is not facet defining either because S is not orderly, or because S, while orderly, is exceptional, there is another family of facet defining inequalities that dominates S.

6 A Class of Asymmetric Facets of $P(G, b)$

Consider any minimal dominator S of G, not necessarily connected. The inequality

$$u_1(S) \leq |S| - 1 \tag{6.1}$$

is clearly valid for $P(G, b)$, and various liftings of (6.1) may yield facet defining inequalities.

The first question that arises in this context, is when does (6.1) define a facet of $P(G, b)$, i.e. when is it the case that all the lifting coefficients of (6.1) are equal to 0? The next Proposition settles this question.

As before, we assume that $|S| \leq b$, for otherwise (6.1) is implied by (2.5), hence redundant.

Proposition 6.1. *The inequality (6.1), where S is a minimal dominator of G, defines a facet of $P(G, b)$ if and only if conditions (a), (b) and (c) below are satisfied:*

(a) $V \setminus S = \cup_{i \in S} P(i)$

(b) S contains no self-dominator

(c) S is an independent set.

Proof. Let F be the face of $P(G,b)$ defined by (6.1), i.e. $F := \{u \in P(G,B) : u_1(S) = |S|-1\}$.

Necessity. Suppose S violates (a), i.e. there exists $v \in W := (V \setminus S) \setminus (\cup_{i\in S} P(i))$. This means that $|\mathrm{Adj}(v) \cap S| \geq 2$, say $\{(v,i),(v,j)\} \subseteq E$ for some $i,j \in S$. Then $u_{i1}+u_{v2} \leq 1$, $u_{j1}+u_{v2} \leq 1$, but for any $u \in F$, $u_{i1}+u_{j1} \geq 1$, since $u_{\ell 1}=1$ for all but one index $\ell \in S$. It follows that $u_{v2}=0$ for all $u \in F$, hence F is not a facet.

Suppose now that S violates (b), and let $i \in S$ be a self-dominator. We claim that for any $u \in F$, $u_{i1}+u_{i2}=1$, and thus F is not a facet. Indeed, let C be a separator in F such that $i \in C$; then shore A of C must contain $S \setminus \{i\}$ in order to have $u_1(S) = |S|-1$, and since $S \setminus \{i\}$ is a dominant of $V \setminus \{i\}$, there are no vertices left for B. Hence $i \in A \cup B$, which is equivalent to $u_{i1}+u_{i2}=1$.

Finally, suppose S violates (c), and let (i,j) be an edge with both ends in S. Then $u_{i1}+u_{i2} \leq 1$, $u_{i1}+u_{j2} \leq 1$, and $u_{j1}+u_{j2} \leq 1$, $u_{j1}+u_{i2} \leq 1$. Thus if $u_{i1}=1$ or $u_{j1}=1$, then $u_{i2}=u_{j2}=0$. But since $u_{i1}+u_{j1} \geq 1$ for all $u \in F$, this is always the case, i.e. $u_{i2}=u_{j2}=0$ for all $u \in F$. Hence again F is not a facet.

Sufficiency. Since every $j \in V \setminus S$ belongs to some pendent set, say $P(i)$, and since S is independent, it is easy to see that $A = S \setminus \{i\}$ and $B = \{i,j\}$ are the shores of a separator in F, say C, and that $C' = C \cup \{j\}$ is also a separator in F, with shores $A' = A$ and $B' = B \setminus \{j\}$. Hence, from Proposition 5.3, for any equation $\alpha u = |S|-1$ satisfied by all $u \in F$, $\alpha_{jk}=0$ for all $j \in V \setminus S$, $k=1,2$. Now consider a coefficient α_{j2}, $j \in S$. Since S is an independent set, the point defined by $u_{\ell 1}=1$ for $\ell \in S \setminus \{j\}$, $u_{\ell 1}=0$ for $\ell \in (V \setminus S) \cup \{j\}$, $u_{j2}=1$, $u_{\ell 2}=0$ for $\ell \in V \setminus \{j\}$, is in F. But if $\alpha_{j2} \neq 0$, this point violates $\alpha u = |S|-1$. Hence $\alpha_{j2}=0$ for all $j \in S$.

As to the coefficients α_{j1}, $j \in S$, if we set to 0 all α_{jk}, $j \in V \setminus S$, $k=1,2$, and all α_{j2}, $j \in S$, we obtain the system

$$\sum_{j\in S} \alpha_{j1} u_{j1} = |S|-1$$

in the unknowns α_{j1}, $j \in S$, which must be satisfied for every $u \in F_S$, where $F_S := \{u_1^S \in \mathbb{R}^s : (u_1^S, u_2^S, u^{V\setminus S}) \in F$ for some $(u_2^S, u^{V\setminus S}) \in \mathbb{R}^s \times \mathbb{R}^{2(n-s)}\}$ with $s=|S|$. If we choose the $|S|$ points $u^i \in F_S$, $i \in S$, defined by $u^i_{j1}=1$ for $j \in S \setminus \{i\}$, $u^i_{i1}=0$, we obtain a system whose unique solution is $\alpha_{j1}=1$ for all $j \in S$. □

Figure 3 shows an example of an inequality (6.1) that is facet defining for $P(G,b)$.

From Proposition 6.1 it follows that the inequality (6.1) is facet defining only under the very special conditions (a), (b), (c). When these conditions do not hold, (6.1) can be lifted or otherwise generalized to yield some facet defining inequalities. There are many valid generalizations, but here we will be concerned with three classes of such inequalities: the first class is obtained by lifting the coefficients α_{j2}, $j \in S$, the second class comes from lifting the coefficients α_{j2}, $j \in V \setminus S$, before $j \in S$, while the third class involves a different type of generalization.

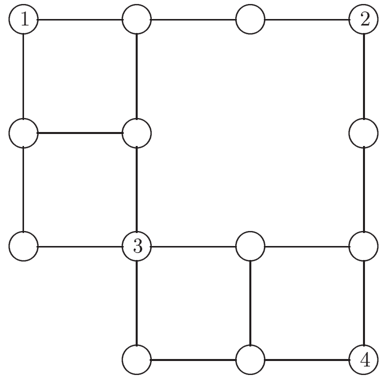


Figure 3: A facet defining inequality (6.1)

7 First Generalization

Let S be a minimal dominator that satisfies conditions (a) and (b), but not condition (c), of Proposition 6.1. Further, let $S_1, \ldots, S_k$ be the vertex sets of the components of $G[S]$. In each component $G[S_\ell]$ such that $|S_\ell| > 1$ we choose an ordered set of vertices $I_\ell = \{v_1, \ldots, v_q\}$ with the following properties (here $\text{Adj}(v_i)$ refers to adjacency in $G[S_\ell]$):

(i) for all $i \in \{2, \ldots, q\}$, $(v_i, v_j) \notin E_\ell$ for all $j \in \{1, \ldots, i-1\}$, i.e. I_ℓ is an independent set;

(ii) for all $i \in \{2, \ldots, q\}$, there exists $j \in \{1, \ldots, i-1\}$ such that $\text{Adj}(v_i) \cap \text{Adj}(v_j) \neq \emptyset$, i.e. v_i is at an edge-distance of 2 from the vertex set $\{v_1, \ldots, v_{i-1}\}$.

(iii) I_ℓ is maximal.

Such a set always exists and is obviously not unique. Figure 4 shows an example of a component $G[S_\ell]$, along with two different sets I_ℓ. Next we define a function $\delta : S_\ell \to \mathbb{Z}$ as follows:

$$\delta(v) = \begin{cases} |\text{Adj}(v_1)| & \text{if } v = v_1 \\ |\text{Adj}(v_i) \setminus \bigcup_{j=1}^{i-1} \text{Adj}(v_j)| + 1 & \text{if } v = v_i \text{ for some } i \geq 2 \\ 0 & \text{if } v \in S_\ell \setminus I_\ell \end{cases} \tag{7.1}$$

The numbers $\delta(v_j)$ for $v_j \in I_\ell$ can be interpreted as the degree of vertex v_j in a spanning tree T_ℓ of $G[S_\ell]$ constructed as follows. For all $v \in S_\ell$, define $\text{Adj}^*(v) := \{v\} \cup \text{Adj}(v)$.
Initialization. Choose some $v \in S_\ell$, set $v_1 := v$ and put v_1 into T_ℓ as a marked vertex.

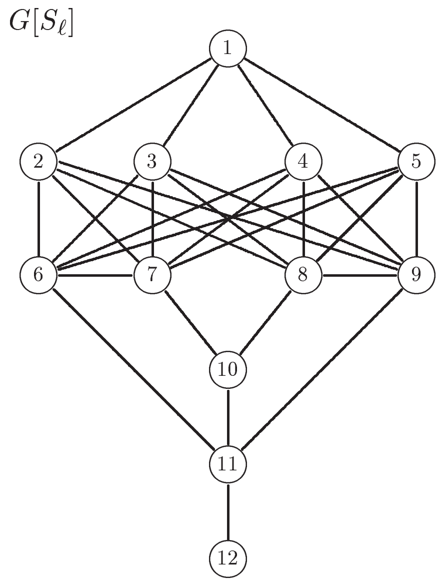
Put into T_ℓ all vertices $v \in \text{Adj}(v_1)$ and all edges joining them to v_1.

Iterative step k**.** Choose some $v \in S_\ell \setminus \bigcup_{j=1}^{k-1} \text{Adj}^*(v_j)$ such that $\text{Adj}(v) \cap \text{Adj}(v_j) \neq \emptyset$ for some $j \in \{1, \ldots, k-1\}$, set $v_k := v$, and put v_k into T_ℓ as a marked vertex, by joining it through an edge to some (arbitrarily chosen) unmarked vertex of T_ℓ.

Put into T_ℓ all vertices $v \in \text{Adj}(v_k) \setminus \bigcup_{j=1}^{k-1} \text{Adj}(v_j)$ and all edges joining them to v_k.

Stop when all vertices of S_ℓ have been included in T_ℓ.

It is not hard to see that the marked vertices of T_ℓ form an ordered set satisfying the conditions defined for I_ℓ. If the set of vertices at edge-distance k from v_1 is considered *level* k of T_ℓ, then the set of all vertices at even levels of T_ℓ, which is the set of all marked vertices of T_ℓ, is precisely the independent set I_ℓ defined by conditions (i), (ii), (iii). Note, however, that the spanning tree T_ℓ is not uniquely defined because of the freedom of choosing the unmarked vertex of T_ℓ to which a newly marked vertex v_k is joined by an edge. Figure 5 shows an example of a spanning tree T_ℓ associated with the set I_ℓ^1 of Figure 4.



$$I_\ell^1 = \{1, 6, 9, 10, 12\}$$
$$I_\ell^2 = \{10, 2, 3, 4, 5, 12\}$$

Figure 4: Two ordered sets I_ℓ^1, I_ℓ^2 in $G[S_\ell]$, satisfying (i), (ii), (iii).

Theorem 7.1. *Let S be a minimal dominator of G satisfying conditions (a) and (b), but not (c), of Proposition 6.1, and let $G[S_\ell]$, $\ell = 1, \dots, k$, be the components of $G[S]$. For each singleton component $S_\ell = \{i\}$, set $\delta_i = 0$, and for all other components $G[S_\ell]$ define $\delta_j = \delta(j)$ for all $j \in S_\ell$ by (7.1). Then the inequality*

$$u_1(S) + \sum_{j \in S} \delta_j u_{j2} \leq |S| - 1 \tag{7.2}$$

is valid and facet defining for $P(G, b)$.

Proof. If the inequality $u_1(S) \leq |S| - 1$ were valid for the polytope $P(G, b)$ restricted to the space of the variables u_{j1}, $j \in S$, then we could lift it to (7.2). However, if we set to 0 all the variables missing from $u_1(S) \leq |S| - 1$, the remaining polytope is just the unit cube

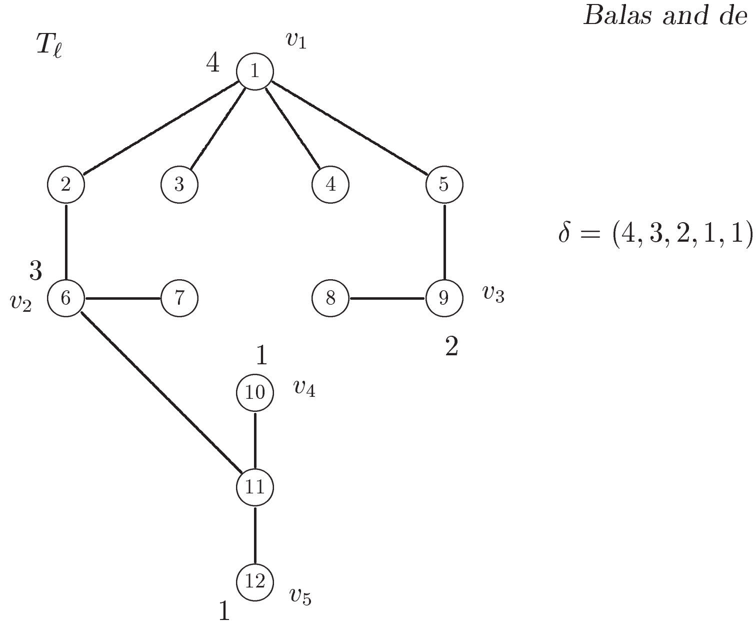


Figure 5: A spanning tree T_ℓ associated with I_ℓ^1 of Figure 4.

in $\mathbb{R}^s$ and the inequality $u_1(S) \leq |S| - 1$ is invalid. However, if instead we project $P(G, b)$ onto $\mathbb{R}^{2s}$,

$$\text{Proj}_S(P(G, b)) := \{u^S \in \mathbb{R}^{2s} : (u^S, u^{V \setminus S}) \in P(G, b) \text{ for some } u^{V \setminus S} \in \mathbb{R}^{2(n-s)}\},$$

then the inequality $u_1(S) \leq |S| - 1$ is valid for the polytope $P^* \subset \mathbb{R}^s$,

$$P^* := \{u^S \in \text{Proj}_S(P(G, b)) : u_{j2} = 0,\ j \in S\},$$

and so it can be lifted by the well known sequential lifting procedure (see e.g. [8]).

Let $j_1, \ldots, j_n$ be any numbering of the vertices in S, such that within each component $G[S_\ell]$ of $G[S]$, vertices in I_ℓ precede those in $S_\ell \setminus I_\ell$ and are numbered according to their position in I_ℓ.

We now calculate the lifting coefficients of the variables u_{j2} for $j = j_1, \ldots, j_s$ in that sequence. But first, we note that $u_1(S) \leq |S| - 1$ is not only valid for P^*, but also facet defining. Indeed, for each $i \in S$ the point u^S defined by $u_{j1} = 1$ for $j \in S \setminus \{i\}$, $u_{i1} = 0$, $u_{j2} = 0$, $j \in S$, is in P^* and satisfies $u_1(S) = |S| - 1$. To see this, note that each such point comes (through projection and restriction) from the incidence vector $u \in P(G, b)$ of a separator C that has as its shores $A = S \setminus \{i\}$ and $B = P(i)$. Clearly, the $|S|$ points u_1^S defined above, one for each $i \in S$, are affinely independent and so $u_1(S) \leq |S| - 1$ defines a facet of P^*.

For the first variable, $u_{j_1 2}$, we solve

$$\max\{u_1(S) : u^S \in \text{Proj}_S(P(G, b)),\ u_{j_1 2} = 1,\ u_{j_r 2} = 0 \text{ for all } r > 1\}$$

and we find that the maximum is $|S| - 1 - |\text{Adj}(j_1)|$, since setting $u_{j_1 2} = 1$ forces to 0 $u_{j_1 1}$, and all the variables u_{j1} for $j \in \text{Adj}(j_1)$. This gives for $u_{j_1 2}$ the coefficient $|\text{Adj}(j_1)|$, which is the difference between the maximum of the objective function with and without $u_{j_1 2}$ set to 1. But this is precisely δ_{j_1} as defined by (7.1). Further, if we denote by $u(\delta_{j_1})$ the maximizing solution (including the variable fixed at 1) that yielded the coefficient δ_{j_1}, namely $u_{j_1 2} = 1$, $u_{j1} = 0$ for $j \in \text{Adj}^*(j_1)$, $u_{j1} = 1$ for $j \in S \setminus \text{Adj}^*(j_1)$, we find that it corresponds to a separator C with shores $A = S \setminus \text{Adj}^*(j_1)$, $B = \{j_1\}$, and satisfies $u_1(S) + \delta_{j_1} u_{j_1 2} \leq |S| - 1$ at equality.

Suppose now that these properties hold for $j = j_1, j_2, \ldots, j_{k-1}$, i.e. that the coefficient of u_{j2} is δ_j for $j = j_1, j_2, \ldots, j_{k-1}$, and that the solution $u(\delta_j)$ for which the maximand attains its bound satisfies the corresponding inequality at equality; and let $j = j_k$. We then have to solve

$$\max\{u_1(S) + \sum_{j=j_1}^{j_{k-1}} \delta_j u_{j2} : u \in \text{Proj}_S(P(G,b)),\ u_{j_k 2} = 1,\ u_{j_r 2} = 0 \text{ for all } r > k\}.$$

To simplify the discussion, assume for the time being that $j_1, \ldots, j_k$ belong to the same component of $G[S]$. Now the maximum of $u_1(S) + \sum_{j=j_1}^{k-1} \delta_j u_{j2}$ without setting $u_{j_k 2} = 1$ is $|S| - 1$. Furthermore, this value is attained for a solution $u(\delta_{k-1})$ that has $u_{j2} = 1$ for $j = j_1, \ldots, j_{k-1}$, $u_{j1} = 0$ for $j \in \bigcup_{r=1}^{k-1} \text{Adj}^*(j_r)$ and $u_{j1} = 1$ for $S \setminus \bigcup_{r=1}^{k-1} \text{Adj}^*(j_r)$. This solution corresponds to a separator C with shores $A = S \setminus \bigcup_{r=1}^{k-1} \text{Adj}^*(j_r)$, $B = \{j_1, \ldots, j_{k-1}\}$, and it satisfies $u_1(S) + \sum_{j=j_1}^{j_{k-1}} \delta_j\, u_{j2} \leq |S| - 1$ at equality. Since $\delta_j \geq 1$ for $j = j_1, \ldots, j_{k-1}$, we may assume wlog that the impact of forcing $u_{j_k 2}$ to 1 on the value of the maximum is measured by the number of variables u_{j1}, $j \in S$, newly forced to 0. But this is precisely the number $|\text{Adj}^*(j_k) \setminus \bigcup_{r=1}^{k-1} \text{Adj}(j_r)| = |\text{Adj}(j_k) \setminus \bigcup_{r=1}^{k-1} \text{Adj}(j_r)| + 1$, which is δ_{j_k} according to (7.1). Thus the value of the maximum is $|S| - 1 - \delta_{j_k}$, and hence the coefficient of $u_{j_k 2}$ is δ_{j_k}.

In the above discussion we have assumed that $j_1, \ldots, j_k$ all belong to the first component. Removing now this assumption, we see that nothing changes. If j_t is the first vertex of a new component, then forcing $u_{j_t 2}$ to 1 will reduce the value of the maximum by $|\text{Adj}(j_t)|$, since all the vertices adjacent to j_t belong to the new component, and everything in the sequel remains the same. This proves that (7.2) is valid for $\text{Proj}_S(P(G,b))$. Furthermore, (7.2) is also facet defining for this polytope, since at every step of the lifting procedure, the solution $u(\delta_{j_k})$ that maximizes the objective function, amended with $u_{j_k 2} = 1$, is independent of all the previous solutions (has a component $u_{j_k 2} = 1$ in a column in which all previous solutions had a coefficient $u_{j_k 2} = 0$).

We can now lift the inequality (7.2) from the subspace of the projection to the full space.

We claim that the lifting coefficients for u_{jk}, $j \in V \setminus S$, $k = 1, 2$, are all equal to 0, and that the lifted inequality obtained this way is facet defining for $P(G,b)$.

The inequality (7.2) is certainly valid for $P(G,b)$. Let $F = \{u \in P(G,b) : u_1(S) + \sum_{j \in S} \delta_j u_{j2} = |S| - 1\}$. Then F is a facet if and only if any equation $\alpha u = |S| - 1$ satisfied by all $u \in F$ has coefficients $\alpha_{j1} = 1$, $\alpha_{j2} = \delta_j$ for $j \in S$, and $\alpha_{jk} = 0$ for $j \in V \setminus S$, $k = 1, 2$. For the coefficient α_{jk}, $j \in S$, $k = 1, 2$, this follows from the fact, proved above, that (7.2) defines a facet of $\text{Proj}_S(P(G,b))$. As to the coefficients α_{jk} for $j \in V \setminus S$, a reasoning similar to that underlying Proposition 5.3 shows that $\alpha_{j1} = 0$ if there exists a separator C with shores A, B such that $j \in A$, and $C' = C \cup \{j\}$ is also a separator. Similarly, $\alpha_{j2} = 0$ if there exists a separator C with shores A, B such that $j \in B$, and $C' = C \cup \{j\}$ is also a separator. Now from the minimality of S and conditions (a), (b) of the theorem, $j \in P(i)$ for some $i \in S$. Let $\bar{u} \in F$ be such that $\bar{u}_{i1} = 0$, and define the separator C as having shores $A = \{\ell \in S : \bar{u}_{\ell 1} = 1\}$ and $B = \{\ell \in S : \bar{u}_{\ell 2} = 1\} \cup \{j\}$. C is clearly feasible, since j is not adjacent to any vertex in $S \setminus \{i\}$. But then $C' = C \cup \{j\}$ is also a separator, with shores $A' = A$ and $B' = B \setminus \{j\}$. This proves that $\alpha_{j2} = 0$ for all $j \in V \setminus S$. Now let $\hat{u} \in F$ be such that $\hat{u}_{i1} = 1$, and define the separator C as having shores $A := \{\ell \in S : \hat{u}_{\ell 1} = 1\} \cup \{j\}$ and $B = \{\ell \in S : \hat{u}_{\ell 2} = 1\}$. Clearly C is feasible, but $C' = C \cup \{j\}$ is also a separator, with shores $A' = A \setminus \{j\}$ and $B' = B$. Thus $\alpha_{j1} = 0$ for all $j \in V \setminus S$. □

8 Second Generalization

Now we turn to the second class of lifted inequalities. Let S be a minimal dominator of G free of self-dominators, and let $T := \{j \in V \setminus S : j \notin \cup_{i \in S} P(i)\}$. In other words, $T := \{j \in V \setminus S : |\text{Adj}(j) \cap S| \geq 2\}$. Consider the graph $G[S \cup T]$, and denote by $G(S,T)$ the bipartite subgraph obtained from $G[S \cup T]$ by deleting all vertices $j \in S \setminus \text{Adj}(T)$ and all edges (i,j) such that $\{i,j\} \subset S$ or $\{i,j\} \subset T$. Let $G(S_\ell, T_\ell)$, $\ell = 1, \ldots, k$ be the components of $G(S,T)$. In each component $\ell \in \{1, \ldots, k\}$, we construct a spanning tree $\mathcal{T}_\ell$ as follows.

Initialization. Choose some $v \in T_\ell$, set $v_1 := v$, and put v_1 into $\mathcal{T}_\ell$ as a marked vertex. Put into $\mathcal{T}_\ell$ all vertices in S_ℓ adjacent to v_1 and all edges joining these vertices to v_1.

Iterative Step k**.** Choose some $v \in T_\ell \setminus \{v_1, \ldots, v_{k-1}\}$ such that $\text{Adj}(v) \cap \text{Adj}(v_j) \cap S_\ell \neq \emptyset$ for some $j \in \{1, \ldots, k-1\}$ (i.e. v has a common neighbor with some marked vertex v_j, $j \in \{1, \ldots, k-1\}$), set $v_k := v$, and put v_k into $\mathcal{T}_\ell$ as a marked vertex by joining it through an edge to some arbitrarily chosen unmarked vertex of $\mathcal{T}_\ell$.

Put into $\mathcal{T}_\ell$ all vertices in $S_\ell \setminus T_\ell$ adjacent to v_k and all edges joining these vertices to v_k.

Stop when all vertices of $G(S_\ell, T_\ell)$ have been put into $\mathcal{T}_\ell$.

Clearly, the marked vertices of $\mathcal{T}_\ell$ are precisely those in T_ℓ, and they form an ordered set $\{v_1, \ldots, v_q\}$, where $q = |T_\ell|$. Furthermore, if $\text{Adj}(v)$ denotes the set of vertices adjacent to v in $G(S_\ell, T_\ell)$, then the degree in $\mathcal{T}_\ell$ of $v \in T_\ell$ is

$$\deg(v) = \begin{cases} |\text{Adj}(v_1)| & \text{if } v = v_1 \\ |\text{Adj}(v_i) \setminus \bigcup_{j=1}^{i-1} \text{Adj}(v_j)| + 1 & \text{if } v = v_i \text{ for some } i > 1. \end{cases}$$

The spanning tree $\mathcal{T}_\ell$ depends on the sequence in which the vertices $v_1, \ldots, v_q$ are selected for marking, and on the choice of the edge that joins the newly selected vertex to some unmarked vertex of $\mathcal{T}_\ell$, i.e. to some vertex of S_ℓ

Next we notice a remarkable property of the spanning trees $\mathcal{T}_\ell$.

Proposition 8.1. *Let $\mathcal{T}_\ell$ be a spanning tree of $G(S_\ell, T_\ell)$ constructed as above, let $v_1, \ldots, v_q$ be the associated sequence of vertices in T_ℓ, with $q = |T_\ell|$. Define*

$$\gamma(v) := \deg(v) - 1 \tag{8.1}$$

for all $v \in T_\ell$. Then for any contiguous subsequence of $\{v_1, \ldots, v_q\}$ starting with v_1, say $\{v_1, \ldots, v_r\}$, $r \leq q$, we have

$$\sum_{i=1}^{r} \gamma(v_i) = |\mathrm{Adj}(\{v_1, \ldots, v_r\})| - 1. \tag{8.2}$$

Proof. By induction. For $r = 1$ (8.2) holds by definition. Suppose (8.2) holds for $r = 1, \ldots, t-1$, and let $r = t \leq q$. Then

$$\begin{aligned} \sum_{i=1}^{t} \gamma(v_i) &= \sum_{i=1}^{t-1} \gamma(v_i) + \gamma(v_t) \\ &= |\mathrm{Adj}(\{v_1, \ldots, v_{t-1}\})| - 1 + |\mathrm{Adj}(v_t) \setminus \bigcup_{j=1}^{t-1} \mathrm{Adj}(v_j)| \\ &= |\mathrm{Adj}(\{v_1, \ldots, v_t\})| - 1. \end{aligned}$$

□

Figure 6 shows an instance of a bipartite graph $G(S_\ell, T_\ell)$, along with two spanning trees corresponding to different orderings of the vertices of T_ℓ, and the associated numbers γ.

Notice that in the spanning tree $\mathcal{T}_\ell^1$, $\sum_{i=1}^{2} \gamma(i) = \sum_{i=1}^{3} \gamma(i) = 5$ and both $|\mathrm{Adj}(\{1,2\})|$ and $|\mathrm{Adj}(\{1,2,3\})|$ are equal to 6, as required by (8.2). Also, $\sum_{i=1}^{4} \gamma(i) = \sum_{i=1}^{8} \gamma(i) = 6$ and $|\mathrm{Adj}(\{1, \ldots, 4\})| = |\mathrm{Adj}(\{1, \ldots, 8\})| = 7$

In $\mathcal{T}_\ell^2$,

$$\begin{aligned} &\sum_{i=1}^{3} \gamma(i) = 3 && \text{and} \quad |\mathrm{Adj}(\{1,2,3\})| = 4, \\ &\sum_{i=1}^{4} \gamma(i) = 4 && \text{and} \quad |\mathrm{Adj}(\{1, \ldots, 4\})| = 5, \\ &\sum_{i=1}^{5} \gamma(i) = \sum_{i=1}^{6} \gamma(v_i) = 5 && \text{and} \quad |\mathrm{Adj}(\{1, \ldots, 5\})| = |\mathrm{Adj}(\{1, \ldots, 6\})| = 6 \end{aligned}$$

In all of these cases (8.2) is satisfied. On the other hand, subsets of T_ℓ that do not represent a contiguous subsequence of $\{v_1, \ldots, v_q\}$, or do not contain v_1, may violate (8.2). In the spanning tree $\mathcal{T}_\ell^1$, for instance, $\gamma(2) + \gamma(4) = 4$, but $|\mathrm{Adj}(\{2,4\})| = 6$. Also, in $\mathcal{T}_\ell^2$, $\gamma(6) + \gamma(7) = 2$, but $|\mathrm{Adj}(\{6,7\})| = 4$.

We are now ready to state our second lifting theorem. In order for the lifted inequality to be facet defining for $P(G, b)$, certain conditions need to be satisfied. Without these conditions, the lifted inequality is still valid for $P(G, b)$, but it may not be facet defining.

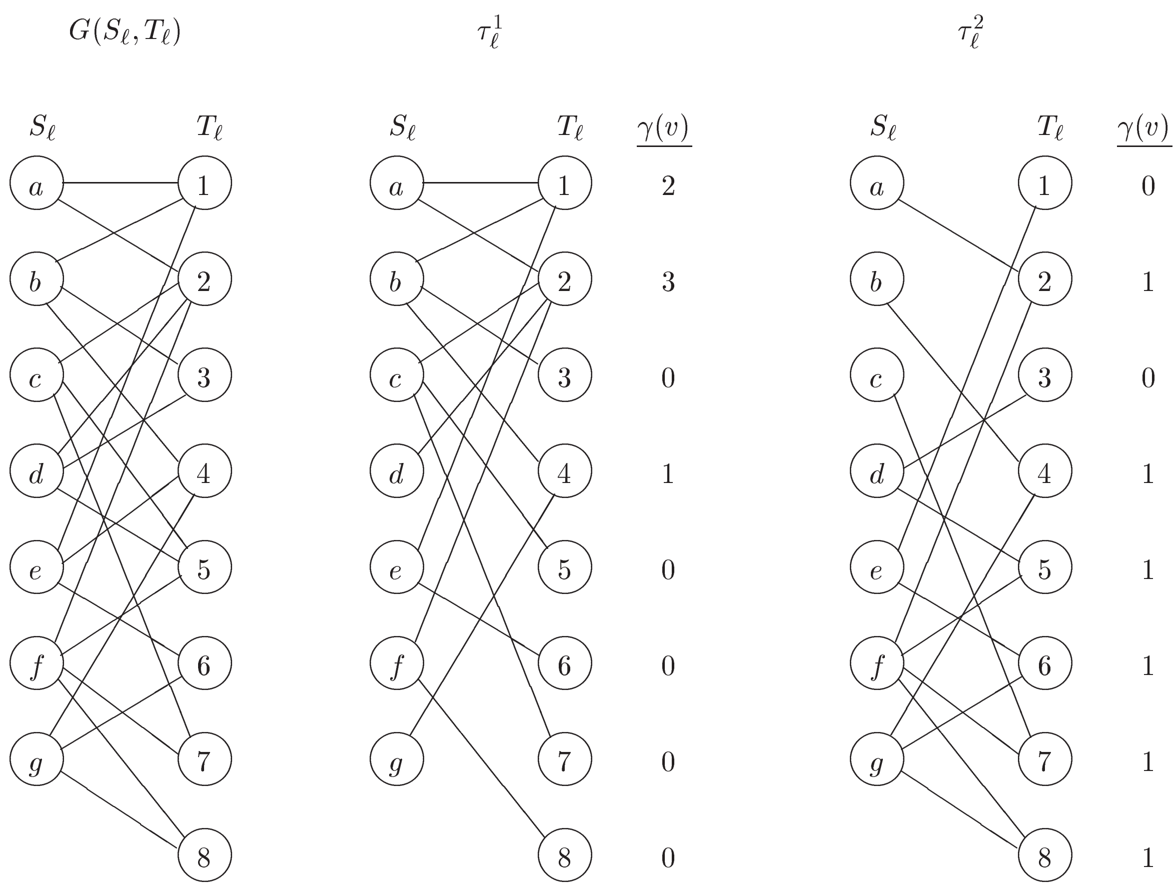


Figure 6: An instance of $G(S_\ell, T_\ell)$, with two spanning trees $\mathcal{T}_\ell^1, \mathcal{T}_\ell^2$ and associated numbers $\gamma(v)$. The ordering of the vertices of T_ℓ is $(1, \ldots, 8)$ for $\mathcal{T}_\ell^1$, and $(8, \ldots, 1)$ for $\mathcal{T}_\ell^2$.

Theorem 8.2. *Let S be a minimal dominator of G not containing any self-dominator, let*

$$T := \{j \in V \setminus S : |\mathrm{Adj}(j) \cap S| \geq 2\} \neq \emptyset,$$

and suppose the following conditions are satisfied:

(a) for every $j \in P(i)$, $i \in S$, there exists $\ell \in P(k)$, $k \in S$, such that $\ell \notin \mathrm{Adj}(j)$.

(b) for every $j \in P(i)$ with $i \in S \setminus \mathrm{Adj}(T)$, there exists $\ell \in P(k)$ with $k \in S \cap \mathrm{Adj}(T)$, such that $\ell \notin \mathrm{Adj}(T \cup \{j\})$.

(c) if $S \subseteq \mathrm{Adj}(T)$, then for every $j \in P(i)$, $i \in S$, there exists $\ell \in \cup_{i \in S} P(i) \setminus \mathrm{Adj}(T)$ such that $\ell \notin \mathrm{Adj}(j)$.

For $j \in T$, let $\gamma_j = \gamma(j)$ be defined as in (8.1), and for $j \in S^* := S \setminus \mathrm{Adj}(T)$, let $\delta_j = \delta(j)$ be defined as in (7.1), with S^* substituted for S. Then the inequality

$$u_1(S) + \sum_{j \in T} \gamma_j u_{j2} + \sum_{j \in S \setminus Adj(T)} \delta_j u_{j2} \leq |S| - 1 \tag{8.3}$$

is valid and facet defining for $P(G, b)$.

Proof. Since the inequality $u_1(S) \leq |S| - 1$ is not valid for $P(G, b)$ restricted to the space of u_{j1}, $j \in S$, we use projection to obtain a polytope for which it is valid. Consider the projection of $P(G, b)$ onto the subspace of the variables indexed by $S \cup T$:

$$\mathrm{Proj}_{S \cup T}(P(G, b)) := \{u^{S \cup T} : u \in P(G, b) \text{ for some } u^{V \setminus (S \cup T)}\}.$$

Clearly, $u_1(S) \leq |S| - 1$ is valid for P^{**} defined as

$$P^{**} := \{u^{S \cup T} \in \mathrm{Proj}_{S \cup T}(P(G, b)) : u_{j1} = 0,\ j \in T,\ u_{j2} = 0,\ j \in S \cup T\}.$$

Furthermore, $u_1(S) \leq |S| - 1$ is facet defining for P^{**}. Indeed, for each $i \in S$, the point $u^{S \cup T}$ defined by $u_{j1} = 1$, $j \in S \setminus \{i\}$, $u_{j1} = 0$, $j \in T \cup \{i\}$, $u_{j2} = 0$, $j \in S \cup T$, is in P^{**} and satisfies $u_1(S) = |S| - 1$. To see that this point is in P^{**}, notice that the separator C with shores $A = S \setminus \{i\}$ and $B = P(i)$ is feasible. Since the $|S|$ points defined this way, one for each $i \in S$, are clearly affinely independent, $u_1(S) \leq |S| - 1$ defines a facet of P^{**}.

We will start the lifting with the coefficients of the variables u_{j2}, $j \in T$. Let $j_1, \ldots, j_t$ be any ordering of T such that, within each component of $G(S, T)$, the sequence of indices is the one given by the spanning tree $\mathcal{T}_\ell$ used to define the coefficients γ. The sequence of the components of $G(S, T)$ themselves is immaterial.

To calculate the coefficient of $u_{j_1 2}$, we solve

$$\max\{u_1(S) : u^{S \cup T} \in \mathrm{Proj}_{S \cup T}(P(G, b)),\ u_{j_1 2} = 1,\ u_{j2} = 0,\ j \in S \cup T \setminus \{j_1\},\ u_{j1} = 0,\ j \in T\}$$

and find the value of the maximum to be $|S| - |\mathrm{Adj}(j_1)|$, since setting $u_{j_1 2} = 1$ forces to 0 all the variables u_{j1} for $j \in \mathrm{Adj}(j_1)$. Here $\mathrm{Adj}(j_1)$ stands for the set of vertices of S adjacent to $j_1 \in T$ in $G(S, T)$. Thus the coefficient of $u_{j_1 2}$, which is equal to the difference between the maximum of $u_1(S)$ with or without $u_{j_1 2}$ set to 1, is $|\mathrm{Adj}(j_1)| - 1$, which is precisely the coefficient $\gamma_{j_1} = \gamma(j_1)$ given by (8.1). Also, the solution $u(\gamma_{j_1})$ yielding the maximum, namely $u_{j_1 1} = 0$, $u_{j1} = 0$, $j \in \mathrm{Adj}(j_1)$, $u_{j1} = 1$, $j \in S \setminus \mathrm{Adj}(j_1)$, amended with $u_{j_1 2} = 1$, satisfies $u_1(S) + \gamma_{j_1} u_{j_1 2} \le |S| - 1$ at equality. This solution is obviously feasible, the associated separator C having shores $A = S \setminus \mathrm{Adj}(j_1)$, $B = \{j_1\}$.

Assume now that these properties hold for $j = j_1, \ldots, j_{k-1}$, and let $j = j_k$. We then have to solve

$$\max\{u_1(S) + \sum_{j=j_1}^{j_{k-1}} \gamma_j u_{j2} : u^{S\cup T} \in \mathrm{Proj}_{S\cup T}(P(G,b)), u_{j_k 2} = 1, u_{j2} = 0,$$
$$j \in S \cup T \setminus \{j_1, \ldots, j_k\}, u_{j1} = 0, j \in T\}.$$

Assume first that $j_1, \ldots, j_k$ belong to the same component of $G(S,T)$. Without setting $u_{j_k 2}$ to 1, the maximum of the above expression is $|S| - 1$, and it is attained for a solution $u(\gamma_{j_{k-1}})$ in which $u_{j2} = 1$ for $j = j_1, \ldots, j_{k-1}$, $u_{j1} = 0$ for $j \in \bigcup_{r=1}^{k-1} \mathrm{Adj}(j_r)$, and $u_{j1} = 1$ for $j \in S \setminus \bigcup_{r=1}^{k-1} \mathrm{Adj}(j_r)$. This solution, which satisfies $u_1(S) + \sum_{j=j_1}^{j_{k-1}} \gamma_j u_{j2} \le |S| - 1$ at equality, corresponds to a separator C with shores $A = S \setminus \bigcup_{r=1}^{k-1} \mathrm{Adj}(j_r)$, $B = j_1, \ldots, j_{k-1}$. Should the set assigned to A be empty (which may happen at the last step), we set $A = \{\ell\}$ for some $\ell \in P(k)$ with $k \in S \cap \mathrm{Adj}(T)$ such that $\ell \notin \mathrm{Adj}(T)$, whose existence follows from condition (b) of the Theorem. Now setting $u_{j_k 2} = 1$ forces to 0 all variables u_{j1} such that $j \in \mathrm{Adj}(j_k) \setminus \bigcup_{r=1}^{k-1} \mathrm{Adj}(j_r)$. Hence the value by which $|S| - 1$ is reduced, is precisely $|\mathrm{Adj}(j_k) \setminus \bigcup_{r=1}^{k-1} \mathrm{Adj}(j_r)| = \gamma_{j_k}$. This completes the induction.

Here we have assumed that the vertices $j_1, \ldots, j_t$ belong to the same component. Throwing out this assumption does not change anything, since the first vertex j of a new component has its coefficient γ_j defined in a way that takes this situation into account.

Next we lift the coefficients of the variables u_{j2}, $j \in S$. Defining $S^* := S \setminus \mathrm{Adj}(T)$ and letting $S_1^*, \ldots, S_k^*$ be the vertex sets of the components of $G[S^*]$, we order the vertices of each component and define the coefficients $\delta_j = \delta(j)$ for all $j \in S^*$ as in Theorem 7.1. These are valid lifting coefficients for our inequality, since variables $u_{\ell 2}$, $\ell \in S^*$, are not affected by the values of u_{j2}, $j \in T$. Thus if S^* is ordered as $\ell_1, \ldots, \ell_{|S^*|}$, we start by solving $\max\{u_1(S) + \sum_{j\in T} \gamma_j u_{j2} : u^{S\cup T} \in \mathrm{Proj}_{S\cup T}(P(G,b)), u_{\ell_1 2} = 1, u_{\ell 2} = 0, \ell \in S \setminus \{\ell_1\}, u_{j1} = 0, j \in T\}$, and obtain the coefficient $\delta_{\ell_1} = |\mathrm{Adj}(\ell_1)|$, where adjacency refers to S^*. At the k-th step we solve

$$\max\{u_1(S) + \sum_{j\in T} \gamma_j u_{j2} + \sum_{\ell=\ell_1}^{\ell_{k-1}} \delta_\ell u_{\ell 2} : u^{S\cup T} \in \mathrm{Proj}_{S\cup T}(P(G,b)), u_{\ell_k 2} = 1,$$
$$u_{\ell 2} = 0, \ell \in S \setminus \{\ell_1, \ldots, \ell_k\}, u_{j1} = 0, j \in T\}$$

and obtain δ_{ℓ_k} as the coefficient of $u_{\ell_k 2}$. The solution yielding this value corresponds to a separator C with shores $B = T \cup \{\ell_1, \ldots, \ell_{k-1}\}$, $A = S^* \setminus \mathrm{Adj}\{\ell_1, \ldots, \ell_{k-1}\}$. Should the set assigned to shore A be empty, which may occur at the last step, we set $A = \{\ell\}$ for some $\ell \in \bigcup_{i \in S \setminus S^*} P(i) \setminus \mathrm{Adj}(T)$, which always exists by condition (b). It is not hard to see that the solution defined this way satisfies $u_1(S) + \sum_{j \in T} \gamma_j u_{j2} + \sum_{\ell=\ell_1}^{\ell_{k-1}} \delta_\ell u_{\ell 2} \leq |S| - 1$ at equality. At the end of this procedure, we obtain inequality (8.3).

We may now continue the lifting procedure for the coefficients u_{j2}, $j \in S \cap \mathrm{Adj}(T)$, but it is obvious that these coefficients are all equal to 0, irrespective of the order in which they are lifted. This is so because setting to 1 any number of variables u_{j2}, $j \in S \cap \mathrm{Adj}(T)$ does not force to 0 any new variables u_{j1}, $j \in S$, beyond those already forced to 0 by the previous liftings, and thus cannot reduce the value of the maximand from $|S| - 1$. Hence $\alpha_{j2} = 0$ for all $j \in S \cap \mathrm{Adj}(T)$ irrespective of the order of lifting. Similarly, lifting the coefficients α_{j1} for $j \in T$ in whatever sequence yields $\alpha_{j1} = 0$, $j \in T$, since setting any of these variables to 1 does not force to 0 any of the variables u_{j1}, $j \in S$.

We have lifted the inequality $u_1(S) \leq |S| - 1$ to the space of $\mathrm{Proj}_{S \cup T}(P(G, b))$ and shown that the resulting inequality (8.3) is valid for that polytope. Moreover, (8.3) is also facet defining for $\mathrm{Proj}_{S \cup T}(P(G, b))$, since at every step of the lifting procedure, the solution that maximizes the objective function, amended with the variable fixed at 1, is independent of all the previous solutions as it has a component equal to 1 in a position where all earlier solutions had a component equal to 0.

We now lift the inequality (8.3) from the subspace of the projection to the full space. Our claim is that the lifting coefficients for u_{j1}, $j \in V \setminus (S \cup T)$, are all equal to 0, and that the inequality (8.3) obtained in this way is facet defining for $P(G, b)$.

The validity of (8.3) for $P(G, b)$ is obvious. If we denote $F := \{u \in P(G, b) : u_1(S) + \sum_{j \in T} \gamma_j u_{j2} + \sum_{j \in S \setminus Adj(T)} \delta_j u_{j2} = |S| - 1\}$, then F is a facet if and only if any equation $\alpha u = |S| - 1$ satisfied by all $u \in F$ has coefficients equal to those of (8.3). As far as the coefficients α_{jk} for $j \in S \cup T$, $k = 1, 2$ are concerned, this condition is satisfied, since (8.3) defines a facet of $\mathrm{Proj}_{S \cup T}(P(G, b))$. For the coefficients α_{jk}, $j \in V \setminus (S \cup T)$, notice that each such j belongs to some pendent set, say $P(i)$, since S is a minimal dominator that contains no self-dominators. Thus $V \setminus (S \cup T) = \bigcup_{i \in S} P(i)$. For α_{j2}, consider three cases. Case 1: $S \setminus \mathrm{Adj}(T) \neq \emptyset$, and $j \in P(i)$ such that $i \in \mathrm{Adj}(T)$. Then there is a feasible separator C in F with shores $B = T \cup \{j\}$ and $A = S \setminus \mathrm{Adj}(T)$, such that $C' = C \cup \{j\}$ is also a feasible separator, with shores $A' = A$ and $B' = B \setminus \{j\}$; thus $\alpha_{j2} = 0$. Case 2: $S \setminus \mathrm{Adj}(T) \neq \emptyset$ and $j \in P(i)$ such that $i \in S \setminus \mathrm{Adj}(T)$. Consider the separator C with shores $B = T \cup \{j\}$, $A = \{\ell\} \cup (S \setminus \mathrm{Adj}(T))$ for some $\ell \in P(k)$ with $k \in S \cap \mathrm{Adj}(T)$ such that $\ell \notin \mathrm{Adj}(T \cup \{j\})$ (the existence of such ℓ is guaranteed by condition (b) of the Theorem). Clearly, C is in F, and $C' = C \cup \{j\}$ is also a feasible separator, with shores $A' = A$ and $B' = B \setminus \{j\}$; thus $\alpha_{j2} = 0$. Case 3: $S \setminus \mathrm{Adj}(T) = \emptyset$. In this case there is a separator C in F with shores $B = T \cup \{j\}$ and $A = \{\ell\}$, where $\ell \in \cup_{i \in S} P(i) \setminus \mathrm{Adj}(T \cup \{j\})$ (the existence of such ℓ follows from condition (c) of the Theorem). Again, $C' = C \cup \{j\}$, with shores $A' = A$, $B' = B \setminus \{j\}$, is also a feasible separator in F; hence $\alpha_{j2} = 0$ in this case too.

For α_{j1}, if $j \in P(i)$, consider the separator C with $A = (S \setminus \{k\}) \cup \{j\}$ with $k \neq i$, and $B = \{\ell\}$ for some $\ell \in P(k)$ such that $\ell \notin \mathrm{Adj}(j)$. The existence of such k, ℓ is guaranteed by condition (a) of the Theorem. Clearly, C is in F. But $C' = C \cup \{j\}$ is also a separator in F, with shores $A' = A \setminus \{j\}$ and $B' = B$, which proves that $\alpha_{j1} = 0$.

We have thus proved that given the conditions of the Theorem, the inequality (8.3) is facet defining for $P(G, b)$. □

9 Third Generalization

The inequalities (7.1) and (8.3) were derived from (6.1) by sequential lifting. Next we generalize the inequality (6.1) in another direction. Consider an arbitrary vertex set S.

Theorem 9.1. *Let $S \subset V$, $2 \leq |S| \leq b$, and let $W(S) := V \setminus (S \cup Adj(S))$. The inequality*

$$u_1(S) - u_2(W(S)) \leq |S| - 1 \tag{9.1}$$

is valid for $P(G, b)$. Moreover, (9.1) is facet defining for $P(G, b)$ if and only if

(a) S is independent

(b) S is a minimal dominator of $\mathrm{Adj}(S)$

(c) Every $v \in$ $\mathrm{Adj}(S)$ *is adjacent to exactly one vertex in S.*

Proof. (i) (9.1) is valid. Let $\bar{u} \in \{0,1\}^{2n}$ be such that $\bar{u}_1(S) - \bar{u}_2(W(S)) > |S| - 1$. Then $\bar{u}_{j1} = 1$ for $j \in S$ and $\bar{u}_{j2} = 0$ for $j \in W(S)$, i.e. $\bar{u}$ corresponds to a separator C whose shore A contains all vertices in S, and whose shore B contains none of the vertices in $W(S)$. But since $V \setminus (S \cup W(S)) = \mathrm{Adj}(S)$, there are no vertices left for B, a contradiction.

(ii) Conditions (a), (b) and (c) for (9.1) to be facet defining. Let $F := \{u : u_1(S) - u_2(W(S)) = |S| - 1\}$.

Necessity. If S is not independent, there exists $(i, j) \in E$ with $\{i, j\} \subseteq S$. But this together with $u \in F$ implies $u_{i2} = u_{j2} = 0$, since otherwise $u_1(S) \leq |S| - 2$. Thus (a) is necessary. Further, if S is not a minimal dominator of $\mathrm{Adj}(S)$, then it contains a vertex $i \in S$ such that $\mathrm{Adj}(S \setminus \{i\}) = \mathrm{Adj}(S)$. But then for any $u \in F$, $u_{i1} + u_{i2} = 1$, i.e. i cannot belong to the separator. Indeed, if $i \in C$, then $|A| = u_1(S) \leq |S| - 1$. But for $u \in F$ we need $u_1(S) = |S| - 1$ and $u_2(W(S)) = 0$, which would imply $B \subseteq \mathrm{Adj}(S \setminus \{i\})$, a contradiction. Since every $u \in F$ satisfies $u_{i1} + u_{i2} = 1$, F is not a facet. Thus (b) is necessary. Now suppose some $j \in \mathrm{Adj}(S)$ is adjacent to both $i \in S$ and $k \in S$, $i \neq k$. Then $u_{i1} + u_{j2} \leq 1$, $u_{k1} + u_{j2} \leq 1$, and since $u \in F$ implies $u_{i1} + u_{k1} \geq 1$, it follows that $u_{j2} = 0$ for all $u \in F$, i.e. F is not a facet. Therefore, (c) is necessary.

Sufficiency. We show that any equation $\alpha u = |S| - 1$ satisfied by every $u \in F$ must have coefficients

$$\alpha_{j1} = \begin{cases} 1 & j \in S \\ 0 & j \in V \setminus S \end{cases} \qquad \alpha_{j2} = \begin{cases} -1 & j \in W(S) \\ 0 & j \in V \setminus W(S). \end{cases}$$

First, consider $j \in \text{Adj}(S)$. From condition (c), j is adjacent to exactly one vertex in S, say i. Consider the separator C with shores $A = S \setminus \{i\}$, $B = \{i, j\}$. Since S is independent (condition a), this separator is feasible. But $C' = C \cup \{j\}$ is also a feasible separator, with shores $A' = A$ and $B' = B \setminus \{j\}$; hence $\alpha_{j2} = 0$ for all $j \in \text{Adj}(S)$. On the other hand, consider the separator C with shores $A = (S \setminus \{k\}) \cup \{j\}$ for some $k \notin \text{Adj}(j)$, and $B = \{k\}$. Again, C is feasible, and so is $C' = C \cup \{j\}$, with shores $A' = A \setminus \{j\}$, $B' = B$. Hence $\alpha_{j1} = 0$ for all $j \in \text{Adj}(S)$.

Consider next the coefficients α_{j1}, $j \in W(S)$. For each such j, the separator C with shores $A = (S \setminus \{i\}) \cup \{j\}$ for some $i \in S$, and $B = \{i\}$ is feasible; but so is $C' = C \cup \{j\}$, with shores $A' = A \setminus \{j\}$, $B' = B$; hence $\alpha_{j1} = 0$ for all $j \in W(S)$. Further, consider the coefficients α_{j2} for $j \in S$. From condition (b), there exists some $\ell \in \text{Adj}(S)$ such that $\{j\} = S \cap \text{Adj}(\ell)$. Thus the separator C with shores $A = S \setminus \{j\}$ and $B = \{j, \ell\}$ is feasible; but so is $C' = C \cup \{j\}$, with shores $A' = A$ and $B' = B \setminus \{j\}$. Hence $\alpha_{j2} = 0$ for all $j \in S$.

Now setting $\alpha_{j1} = 0$ for all $j \in V \setminus S$ and $\alpha_{j2} = 0$ for all $j \in V \setminus W(S)$, we are left with a system of equations in the space of the α_{j1}, $j \in S$ and α_{j2}, $j \in W(S)$,

$$\sum_{j \in S} \alpha_{j1} u_{j1} - \sum_{j \in W(S)} \alpha_{j2} u_{j2} = |S| - 1$$

which has to be satisfied for all $u \in F$. The following is a list of $|S \cup W(S)|$ affinely independent points u in $F_{S \cup W(S)}$, the projection of F onto the subspace of $\{u_{j1}, j \in S\} \cup \{u_{j2}, j \in W(S)\}$, whose unique solution is the required $\alpha_{j1} = 1$, $j \in S$, $\alpha_{j2} = -1$, $j \in W(S)$:

$$\begin{array}{lll} u^i_{j1} = \begin{cases} 1 & j \in S \setminus \{i\} \\ 0 & j = i \end{cases} & u^i_{j2} = 0, j \in W(S), & i \in S \\ u^i_{j1} = 1 \quad j \in S & u^i_{j2} = \begin{cases} -1 & j = i \\ 0 & j \in W(S) \setminus \{i\} \end{cases} & i \in W(S). \end{array}$$

This proves the sufficiency of the conditions in the Theorem. □

Corollary 9.2. *Let S be a maximal set satisfying the conditions of Theorem 9.1, i.e. such that there exists no $T \supsetneqq S$ satisfying them. Then for every $S' \subseteq S$, $|S'| \geq 2$, the inequality (9.1), with S' substituted for S, is valid and facet defining for $P(G, b)$.*

Proof. If S satisfies the conditions of Theorem 9.1, so does every $S' \subseteq S$, $|S'| \geq 2$. □

Notice that the inequality (6.1) is a special case of (9.1).

Example. Consider the graph G of Figure 7, obtained from the Petersen graph by subdividing the edges of the outer 5-cycle. The set $S = \{1, 4, 7, 15\}$ satisfies the conditions of Theorem 9.1 for any $b \geq 4$, with $S \cup \text{Adj}(S) = V$. Consequently, the inequality

$$u_{11} + u_{41} + u_{71} + u_{15,1} \leq 3$$

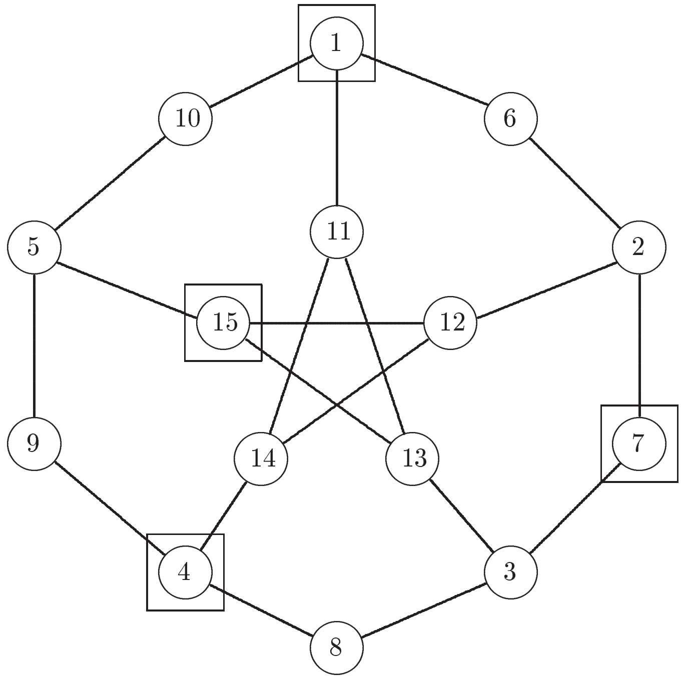


Figure 7: Graph illustrating the example.

is valid and facet defining for $P(G, b)$. From Corollary 9.2, the following inequalities are also valid and facet defining for $P(G, b)$:

$$u_{11} + u_{41} + u_{71} - u_{52} - u_{12,2} - u_{13,2} - u_{15,2} \leq 2$$
$$u_{11} + u_{41} + u_{15,1} - u_{22} - u_{32} - u_{72} \leq 2$$
$$u_{11} + u_{71} + u_{15,1} - u_{42} - u_{82} - u_{92} - u_{14,2} \leq 2$$
$$u_{41} + u_{71} + u_{15,1} - u_{12} - u_{62} - u_{10,2} - u_{11,2} \leq 2$$
$$u_{11} + u_{41} - u_{22} - u_{32} - u_{52} - u_{72} - u_{12,2} - u_{13,2} \leq 1$$
$$u_{11} + u_{71} - \sum(u_{j2} : j \notin \{1, 7\} \cup \text{Adj}(\{1, 7\})) \leq 1$$
$$u_{11} + u_{15,1} - \sum(u_{j2} : j \notin \{1, 15\} \cup \text{Adj}(\{1, 15\})) \leq 1$$
$$u_{41} + u_{71} - \sum(u_{j2} : j \notin \{4, 7\} \cup \text{Adj}(\{4, 7\})) \leq 1$$
$$u_{41} + u_{15,1} - \sum(u_{j2} : j \notin \{4, 15\} \cup \text{Adj}(\{4, 15\})) \leq 1$$
$$u_{71} + u_{15,1} - \sum(u_{j2} : j \notin \{7, 15\} \cup \text{Adj}(\{7, 15\})) \leq 1.$$

10 Concluding Remarks

We have given a mixed integer programming formulation of the VS problem, and a partial polyhedral description of the convex hull $P(G)$ of feasible points. In the process, we have identified several classes of valid inequalities and derived conditions under which they are facet defining, sometimes using novel proof techniques.

In a companion paper [1], we describe a branch-and-cut algorithm for the VSP, using several of the inequalities developed here, based on efficient separation routines and bounding heuristics. The algorithm was tested on a large variety of VSP instances. One of the highlights of our computational experiments is the major role of cut density (as distinct from cut strength) in the overall efficiency of the algorithm.

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