INSTITUTO DE COMPUTAÇÃO
UNIVERSIDADE ESTADUAL DE CAMPINAS

Pfaffian Graphs

Alberto Alexandre Assis Miranda
Cláudio Leonardo Lucchesi

Technical Report - IC-04-003 - Relatório Técnico

April - 2004 - Abril

Pfaffian Graphs

Alberto Alexandre Assis Miranda   Cláudio Leonardo Lucchesii

26th April 2004

Abstract

It is well known that, in general, the problem of determining the number of perfect matchings of a graph is NP-hard. Some graphs, called *Pfaffian*, have a special type of orientation that is also called *Pfaffian*. Given a Pfaffian orientation of a graph G, the number of perfect matchings of G may be evaluated in polynomial time. Two (not necessarily Pfaffian) orientations are *similar* if they differ by a cut. Similarity is an equivalence relation on the collection of orientations of a graph. If an orientation is Pfaffian then every similar orientation is also Pfaffian.

For any Pfaffian matching covered graph G, the number of similarity classes of Pfaffian orientations of G is equal to 2^b, where b denotes the number of bricks of the tight cut decomposition of G.

The following four problems are polynomially equivalent, that is, given a polynomial algorithm for any of the four problems it is then possible to find polynomial algorithms for the other three problems: (i) determine whether or not a graph is Pfaffian, (ii) determine whether or not a given orientation of a graph is Pfaffian, (iii) determine whether or not a graph is Pfaffian, in case it is Pfaffian, find a Pfaffian orientation for the graph, and (iv) determine the number of similarity classes of Pfaffian orientations of a graph.

Contents

1 Pre-Requisites from Linear Algebra

1.1 Permutations

For any positive integer n, we denote by Π_n the set of permutations on $\{1, 2, \cdots, n\}$. Let π be a permutation in Π_n. Quite often we describe π by the n-tuple $(\pi(1), \pi(2), \cdots, \pi(n))$. An *inversion* of π is a pair (i, j) such that $i < j$ and $\pi(i) > \pi(j)$. The *parity* of π is defined to be equal to the parity of the number of inversions of π. The *sign* of π is defined to be

$$\mathrm{sgn}(\pi) := \begin{cases} 1, & \text{if } \pi \text{ is even,} \\ -1, & \text{otherwise} \end{cases}$$

The *interchange* of i and j in π, where $1 \leq i < j \leq n$, is the permutation π' such that

$$\pi'(k) = \begin{cases} \pi(j), & \text{if } k = i \\ \pi(i), & \text{if } k = j \\ \pi(k), & \text{otherwise} \end{cases}$$

The following two properties of permutations are well-known and will be used throughout this text without explicit mention: (i) the operation of interchange alters the parity of a permutation, and (ii) the sign of the permutation $\pi_1 \cdot \pi_2$, obtained by the composition of permutations π_1 and π_2, satisfies the equality $\mathrm{sgn}(\pi_1 \cdot \pi_2) = \mathrm{sgn}(\pi_1) \cdot \mathrm{sgn}(\pi_2)$.

A directed graph, D, *represents a permutation* in Π_n if $V(D) = \{1, 2, \cdots, n\}$ and each vertex has both its indegree and outdegree equal to one. We denote by $\mathcal{G}_n$ the set of directed graphs that represent permutations in Π_n. There is a natural bijection relating $\mathcal{G}_n$ and Π_n. For any graph G in $\mathcal{G}_n$, the corresponding permutation in Π_n, denoted π_G, is defined to be the permutation that associates with each vertex i the integer $\pi_G(i) = j$, where j is the head of the edge of G whose tail is equal to i. Conversely, for any permutation π in Π_n, the corresponding directed graph in $\mathcal{G}_n$, denoted G_π, is defined so that for each vertex i, there exists precisely one edge in G_π whose tail is equal to i; moreover, the head of that edge is equal to $\pi(i)$. Figure 1 illustrates the graph G_π, where $\pi = (2, 3, 1, 6, 5, 4)$.

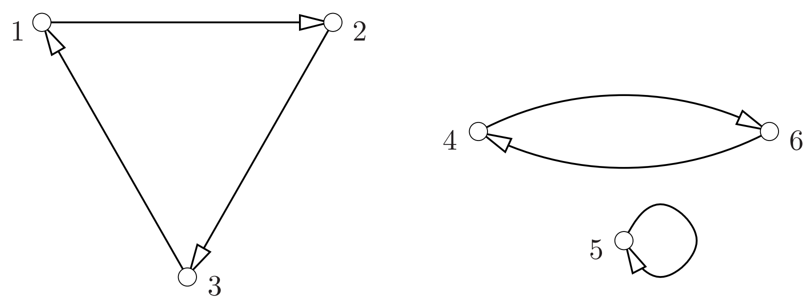

Figure 1: The graph in $\mathcal{G}_6$ associated with permutation $(2, 3, 1, 6, 5, 4)$.

The following elementary result will be used in the proof of a fundamental result involving Pfaffians and determinants of skew symmetric matrices:

Lemma 1.1
Let π be a permutation in Π_n, C a directed circuit in the associated graph G_π, G' the graph in $\mathcal{G}_n$ obtained from G_π by reversing the edges of C, π' the corresponding permutation in Π_n. Then, $\operatorname{sgn}(\pi') = \operatorname{sgn}(\pi)$.

Proof: By induction on the length r of C. If $r = 1$ then $G' = G_\pi$ and $\pi' = \pi$, whence the assertion holds immediately. We may thus assume that $r > 1$. Let $(v_0, v_1, \cdots, v_{r-1})$ denote the vertices of C, so that $v_{i+1} = \pi(v_i)$, for $i = 0, \cdots r - 1$, and the indices are computed modulo r. Let π_1 denote the permutation obtained from π by interchanging v_{r-2} and v_{r-1}. Likewise, let π'_1 denote the permutation obtained from π' by interchanging v_0 and v_{r-1}. Then,

$$\operatorname{sgn}(\pi_1) = -\operatorname{sgn}(\pi) \text{ and } \operatorname{sgn}(\pi'_1) = -\operatorname{sgn}(\pi').$$

The graph G_1 that corresponds to π_1 may then be obtained from G_π by replacing circuit C by directed circuit $C_1 := (v_0, v_1, \cdots, v_{r-2})$ and loop (v_{r-1}). Likewise, the graph G'_1 that corresponds to permutation π'_1 may be obtained from G' by replacing the reversal of circuit C by the reversal of circuit C_1 and loop (v_{r-1}). But G'_1 is the graph obtained from G_1 by the reversal of circuit C_1, of length $r - 1$. By induction hypothesis, $\operatorname{sgn}(\pi'_1) = \operatorname{sgn}(\pi_1)$. We conclude that

$$\operatorname{sgn}(\pi') = -\operatorname{sgn}(\pi'_1) = -\operatorname{sgn}(\pi_1) = \operatorname{sgn}(\pi),$$

and the assertion holds. □

1.2 Permanents, Determinants and Pfaffians

Let $A := (a_{ij})_{n\times n}$ be a square, $n \times n$ matrix, whose entries lie in $\Re$. Matrix A is *skew symmetric* if $a_{ij} = -a_{ji}$ for every pair i, j of indices, $1 \le i, j \le n$. For each permutation π in Π_n, let

$$\langle a_\pi \rangle := a_{1\pi(1)}\, a_{2\pi(2)} \cdots a_{n\pi(n)} \qquad \text{and} \qquad a_\pi := \operatorname{sgn}(\pi)\, \langle a_\pi \rangle.$$

The *permanent* $\operatorname{prm}(A)$ of A and the *determinant* $\det(A)$ of A are defined as follows:

$$\operatorname{prm}(A) := \sum_{\pi\in\Pi_n} \langle a_\pi \rangle \qquad \text{and} \qquad \det(A) := \sum_{\pi\in\Pi_n} a_\pi.$$

It is well known that the determinant of an $n \times n$ matrix may be evaluated in time $O(n^3)$. Valiant [10] has shown that the problem of determining the value of the permanent of a matrix of 0's and 1's is NP-hard.

Let us now assume further that A is skew symmetric. Denote by $\mathcal{P}_n$ the set of partitions of $\{1, 2, \cdots, n\}$ in pairs. Clearly, if n is odd then $\mathcal{P}_n$ is empty. Let

$$P := \{\{i_1, i_2\}, \{i_3, i_4\}, \cdots, \{i_{n-1}, i_n\}\}$$

be a partition in $\mathcal{P}_n$. Let

$$\pi_P := (i_1, i_2, \cdots, i_{n-1}, i_n)$$

and

$$\langle a_P \rangle := a_{i_1 i_2} a_{i_3 i_4} ... a_{i_{n-1} i_n}.$$

Observe that π_P and $\langle a_P \rangle$ are not uniquely defined, as both depend on the order each pair in P is enumerated. Moreover, π_P depends also on the order the pairs of P are enumerated. Let

$$a_P := \operatorname{sgn}(\pi_P)\langle a_P \rangle.$$

We remark that a_P is well defined. The value of a_P does not depend on the order the pairs of P are enumerated, for all such enumerations preserve both $\operatorname{sgn}(\pi_P)$ and $\langle a_P \rangle$. It does not depend on the order each pair of P is enumerated either, since the exchange of two elements in the pair alters the sign of π_P, but as A is a skew symmetric matrix, the sign of $\langle a_P \rangle$ is also changed. We define the *Pfaffian* of A to be

$$\operatorname{pf}(A) := \sum_{P\in\mathcal{P}_n} a_P.$$

THEOREM 1.2
For every skew symmetric matrix A the following equality holds:

$$\det(A) = (\operatorname{pf}(A))^2.$$

Proof: Let $\Pi^{(0)}$ denote the collection of permutations $\pi \in \Pi_n$ such that the corresponding graph G_π contains a loop. Let $\Pi^{(1)}$ denote the collection of permutations $\pi \in \Pi_n$ such that the corresponding graph G_π does not contain loops, but is not bipartite. Finally, let $\Pi^{(2)}$ denote the collection of permutations $\pi \in \Pi_n$ such that the corresponding graph G_π is bipartite. Clearly, $\{\Pi^{(0)}, \Pi^{(1)}, \Pi^{(2)}\}$ is a partition of Π_n. By definition of determinant,

$$det(A) = \sum_{i=0}^{2} \left(\sum_{\pi \in \Pi^{(i)}} a_\pi \right).$$

By Propositions 1.3 and 1.4, asserted below, it follows that

$$\det(A) = \sum_{\pi \in \Pi^{(2)}} a_\pi.$$

Consider first the case in which n is odd. Then, $\Pi^{(2)}$ is empty, whence $\det(A) = 0$. Moreover, $\mathcal{P}_n$ is also empty, whence $\text{pf}(A) = 0$. We deduce that the asserted equality holds in this case. We may thus assume that n is even. By Lemma 1.5, asserted below, we have that

$$\det(A) = \sum_{P \in \mathcal{P}_n, Q \in \mathcal{P}_n} a_P \, a_Q = \sum_{P \in \mathcal{P}_n} a_P \sum_{Q \in \mathcal{P}_n} a_Q = (\text{pf}(A))^2.$$

The proof of the theorem is complete, pending the proofs of Propositions 1.3 and 1.4, and the proof of Lemma 1.5.

PROPOSITION 1.3
The contribution to the value of $\det(A)$ *of the summands that correspond to partitions in* $\Pi^{(0)}$ *is equal to zero. That is,*

$$\sum_{\pi \in \Pi^{(0)}} a_\pi = 0.$$

Proof: Let π be a permutation in $\Pi^{(0)}$. By definition of $\Pi^{(0)}$, π has a fixed point. Thus, for some index i, the multiplicand a_{ii} is used by a_π. As A is skew symmetric, the entries in the diagonal of A are all equal to zero. Therefore, $a_\pi = 0$. This conclusion holds for each permutation π in $\Pi^{(0)}$. We deduce that $\sum_{\pi \in \Pi^{(0)}} a_\pi = 0$. □

PROPOSITION 1.4
The contribution to the value of $\det(A)$ *of the summands that correspond to partitions in* $\Pi^{(1)}$ *is equal to zero. That is,*

$$\sum_{\pi \in \Pi^{(1)}} a_\pi = 0.$$

Proof: Let us begin by defining a mapping φ from $\Pi^{(1)}$ into itself. Let π denote a permutation in $\Pi^{(1)}$. By definition of $\Pi^{(1)}$, the corresponding graph G_π is non-bipartite. let v denote the smallest vertex of G_π that lies in an odd circuit, let C denote that circuit. Let G' denote the graph obtained from G_π by reversing the edges of C, let $\varphi(\pi) := \pi_{G'}$. Clearly, the mapping φ is an involution. Moreover, by definition of $\Pi^{(1)}$, G_π has no loops, whence φ has no fixed points.

By Lemma 1.1, $\mathrm{sgn}(\pi) = \mathrm{sgn}(\varphi(\pi))$. Let $(i, \pi(i))$ be any edge of C. The multiplicand $a_{i\pi(i)}$ of a_π is replaced in $a_{\varphi(\pi)}$ by the multiplicand $a_{\pi(i),i}$. As A is skew and symmetric, the latter is equal to $-a_{i\pi(i)}$. Thus, each multiplicand in a_π that corresponds to an edge of C is replaced by its complement. As C is odd, we conclude that $a_\pi = -a_{\varphi(\pi)}$. This conclusion holds for each permutation π in $\Pi^{(1)}$. As φ is bijective without fixed points, it follows that $\sum_{\pi \in \Pi^{(1)}} a_\pi = 0$. □

Lemma 1.5
If n is even then there exists a bijection φ from $\Pi^{(2)}$ onto $\mathcal{P}_n \times \mathcal{P}_n$ such that for every permutation π in $\Pi^{(2)}$,

$$a_\pi = a_P \, a_Q,$$

where $(P, Q) = \varphi(\pi)$.

Proof: Let π be a permutation in $\Pi^{(2)}$. We now define the pair of partitions (P, Q) such that $\varphi(\pi) = (P, Q)$. Let G_π be the directed graph that represents π. Then, G_π is the union of disjoint directed circuits. By definition of $\Pi^{(2)}$, graph G_π is bipartite. Thus, each directed circuit of G_π has even length. We now enumerate the pairs of P and Q in a particular order, so as to facilitate the argument. Arbitrarily enumerate the directed circuits of $G\pi$: $(C_1, C_2, \cdots, C_r)$. The pairs of P and Q will be enumerated in such a way that the first pairs of both P and Q correspond to pairs of vertices of C_1, the next set of pairs correspond to pairs of vertices of C_2 and so on.

We now define the enumeration of pairs in P and Q of pairs of vertices of a directed circuit $C \in \{C_1, C_2, \cdots, C_r\}$. Let v_C be the minimum vertex of C. The pairs of vertices of C in P are then enumerated in the following order:

$$\{\{v_C, \pi(v_C)\}\ , \{\pi^2(v_C), \pi^3(v_C)\}, \cdots, \{\pi^{2s-2}(v_C), \pi^{2s-1}(v_C)\}\},$$

where $2s$ denotes the length of C. Likewise, the pairs of vertices of C in Q are enumerated in the following order:

$$\{\{\pi(v_C), \pi^2(v_C)\}, \{\pi^3(v_C), \pi^4(v_C)\}, \cdots, \{\pi^{2s-1}(v_C), \pi^{2s}(v_C) = v_C\}\}.$$

It follows immediately from the definition of P and Q that both are partitions in $\mathcal{P}_n$. Each edge $(v, \pi(v))$ of G_π corresponds to a multiplicand $a_{v\pi(v)}$ of a_π and also to the same multiplicand in precisely one of a_P and a_Q. Thus, $\langle a_\pi \rangle = \langle a_P \rangle \langle a_Q \rangle$. In order to prove the asserted equality we now show that $\mathrm{sgn}(\pi) = \mathrm{sgn}(\pi_P)\mathrm{sgn}(\pi_Q)$. For this, note that for any integer i, $1 \le i \le n$, if v denotes $\pi_P(i)$ then $\pi_Q(i) = \pi(v)$. It follows that for any vertex v of G_π, $\pi(v) = \pi_Q(\pi_P^{-1}(v))$, whence $\pi = \pi_Q \cdot \pi_P^{-1}$. Consequently, denoting the identity permutation of Π_n by λ, we get that

$$\mathrm{sgn}(\pi) = \mathrm{sgn}(\pi_Q)\mathrm{sgn}(\pi_P^{-1}) = \mathrm{sgn}(\pi_Q)\mathrm{sgn}(\lambda)/\mathrm{sgn}(\pi_P) = \mathrm{sgn}(\pi_Q)\mathrm{sgn}(\pi_P).$$

We conclude that $a_\pi = a_P\, a_Q$, as asserted.

In order to complete the proof we now show that φ is bijective. We do this by exhibiting the inverse σ of φ. Let P and Q denote two partitions in $\mathcal{P}_n$. Let H_P be the undirected graph whose set of vertices is $\{1, 2, \cdots, n\}$ and such that vertices v and w are joined by an edge if and only if pair $\{v, w\}$ lies in P. Likewise, define H_Q. Adjust notation, by renaming the edges of H_Q if necessary, so that the set of edges of H_P and H_Q are disjoint. Let $H := H_P \cup H_Q$. The pairs in P correspond to a perfect matching of H. Likewise, the pairs in Q correspond to a perfect matching of H. Thus, H is bipartite and regular, each of its vertices has degree two.

Let us now give an orientation to graph H, thereby obtaining a graph G that represents a permutation in Π_n. For this, recall first that H is regular of degree two, therefore, we must ensure that each of the circuits in H is a directed circuit in G. Let C be any circuit in H, let v_C denote the smallest vertex of C. Let w denote the neighbor of H such that $\{v_C, w\}$ lies in P. Direct C so that C becomes a directed circuit and (v_C, w) lies in $E(G)$. Repeat this procedure for each circuit C in H. We then conclude that in the resulting directed graph G, each vertex has both its indegree and outdegree equal to one. That is, G represents a permutation. Let $\sigma(P, Q) := \pi_G$. It is easy to see that σ is the inverse of φ. □

The proof of Lemma 1.5 completes the proof of Theorem 1.2. □

2 Pfaffian Graphs

For any graph G, let $\mathcal{M}(G)$ denote the set of perfect matchings of G.

Proposition 2.1
The problem of determining $|\mathcal{M}(G)|$, given G, is NP-hard.

Proof: Consider an $n \times n$ matrix A of 0's and 1's. Define a simple graph G with bipartition $\{B', B''\}$, as follows: $B' := \{b'_1, b'_2, \cdots, b'_n\}$, $B'' := \{b''_1, b''_2, \cdots, b''_n\}$; for each pair i, j, $1 \leq i, j \leq n$, let b'_i be joined to b''_j by an edge if and only if $A_{ij} = 1$. Clearly, the number of perfect matchings of G is equal to $\mathrm{prm}(A)$. Valiant [10] has shown that the problem of determining the value of the permanent of a matrix of 0's and 1's is NP-hard. Thus, determining the number of perfect matchings of a bipartite graph is NP-hard. □

For a certain class of graphs called *Pfaffian*, it is possible to determine the number of perfect matchings in polynomial time, provided a *Pfaffian* orientation of the graph is given.

Let G be a graph on n vertices. Adjust notation, by renaming the vertices of G if necessary, so that $V(G) = \{1, 2, \cdots, n\}$. Consider an orientation $\vec{G}$ of G. For each pair (v, w) of (not necessarily distinct) vertices of $\vec{G}$, let $E(v, w)$ denote the set of edges of $\vec{G}$ whose tail is equal to v and whose head is equal to w. Define the *adjacency matrix* $A := (a_{ij})_{n\times n}$ of $\vec{G}$ as follows:

$$a_{ij} := |E(i, j)| - |E(j, i)|.$$

Clearly, matrix A is skew symmetric. Moreover, A depends on the enumeration of the vertices of G and also on the orientation of G. As we shall see, there are however certain properties of A that are invariant.

Distinct enumerations of the vertices of G yield distinct adjacency matrices A. However, if the orientation of G is preserved, then the absolute value of the determinant of all those matrices are identical. It then follows from Theorem 1.2 that $|\mathrm{pf}(A)|$ does not depend on the enumeration of the vertices of G. It depends solely on the orientation $\vec{G}$.

Let M be a perfect matching of G. Set M induces naturally a partition P in $\mathcal{P}_n$: let the pairs of P consist of the ends of each edge of M. For each partition P in $\mathcal{P}_n$, let $\mathcal{M}_P$ denote the set of perfect matchings of G that induce partition P. Let $M := \{e_1, e_2, \cdots, e_{n/2}\}$ be a perfect matching in $\mathcal{M}_P$. For each edge e_i in M, let t_i and h_i denote the tail and head of e_i, respectively. Let π_M denote the permutation $(t_1, h_1, t_2, h_2, \cdots, t_{n/2}, h_{n/2})$. Note that π_M depends on the enumeration of the edges of M, its sign depends solely on M and on the enumeration of $V(G)$. We thus define the *sign* of M to be $\mathrm{sgn}(M) := \mathrm{sgn}(\pi_M)$.

Lemma 2.2
The number of perfect matchings of G and the Pfaffian of A satisfy the following inequality:

$$|\mathcal{M}(G)| \geq |\mathrm{pf}(A)|. \tag{1}$$

Moreover, equality holds if and only if all perfect matchings of G have the same sign.

Proof: For each partition P in $\mathcal{P}_n$, we have that $a_P = \sum_{M \in \mathcal{M}_P} \mathrm{sgn}(M)$. From this, we deduce that

$$\begin{aligned} |\mathcal{M}(G)| &= \sum_{P \in \mathcal{P}_n} |\mathcal{M}_P| = \sum_{P \in \mathcal{P}_n} \sum_{M \in \mathcal{M}_P} |\mathrm{sgn}(M)| \geq \left| \sum_{P \in \mathcal{P}_n} \sum_{M \in \mathcal{M}_P} \mathrm{sgn}(M) \right| \\ &= \left| \sum_{P \in \mathcal{P}_n} a_P \right| = |\mathrm{pf}(A)|, \end{aligned}$$

where the inequality holds with equality if and only if all perfect matchings of G have the same sign. □

We say that orientation $\vec{G}$ of G is *Pfaffian* if and only if equality holds in (1). We remark that the interchange of any two vertices of G in the enumeration of the vertices of G changes the sign of all perfect matchings of G. Consequently, an orientation being Pfaffian does not depend on the particular enumeration of the vertices of G. Graph G is *Pfaffian* if it has a Pfaffian orientation. From Theorem 1.2, it follows that given a Pfaffian orientation of G it is then possible to evaluate $|\mathcal{M}(G)|$ in polynomial time.

LEMMA 2.3
Let M_1 and M_2 denote two perfect matchings of G. Let k denote the number of evenly oriented M_1, M_2-alternating circuits. Then,

$$\operatorname{sgn}(M_1)\operatorname{sgn}(M_2) = (-1)^k \tag{2}$$

Proof: Reversing any edge of any M_1,M_2-alternating circuit changes the sign of whichever of M_1 and M_2 contains the edge and preserves the sign of the other; it also changes the number k by one. Thus, the operation changes the sign of both sides of (2). Fix a direction for each M_1,M_2-alternating circuit. Reversal of each reverse edge of each M_1,M_2-alternating circuit yields an orientation in which each M_1,M_2-alternating circuit is a directed circuit. We may thus assume that each M_1,M_2-alternating circuit is directed. In particular, each M_1,M_2-alternating circuit is evenly oriented.

We use induction on the number k of M_1,M_2-alternating circuits in order to reduce to the case in which there is precisely one M_1,M_2-alternating circuit. If $k = 0$ then $M_1 = M_2$ and the assertion holds immediately. Consider next the case in which $k > 1$, let C be an M_1,M_2-alternating circuit. Let $M_3 := M_2 \oplus E(C)$, where $\oplus$ denotes the operation of symmetric difference. By induction hypothesis, $\operatorname{sgn}(M_1)\operatorname{sgn}(M_3) = (-1)^{k-1}$ and $\operatorname{sgn}(M_2)\operatorname{sgn}(M_3) = -1$. We deduce that

$$\operatorname{sgn}(M_1)\operatorname{sgn}(M_2) = \operatorname{sgn}(M_1)\operatorname{sgn}(M_2)\,(\operatorname{sgn}(M_3))^2 = (-1)^k.$$

We may thus assume that there exists precisely one M_1,M_2-alternating (directed) circuit, C. Let $2s$ denote the length of C. Interchange of two indices of vertices of G in the enumeration of its vertices changes the sign of both M_1 and M_2, whence both sides of (2) are preserved. We may therefore assume that $C = (1, 2, \cdots, 2s)$. We may also assume that for each edge e of $M_1 \cap M_2$, the tail of e is odd, say $2t - 1$, and the head of e is $2t$. Therefore, one of $\operatorname{sgn}(M_1)$ and $\operatorname{sgn}(M_2)$ is equal to $\operatorname{sgn}(1, 2, \cdots, n) = 1$, the other is equal to $\operatorname{sgn}(2, 3, 4, 5, \cdots, 2s-2, 2s-1, 2s, 1, 2s+1, \cdots, n-1, n) = -1$. The assertion holds. □

2.1 A Characterization of Pfaffian Orientations

A subgraph H of graph G is *nice* if $G - V(H)$ has a perfect matching. Let P be a path in G. We denote by $\operatorname{fw}(P)$ and by $\operatorname{rv}(P)$ the set of forward and reverse edges of P, respectively. Assume that the length of P is even and also that P does not repeat any edge. Then, $|\operatorname{fw}(P)|$ and $|\operatorname{rv}(P)|$ have the same parity; if that common parity is even then we say that P is *evenly oriented*, otherwise it is *oddly oriented*. We remark that P is evenly oriented if and only if its reverse is evenly oriented.

THEOREM 2.4
Let G be a graph with an even number of vertices, $\vec{G}$ an orientation of G. The following statements are equivalent:

(i) *Orientation $\vec{G}$ of G is Pfaffian.*

(ii) *Every nice circuit of G is oddly oriented.*

(iii) *Fix a perfect matching M of G. Every M-alternating circuit is oddly oriented.*

Proof: Statement (i) implies statement (ii). Fix an enumeration of the vertices of G. Assume that $\vec{G}$ is a Pfaffian orientation of G. Let C be a nice circuit of G. Let F be a perfect matching of $G - V(C)$. Let C_1 and C_2 be the two perfect matchings of C. Then, $M_1 := F \cup C_1$ and $M_2 := F \cup C_2$ are both perfect matchings of G. By hypothesis, $\vec{G}$ is Pfaffian. By Lemma 2.2, M_1 and M_2 have the same sign. Circuit C is the only M_1,M_2-alternating circuit. By Lemma 2.3, C is oddly oriented. This conclusion holds for each nice circuit C of G.

Statement (ii) implies statement (iii). Assume that every nice cycle of G is oddly oriented. Let C be an M-alternating circuit. Then, $M - E(C)$ is a perfect matching of $G - V(C)$, whence C is nice. By hypothesis, C is oddly oriented. This conclusion holds for each M-alternating circuit of G.

Statement (iii) implies statement (i). Assume that every M-alternating circuit is oddly oriented. Fix an enumeration of the vertices of G. Let N be any perfect matching of G. By hypothesis, each M,N-alternating circuit is oddly oriented. By Lemma 2.3, matchings M and N have the same sign. This conclusion holds for each perfect matching N of G. We deduce that all perfect matchings of G have the same sign. By Lemma 2.2, it follows that $\vec{G}$ is Pfaffian. □

2.2 Similarity of Orientations

Let $\oplus$ denote the operation of symmetric difference. For any orientation $\vec{G}$ of a graph G and any set C of edges of G, we shall denote by $\vec{G} \oplus C$ the orientation of G obtained from $\vec{G}$ by reversing the edges of set C.

Given a set X of vertices of graph G, the *cut* associated with X, denoted $\partial(X)$, is the set of edges of G that have one end in X, the other in the complement $\overline{X}$ of X. Set X is a *shore* of cut C.

Two orientations of G are *similar* if they differ by a cut. If two orientations, $\vec{G_1}$ and $\vec{G_2}$, are similar, this is represented by $\vec{G_1} \sim \vec{G_2}$. The following result is easily proved:

Proposition 2.5
For any two sets X and Y of vertices of graph G, the following equality holds: $\partial(X \oplus Y) = \partial(X) \oplus \partial(Y)$. □

Corollary 2.6
The relation of similarity of orientations is an equivalence relation.

Proof: Certainly $\sim$ is reflexive and symmetric. To show that it is also transitive, let $\vec{G_1}$, $\vec{G_2}$ and $\vec{G_3}$ be three orientations of G such that $\vec{G_1} \sim \vec{G_2}$ and $\vec{G_2} \sim \vec{G_3}$. By definition of similarity, there exist cuts C_{12} and C_{23} such that $\vec{G_1}$ and $\vec{G_2}$ differ precisely at the edges of C_{12}, and $\vec{G_2}$ and $\vec{G_3}$ differ precisely at the edges of C_{23}. Thus, $\vec{G_1}$ and $\vec{G_3}$ differ precisely at the edges of $C_{12} \oplus C_{23}$, which is also a cut, by Proposition 2.5. We conclude that $\vec{G_1} \sim \vec{G_3}$. □

Lemma 2.7
For any Pfaffian orientation $\vec{G}$ of G and any cut C of G, $\vec{G} \oplus C$ is also Pfaffian.

Proof: For any nice circuit D, $|C \cap D|$ is always even, as C is a cut and D is a circuit. Therefore, $|\mathrm{fw}(D) \cap C|$ and $|\mathrm{rv}(D) \cap C|$ have the same parity. We conclude that, in $\vec{G} \oplus C$, the number of forward edges of D has parity equal to that of the number of forward edges of D in $\vec{G}$. As $\vec{G}$ is Pfaffian, it follows that D is oddly oriented in $\vec{G}$, whence it is also oddly oriented in $\vec{G} \oplus C$. This conclusion holds for each nice circuit D of G. Thus, $\vec{G} \oplus C$ is Pfaffian. □

We shall denote by $\Phi(G)$ the set of similarity classes of orientations of G that consist of Pfaffian orientations. We remark that, by Lemma 2.7, each similarity class either consists of Pfaffian orientations of G or does not contain any Pfaffian orientation of G.

Lemma 2.8
For any Pfaffian orientation $\vec{G}$ of G and any cut C of G, $\vec{G} \oplus \overline{C}$ is also Pfaffian.

Proof: Reversal of every edge of Pfaffian orientation $\vec{G}$ results in another Pfaffian orientation. By Lemma 2.7, reverting back the edges of C also yields another Pfaffian orientation. But the latter is $\vec{G} \oplus \overline{C}$. □

Proposition 2.9
Two orientations of G that differ precisely at the edges of the complement $\overline{C}$ of a cut C are similar if and only if G is bipartite.

Proof: If G is bipartite then the set $E(G)$ of edges of G is a cut, whence so too is $\overline{C} = E(G) \oplus C$, by Proposition 2.5. Conversely, if $\overline{C}$ is a cut, then so too is $E(G) = C \oplus \overline{C}$, by Proposition 2.5. In that case, G is bipartite. We conclude that the two orientations are similar if and only if G is bipartite, as asserted. □

Corollary 2.10
If a nonbipartite graph G is Pfaffian then $|\Phi(G)| \geq 2$. □

3 Matching Covered Graphs

An edge of a graph G is *admissible* if it lies in some perfect matching of G. Graph G is *matching covered* if it is connected, has at least one edge and each edge of G is admissible. In a Pfaffian orientation of a graph G, nonadmissible edges may have arbitrary orientations. Thus, in order to determine properties of Pfaffian orientations of a graph, we may restrict ourselves to the subgraph of G spanned by the set of admissible edges. Also, if every edge of G is admissible, then each nontrivial connected component of G is matching covered; moreover, an orientation of G is Pfaffian if and only its restriction to each component of G is Pfaffian. So, rather trivially, the problem of determining properties of Pfaffian orientations for general graphs is reduced to the study of properties of Pfaffian orientations of matching covered graphs. We give below a concise summary of important properties of matching covered graphs and in the remainder of the section deduce fundamental properties concerning Pfaffian orientations of matching covered graphs. For details on matching covered graphs, see [1, 2, 3, 7].

The classical paper by Edmonds [4] describes a polynomial algorithm for determining whether or not a graph has a perfect matching. An edge e of a graph G is admissible if and only if it is not a loop and $G - v - w$ has a perfect matching, where v and w are the ends of e. It follows that it is possible to determine in polynomial time the set of admissible edges of a graph. Consequently, it is possible to determine in polynomial time whether or not a graph is matching covered.

Let X be a set of vertices of a matching covered graph G such that $|X|$ is odd. Let $C := \partial(X)$ be the cut associated with X. We denote by $G\{X \to x\}$ the graph obtained from G by contracting the vertices of X to a single vertex x. Likewise, if $\vec{G}$ is an orientation of G, we denote by $\vec{G}\{X \to x\}$ the directed graph obtained from $\vec{G}$ by contracting X to a single vertex x. The *C-contractions* of G are the two graphs obtained from G by contracting X to a single vertex or $\overline{X}$ to a single vertex. Likewise, the *C-contractions* of $\vec{G}$ are the two directed graphs obtained from $\vec{G}$ by contracting X or $\overline{X}$ to a single vertex. Cut C is *separating* if both C-contractions of G are matching covered.

Cut C is *tight* if each perfect matching of G contains precisely one edge in C. Every tight cut is separating. The cuts whose shores are vertices are trivially tight cuts. A tight cut is *nontrivial* if neither of its two shores is a singleton. If a matching covered graph G is free of nontrivial tight cuts, then it is a *brace* if G is bipartite and a *brick* if G is nonbipartite. Given a matching covered graph G, a *tight cut decomposition* of G is a family $\mathcal{C}$ of bricks and braces obtained by the following *tight cut decomposition procedure*: start with the singleton $\mathcal{C} := (G)$. If any graph H in $\mathcal{C}$ has a nontrivial tight cut, say C, replace H in $\mathcal{C}$ by the two C-contractions of H. Repeat this procedure until $\mathcal{C}$ consists solely of bricks and braces. The following remarkable result is due to Lovász [6]:

Theorem 3.1
The result of the application of two tight cut decomposition procedures results in two families that have the same graphs, up to multiplicities of edges.

A matching covered graph G thus has an invariant, denoted $b(G)$, which is defined to be the number of bricks in the tight cut decomposition of G. Another invariant, denoted $p(G)$, is the number of bricks in the tight cut decomposition of G whose underlying simple graphs are isomorphic to the Petersen graph. We denote by $(b+p)(G)$ the invariant $b(G)+p(G)$. We will need the following simple fact about bipartite matching covered graphs:

Proposition 3.2 (see [1])
In a bipartite matching covered graph, every C-contraction of a tight cut C is bipartite.

The following simple consequence of Hall's Theorem characterizes braces:

Theorem 3.3 (see [7])
Let G be a matching covered graph with bipartition $\{A, B\}$. Then, G is a brace if and only if $G - v_1 - v_2 - w_1 - w_2$ has a perfect matching, for every pair $\{v_1, v_2\}$ of distinct vertices

of A and every pair $\{w_1, w_2\}$ of distinct vertices of B.

It is then possible to determine in polynomial time whether or not a bipartite matching covered graph G is a brace. Moreover, from the proof of the above result it is possible to find a nontrivial tight cut of G, if G is not a brace. Consequently, tight cut decompositions of bipartite matching covered graphs may be determined in polynomial time.

A matching covered graph G is *bicritical* if $G - v - w$ has a perfect matching, for every pair $\{v, w\}$ of distinct vertices of G. The following fundamental result of Edmonds, Lovász and Pulleyblank [5] characterizes bricks:

Theorem 3.4
A nonbipartite matching covered graph G is a brick if and only if it is bicritical and 3-connected.

Again, it is possible to determine in polynomial time whether or not a matching covered graph G is bicritical. Consequently, it is possible to determine in polynomial time whether a nonbipartite matching covered graph G is a brick. Moreover, from the proof of the above result it is possible to determine a nontrivial tight cut of G, if G is not a brick. We conclude that it is then possible to determine a tight cut decomposition of any matching covered graph G in polynomial time. Also, the invariants $b(G)$, $p(G)$ and $(b + p)(G)$ may thus be evaluated in polynomial time.

3.1 Pfaffian Orientations and Tight Cuts

In this section we relate the existence of Pfaffian orientations of a matching covered graph G to those of the bricks and braces of a tight cut decomposition of G.

An orientation $\vec{G}$ of a graph G is *edge-simple* if, for every pair of multiple edges e and f of G, the head of e in $\vec{G}$ is equal to the head of f in $\vec{G}$.

Proposition 3.5
Every Pfaffian orientation of a matching covered graph is edge-simple.

Proof: Let G be a matching covered graph, $\vec{G}$ a Pfaffian orientation of G, e and f two multiple edges of G. As G is matching covered, there exists a perfect matching of G that contains edge e. Thus, the cycle C spanned by edges e and f is nice in G. By Theorem 2.4, circuit C is oddly oriented in $\vec{G}$, whence the head of e is equal to the head of f in $\vec{G}$. This conclusion holds for each pair e, f of multiple edges of G. □

Corollary 3.6
Let G be a matching covered graph, v a vertex of G, $\vec{G}$ a Pfaffian orientation of G. Then, $\vec{G}$ is similar to an orientation $\vec{H}$ such that v is a source of $\vec{H}$.

Proof: Let X denote the set of neighbors of v that, in $\vec{G}$, are tails of edges incident with v. Let $\vec{H} := \vec{G} \oplus \partial(X)$. As G is matching covered, $\vec{G}$ is edge-simple. Thus, each edge in $\partial(v) \cap \partial(X)$, has its head in $\vec{G}$ equal to v. It follows that v is a source in $\vec{H}$. □

Theorem 3.7

Let G be a matching covered graph, C a tight cut of G, G_1 and G_2 the two C-contractions of G. For $i = 1, 2$, let $\vec{G}_i$ denote any Pfaffian orientation of G_i. Then G has a Pfaffian orientation $\vec{G}$ such that C is directed in $\vec{G}$ and the C-contractions of $\vec{G}$ are similar to $\vec{G}_1$ and $\vec{G}_2$, respectively.

Proof: Let X denote the shore of C such that $G_1 = G\{X \to x\}$, whereupon $G_2 = G\{\overline{X} \to \overline{x}\}$. By Corollary 3.6, $\vec{G_1}$ is similar to an orientation $\vec{H}_1$ of G_1 in which vertex x is a source. Likewise, $\vec{G}_2$ is similar to an orientation $\vec{H}_2$ of G_2 in which vertex $\overline{x}$ is a sink. Let $\vec{G}$ be the orientation of G such that $\vec{H}_1$ and $\vec{H}_2$ are the C-contractions of $\vec{G}$. Clearly, cut C is directed in G and the C-contractions of $\vec{G}$ are similar to $\vec{G}_1$ and $\vec{G}_2$.

We assert $\vec{G}$ is Pfaffian. Let M be any perfect matching of G. As C is tight, it contains precisely one edge in C, say e. For $i = 1, 2$, let M_i denote the restriction of M to G_i. Then, M_i is a perfect matching of G_i. Let Q be any M-alternating circuit of G. We now show that Q is oddly oriented in $\vec{G}$. If Q does not contain any edge of C then it is an M_i-alternating circuit of G_i, for some $i \in \{1, 2\}$. In that case, as $\vec{H}_i$ is Pfaffian, it follows that Q is oddly oriented in $\vec{H}_i$, whence it is also oddly oriented in $\vec{G}$. We may thus assume that Q contains edges in C. As C is tight, it contains precisely two edges: one of them is e, the edge of M in C; let f denote the other edge. Then,

$$Q = (v_2, e, v_1) \cdot Q_1 \cdot (w_1, f, w_2) \cdot Q_2,$$

where v_1 is the end of e in $\overline{X}$, v_2 its end in X, w_1 is the end of f in $\overline{X}$, w_2 its end in X, Q_1 is an M_1-alternating path of even length in $G[\overline{X}]$ and Q_2 is an M_2-alternating path of even length in $G[X]$. Then,

$$(x, e, v_1) \cdot Q_1 \cdot (w_1, f, x)$$

is an M_1-alternating circuit in G_1. As $\vec{H}_1$ is Pfaffian and x is a source in $\vec{H}_1$, it follows that path Q_1 is evenly oriented in $\vec{H}_1$. Similarly, path Q_2 is evenly oriented in $\vec{H}_2$. It follows that Q is oddly oriented in $\vec{G}$. This conclusion holds for each M-alternating circuit Q of G. We deduce that $\vec{G}$ is Pfaffian, as asserted. □

From the above result we deduce that if every brick and brace of a matching covered graph G is Pfaffian then G is also Pfaffian. We now prove a generalization of the converse of the result above.

Theorem 3.8

Let G be a matching covered graph, C a separating cut of G. Then, for each Pfaffian orientation of G there exists a similar orientation $\vec{G}$ of G such that C is a directed cut in $\vec{G}$ and each C-contraction of $\vec{G}$ is Pfaffian.

Proof: Let X be a shore of C. For each edge e in C, let $v(e)$ denote the end of e in X, $\overline{v}(e)$ its end in $\overline{X}$. Fix an edge e of C. As C is separating, G has a perfect matching, M, that contains only edge e in C. Let W denote the set of vertices that are ends of edges in C. For each vertex v in $W \cap X$, there exists in $G[X]$ an M-alternating path Q_v from v to $v(e)$ of even length. To see this, note first that if $v = v(e)$ then the path Q_v of length zero has the asserted properties. We may thus assume that v and $v(e)$ are distinct. Let f be any edge of C incident with v. As C is separating, there exists in G a perfect matching N that contains only edge f in C. One of the M,N-alternating circuits of G contains only edges e and f in C. The segment of that circuit in $G[X]$ that extends from v to $v(e)$ is M-alternating and has even length, as asserted. Likewise, for each vertex v of $W \cap \overline{X}$, there exists in $G[\overline{X}]$ an M-alternating path Q_v from v to $\overline{v}(e)$ of even length.

Let $\vec{H}$ be a Pfaffian orientation of G, let Y denote the set of those vertices v of W such that Q_v is oddly oriented in $\vec{H}$. Let

$$\vec{G} := \vec{H} \oplus \partial(Y)$$

Certainly, $\vec{G}$ is Pfaffian, because it is similar to $\vec{H}$. Moreover, for each vertex v in W, path Q_v is evenly oriented in $\vec{G}$.

To show that C is a directed cut of $\vec{G}$, let f denote any edge of $C - e$, consider the M-alternating circuit

$$Q_f := (v(e), e, \overline{v}(e)) \cdot \left(Q_{\overline{v}(f)}\right)^R \cdot (\overline{v}(f), f, v(f)) \cdot Q_{v(f)},$$

where $(Q_{\overline{v}(f)})^R$ denotes the reversal of $Q_{\overline{v}(f)}$. As $\vec{G}$ is Pfaffian, then, by Theorem 2.4, Q_f is oddly oriented. As $Q_{v(f)}$ and $Q_{\overline{v}(f)}$ are both evenly oriented, it follows that $v(e)$ is the tail of e in $\vec{G}$ if and only if $v(f)$ is the tail of f in $\vec{G}$. This conclusion holds for each edge f of $C - e$. We deduce that C is a directed cut in $\vec{G}$, as asserted.

We now show that the C-contraction $\vec{G}_1 := \vec{G}\{X \to x\}$ is Pfaffian. For this, note first that $M_1 := M \cap E(\vec{G}_1)$ is a perfect matching of $\vec{G}_1$. Let Q_1 be any M_1-alternating circuit of $\vec{G}_1$. If Q_1 does not contain vertex x then Q_1 is an M-alternating circuit of $\vec{G}$, whence oddly oriented. We may thus assume that x lies in Q_1. Then, e is the edge of $C \cap M$ that lies in Q_1. Let f denote the edge of $C - M$ that lies in Q_1. Let Q_1' denote the segment of Q_1 from $\overline{v}(e)$ to $\overline{v}(f)$ that does not contain vertex x. Then, Q_1 or its reversal is equal to $(x, e, \overline{v}(e)) \cdot Q_1' \cdot (\overline{v}(f), f, x)$. Consider the M-alternating circuit

$$R := (v(e), e, \overline{v}(e)) \cdot Q_1' \cdot (\overline{v}(f), f, v(f)) \cdot Q_{v(f)}.$$

Then, in $\vec{G}$, $Q_{v(f)}$ is evenly oriented, by definition of $\vec{G}$. As R is an M-alternating circuit, it is oddly oriented in $\vec{G}$. Thus, Q_1' is evenly oriented in $\vec{G}$. It follows that in $\vec{G}_1$ path Q_1' is also evenly oriented. We conclude that Q_1 is oddly oriented in $\vec{G}_1$. This conclusion holds for each M_1-alternating circuit Q_1 of $\vec{G}_1$. As asserted, $\vec{G}_1$ is Pfaffian. Likewise, C-contraction $\vec{G}\{\overline{X} \to \overline{x}\}$ is also Pfaffian. □

Theorem 3.9
Let G be a matching covered graph, C a tight cut of G, G_1 and G_2 the two C-contractions of G. Then there exists a bijection relating $\Phi(G)$ and $\Phi(G_1) \times \Phi(G_2)$.

Proof: Let $(\mathcal{C}_1, \mathcal{C}_2)$ be a pair in $\Phi(G_1) \times \Phi(G_2)$. Let $\mathcal{C}(\mathcal{C}_1, \mathcal{C}_2)$ denote the set of those orientations $\vec{G}$ of G such that C is a directed cut in $\vec{G}$ and the C-contractions of $\vec{G}$ lie in $\mathcal{C}_1$ and in $\mathcal{C}_2$, respectively.

Proposition 3.10
Any two orientations $\vec{G}$ and $\vec{G}'$ in $\mathcal{C}(\mathcal{C}_1, \mathcal{C}_2)$ are similar.

Proof: Let $\vec{G}''$ denote the orientation $\vec{G}' \oplus C$. Then, $\vec{G}''$ lies in $\mathcal{C}(\mathcal{C}_1, \mathcal{C}_2)$ and is similar to $\vec{G}'$. By definition, cut C is directed in each of $\vec{G}$ and $\vec{G}'$. Replace $\vec{G}'$ by $\vec{G}''$ if necessary, so that the orientation of each edge of C is the same in each of $\vec{G}$ and $\vec{G}'$. Let $i \in \{1, 2\}$. Let $\vec{G}_i$ denote the C-contraction of $\vec{G}$ that lies in $\mathcal{C}_i$. Let $\vec{G}'_i$ denote the C-contraction of $\vec{G}'$ that lies in $\mathcal{C}_i$. Then, $\vec{G}_i$ and $\vec{G}'_i$ are similar orientations of G_i. Let D_i denote the cut of G_i that consist of the edges of G_i that have distinct orientations in $\vec{G}_i$ and in $\vec{G}'_i$. For each edge of C, its orientations in $\vec{G}$ and in $\vec{G}'$ are the same. Thus, its orientations are also the same in $\vec{G}_i$ and $\vec{G}'_i$. Therefore, D_i and C are disjoint. Thus D_i is a cut of G_i, disjoint with C. We conclude that D_1 and D_2 are disjoint, whence $D := D_1 \cup D_2 = D_1 \oplus D_2$ is a cut of G. Moreover, $\vec{G} = \vec{G}' \oplus D$. Thus, $\vec{G}$ and $\vec{G}'$ are similar. □

By Theorem 3.7, $\mathcal{C}(\mathcal{C}_1, \mathcal{C}_2)$ contains a Pfaffian orientation. We conclude that $\mathcal{C}(\mathcal{C}_1, \mathcal{C}_2)$ is part of a Pfaffian similarity class of G. We thus define mapping φ from $\Phi(G_1) \times \Phi(G_2)$ to $\Phi(G)$ by setting $\varphi(\mathcal{C}_1, \mathcal{C}_2)$ to be equal to the Pfaffian similarity class of G that includes $\mathcal{C}(\mathcal{C}_1, \mathcal{C}_2)$.

We now show that φ is surjective. For this, let $\mathcal{C}$ be a Pfaffian similarity class of G. By Theorem 3.8, $\mathcal{C}$ contains an orientation, $\vec{G}$, such that C is a directed cut in $\vec{G}$ and each C-contraction of $\vec{G}$ is Pfaffian. For $i = 1, 2$, let $\vec{G}_i$ denote the C-contraction of G that is an orientation of G_i, let $\mathcal{C}_i$ denote the similarity class of G_i that contains $\vec{G}_i$. Then, by definition, $\vec{G}$ lies in $\mathcal{C}(\mathcal{C}_1, \mathcal{C}_2)$, whence $\mathcal{C} = \varphi(\mathcal{C}_1, \mathcal{C}_2)$. Indeed, φ is surjective.

To complete the proof, we now show that φ is injective. For this, let $(\mathcal{C}_1, \mathcal{C}_2)$ and $(\mathcal{C}'_1, \mathcal{C}'_2)$ be two pairs in $\Phi(G_1) \times \Phi(G_2)$ such that $\varphi(\mathcal{C}_1, \mathcal{C}_2) = \varphi(\mathcal{C}'_1, \mathcal{C}'_2)$. Let $\vec{G}$ be an orientation in $\mathcal{C}(\mathcal{C}_1, \mathcal{C}_2)$, $\vec{G}'$ an orientation in $\mathcal{C}(\mathcal{C}'_1, \mathcal{C}'_2)$. Then, $\vec{G}$ and $\vec{G}'$ are similar. Adjust notation, by reversing the edges of C in $\vec{G}'$ if necessary, so that each edge of C has the same orientation in $\vec{G}$ and in $\vec{G}'$. Let $D := \partial(Y)$ be the cut of G that consists of the edges of G that have distinct orientations in $\vec{G}$ and in $\vec{G}'$. For $i = 1, 2$, let $\vec{G}_i$ and $\vec{G}'_i$ denote the C-contractions of $\vec{G}$ and $\vec{G}'$, respectively, that are orientations of G_i.

Assume, to the contrary, that D separates C, that is, there exist edges e and f in C such that the ends of e lie both in Y and the ends of f lie both in $\overline{Y}$. As C is separating, there exist perfect matchings M and N of G such that e is the only edge of C in M and f its only edge in N. Let Q be the M,N-alternating circuit that contains edge e. Then, it contains edge f; moreover, e and f are the only edges of Q in C. We may then write

$$Q = (v_2, e, v_1) \cdot Q_1 \cdot (w_1, f, w_2) \cdot Q_2,$$

where v_1 is the end of e in $\overline{X}$, v_2 its end in X, w_1 is the end of f in $\overline{X}$, w_2 its end in X, Q_1 is an M-alternating path of even length in $G[\overline{X}]$ and Q_2 is an M-alternating path of even length in $G[X]$. Let M_1 denote the restriction of M to G_1. Then, M_1 is a perfect matching of G_1. Moreover,

$$R := (x, e, v_1) \cdot Q_1 \cdot (w_1, f, x)$$

is an M_1-alternating circuit in G_1. Path Q_1 contains an odd number of edges in D. Therefore, it is oddly oriented in precisely one of $\vec{G}_1$ and $\vec{G}'_1$. We deduce that R is evenly oriented in precisely one of $\vec{G}_1$ and $\vec{G}'_1$. This is a contradiction to the hypothesis that $\vec{G}$ lies in $\mathcal{C}(\mathcal{C}_1, \mathcal{C}_2)$ or that $\vec{G}'$ lies in $\mathcal{C}(\mathcal{C}'_1, \mathcal{C}'_2)$. As asserted, cut D does not separate cut C.

Every edge of C has the same orientation in $\vec{G}$ and in $\vec{G}'$. Thus, C and D are disjoint. It follows that either all the ends of all edges of C lie in Y or all the ends of all edges of C lie in $\overline{Y}$. Adjust notation, by interchanging Y with $\overline{Y}$, if necessary, so that all the ends of all edges of C lie in $\overline{Y}$. Let D_1 be the cut of G_1 whose shore is $Y \cap \overline{X}$, let D_2 be the cut in G_2 whose shore is $Y \cap X$. Then D_1 and D_2 are disjoint and their union is equal to D. It follows that $\vec{G}'_1 \oplus D_1 = \vec{G}_1$ and $\vec{G}'_2 \oplus D_2 = \vec{G}_2$. We deduce that the C-contractions of $\vec{G}$ and $\vec{G}'$ are similar. That is, $(\mathcal{C}_1, \mathcal{C}_2) = (\mathcal{C}'_1, \mathcal{C}'_2)$. We conclude that φ is injective. □

Corollary 3.11

Let G be a matching covered graph, C a tight cut of G, G_1 and G_2 the two C-contractions of G. Then, $|\Phi(G)| = |\Phi(G_1)| \cdot |\Phi(G_2)|$. □

Corollary 3.12

The Petersen graph is non-Pfaffian.

Proof: Let G denote the Petersen graph. Let $Q := (v_0, v_1, v_2, v_3, v_4)$ denote a pentagon of G. Then, $G - V(Q)$ is also a pentagon, say $Q' := (v'_0, v'_1, v'_2, v'_3, v'_4)$. Adjust notation, by rotating or reversing Q' if necessary, so that v_i and v'_i are adjacent in G, for $i = 0, \cdots, 4$. Then, $C := \partial(V(Q))$ is a separating cut of G.

Assume, to the contrary, that G is Pfaffian. By Theorem 3.8, G has a Pfaffian orientation $\vec{G}$ such that C is directed in $\vec{G}$ and each C-contraction of $\vec{G}$ is Pfaffian. Let $G_1 := G\{V(Q') \to x\}$ denote a C-contraction of G. Let $\vec{G}_1 := \vec{G}\{V(Q') \to x\}$ denote the corresponding C-contraction of $\vec{G}$. For $i = 0, \cdots, 4$, let M_i denote the perfect matching of G_1 that contains edge xv_i. In particular, M_0 contains edges v_1v_2 and v_3v_4.

The M_0,M_2-alternating circuit of G_1 is oddly oriented in $\vec{G}_1$. Moreover, x is a source or a sink in $\vec{G}_1$. Thus, path $R_{02} := (v_0, v_1, v_2)$ is evenly oriented in $\vec{G}_1$. Likewise, the M_0,M_4-alternating path in G_1 is oddly oriented in $\vec{G}_1$, whence path $R_{04} := (v_0, v_1, v_2, v_3, v_4)$ is evenly oriented in $\vec{G}_1$. As R_{02} is evenly oriented, it follows that $R_{24} := (v_2, v_3, v_4)$ is also evenly oriented. In sum, both R_{02} and R_{24} are evenly oriented in $\vec{G}_1$. Those paths have length equal to two. Therefore, either R_{02} or its reversal is directed. Likewise, either R_{24} or its reversal is directed. By symmetry, paths $R_{03} := (v_0, v_4, v_3)$ and $R_{31} := (v_3, v_2, v_1)$ also have that property. It follows that Q or its reversal is a directed circuit in $\vec{G}_1$. Thus, Q or its reverse is a directed circuit in $\vec{G}$. By symmetry, circuit Q' or its reverse is also a directed circuit in $\vec{G}$.

Note that $M := C$ is a perfect matching of G. Let N be any perfect matching of G that contains precisely one edge in C. The M,N-alternating circuit R contains precisely four edges in C and two edges in each of Q and Q'. Moreover, in each of Q and Q', precisely one of the two edges of R is a forward edge. As C is directed in $\vec{G}$, we conclude that R is evenly oriented. This is a contradiction. As asserted, G is non-Pfaffian. □

Corollary 3.13
Let G be a matching covered graph. If $p(G) > 0$ then G is non-Pfaffian.

Proof: By induction on the size of G. Consider first the case in which G is free of nontrivial tight cuts. Then, the underlying simple graph H of G is the Petersen graph. By Corollary 3.12, H is not Pfaffian, whence neither is G. We may thus assume that G has a nontrivial tight cut, C. Let G_1 and G_2 denote the C-contractions of G. As $p(G_1) + p(G_2) = p(G) > 0$, it follows that either $p(G_1) > 0$ or $p(G_2) > 0$. Adjust notation so that $p(G_1) > 0$. By induction hypothesis, G_1 is not Pfaffian. By Theorem 3.8, neither is G. □

3.2 Pfaffian Orientations and Removable Ears

Let G be a graph. A *single ear* of G is a path

$$R := (v_0, e_1, v_1, \cdots, v_{2r}, e_{2r+1}, v_{2r+1}),$$

without repeated vertices, and of odd length $2r + 1$, $r \geq 0$, such that the internal vertices of R have all degree two in G. Each of the edges $e_1, e_3, \cdots, e_{2r+1}$ is *odd*. Each of the edges $e_2, e_4, \cdots, e_{2r}$, if any, is *even*. Note that reversal of R preserves the parity of each edge of R. A *double ear* is a pair of vertex disjoint single ears. An *ear* is a single ear or a double ear. Let R be an ear of G. The *rank* of R, denoted $\rho(R)$, is zero if R is a single ear, one if R is a double ear. We denote by $G - R$ the graph obtained from G by the removal of the edges and internal vertices of each single ear of R.

Assume that G is matching covered. A single ear R is *removable* if $G - R$ is matching covered. A double ear R is *removable* if $G - R$ is matching covered but neither single ear of R is removable. An ear R is *b-removable* if it is removable and $b(G - R) = b(G) - \rho(R)$. It is $(b+p)$-*removable* if it is removable and $(b+p)(G - R) = (b+p)(G) - \rho(R)$. The following fundamental result was proved in [3].

Theorem 3.14
Every matching covered graph distinct from K_2 has a $(b + p)$-removable ear.

3.2.1 Elementary Properties of Ears

Proposition 3.15
Let G be a matching covered graph distinct from K_2, R a single ear of G. A subset S of $E(R)$ is a cut of G if and only if $|S|$ is even.

Proof: By induction on $|S|$. If S is empty then the assertion holds immediately. If S is a singleton then S is not a cut of G, for K_2 is the only matching covered graph that contains edge cuts. We may thus assume that S has at least two edges. Let S_2 denote a subset of S that contains precisely two edges. Clearly, S_2 is a cut of G. If S is a cut then so too is $S \oplus S_2 = S - S_2$. If $S - S_2$ is a cut of G then so too is $(S - S_2) \oplus S_2 = S$. We conclude that

S is a cut of G if and only if $S - S_2$ is a cut of G. By induction hypothesis, $S - S_2$ is a cut of G if and only if $|S - S_2|$ is even. As asserted, S is a cut of G if and only if $|S|$ is even. □

Proposition 3.16

Let G be a graph, R an ear of G. Then, each nice circuit of $G - R$ is a nice circuit of G. Consequently, if G is Pfaffian then $G - R$ is also Pfaffian.

Proof: Let $H := G - R$. Let Q be a nice circuit of H. Then, $H - V(Q)$ has a perfect matching, say M_0. Let M be the set of edges of G obtained from M_0 by the addition of the set of even edges of R. Then, M is a perfect matching of $G - V(Q)$. We conclude that Q is a nice circuit of G. This conclusion holds for each nice circuit Q of $G - R$.

Assume that G is Pfaffian. Let $\vec{G}$ be a Pfaffian orientation of G, let $\vec{H} := \vec{G} - R$. Then, each nice circuit of G is oddly oriented in $\vec{G}$, whence every nice circuit of H is oddly oriented in $\vec{H}$. We conclude that $\vec{H}$ is a Pfaffian orientation of H, whence H is Pfaffian. □

Proposition 3.17

Let G be a matching covered graph, R a double ear of G, M a perfect matching of G. If R is removable then either M contains all the odd edges of R or it contains none.

Proof: Let R_1 and R_2 denote the two single ears of R. Assume, to the contrary, that M contains some, but not all odd edges of R. Adjust notation so that M contains some odd edge of R_1. Then, M contains all the odd edges of R_1. Thus, M does not contain some odd edge of R_2, whence it contains no odd edge of R_2. Therefore, $M' := M - R_2$ is a perfect matching of $G - R_2$ that contains all the odd edges of R_1. Thus, every odd edge of R_1 is admissible in $G - R_2$. By hypothesis, graph $G - R$ is matching covered. Any perfect matching of $G - R$ may be extended to a perfect matching of $G - R_2$ by the addition of the even edges of R_1. Not only the edges of $G - R$ are admissible in $G - R_2$, but also the even edges of R_1 are admissible in $G - R_2$. It follows that every edge of $G - R_2$ is admissible in $G - R_2$. We deduce that $G - R_2$ is matching covered, whence R_2 is removable in G. This is a contradiction to the definition of removable double ear. □

Corollary 3.18

Let G be a matching covered graph, R a removable ear of G. Then, G has a nice circuit that contains all the edges of R.

Proof: Let e be an odd edge of R. Let M be a perfect matching of G that contains edge e. As $G - R$ is matching covered, it contains at least one edge. Thus, G is distinct from K_2, whence it has a perfect matching, say N, that does not contain edge e. Let Q denote the M,N-alternating circuit that contains edge e. Every M-alternating circuit is nice. Thus, Q is nice. Let M' denote the perfect matching $M' := M \oplus Q$ of G. Perfect matching M contains odd edge e of R; perfect matchings N and M' do not contain e. By Proposition 3.17, M contains all the odd edges of R whereas N and M' contain no odd edge of R. It follows that Q contains all the edges of R. □

3.2.2 The Number of Pfaffian Orientations

In this section we determine the number of Pfaffian orientations of a matching covered graph. For this, we need two simple but important results.

LEMMA 3.19
Let G be a matching covered graph, R a removable single ear of G, Q a nice circuit of G that contains all the edges of R. For $i = 1, 2$, let $\vec{G}_i$ be an orientation of G such that Q is oddly oriented in $\vec{G}_i$, let $\vec{H}_i := \vec{G}_i - R$. If $\vec{H}_1$ and $\vec{H}_2$ are similar, then $\vec{G}_1$ and $\vec{G}_2$ are similar.

Proof: Consider first the case in which $\vec{H}_1 = \vec{H}_2$. Let S denote the set of edges of G that have distinct orientations in $\vec{G}_1$ and $\vec{G}_2$. As $\vec{H}_1 = \vec{H}_2$, it follows that S is a subset of $E(R)$. As Q is oddly oriented in each of $\vec{G}_1$ and $\vec{G}_2$, it follows that the number of forward edges of R in $\vec{G}_1$ has parity equal to that of the number of forward edges of R in $\vec{G}_2$. We conclude that $|S|$ is even. By Proposition 3.15, S is a cut of G. We deduce that $\vec{G}_1 \sim \vec{G}_2$.

We may thus assume that $\vec{H}_1$ and $\vec{H}_2$ are distinct. Assume that $\vec{H}_1$ and $\vec{H}_2$ are similar. Let D denote the cut of $G - R$ that consists of the edges of $G - R$ that have distinct orientations in $\vec{H}_1$ and $\vec{H}_2$. Let X be a shore of D. Let C be the corresponding cut $\partial(X)$ in G. Let $\vec{G}'_2$ denote orientation $\vec{G}_2 \oplus C$. As Q is oddly oriented in $\vec{G}_2$, it is also oddly oriented in $\vec{G}'_2$. Also, $\vec{G}_2$ and $\vec{G}'_2$ are similar. We may thus replace $\vec{G}_2$ by $\vec{G}'_2$, thereby ensuring that $\vec{H}_1 = \vec{H}_2$. By the previous case, $\vec{G}_1 \sim \vec{G}_2$. □

LEMMA 3.20
Let G be a matching covered graph, R a removable double ear of G, Q a nice circuit of G that contains all the edges of R. For $i = 1, 2, 3$, let $\vec{G}_i$ be an orientation of G such that Q is oddly oriented in $\vec{G}_i$, let $\vec{H}_i := \vec{G}_i - R$. If $\vec{H}_1$, $\vec{H}_2$ and $\vec{H}_3$ are similar, then at least two of $\vec{G}_1$, $\vec{G}_2$ and $\vec{G}_3$ are similar.

Proof: Consider first the case in which $\vec{H}_1 = \vec{H}_2 = \vec{H}_3$. Let R_1 and R_2 denote the two single ears of R. The parity of the number of forward edges of R_1 must be the same in at least two of the three orientations $\vec{G}_1$, $\vec{G}_2$ and $\vec{G}_3$. Adjust notation so that the parity of the number of forward edges of R_1 in $\vec{G}_1$ is equal to the parity of the number of forward edges of R_1 in $\vec{G}_2$.

We assert that $\vec{G}_1$ and $\vec{G}_2$ are similar. For this, let S denote the set of edges of G that have distinct orientations in $\vec{G}_1$ and $\vec{G}_2$. By hypothesis of the case, $\vec{H}_1 = \vec{H}_2$. Thus, $S \subseteq E(R)$. For $i = 1, 2$, let $S_i := S \cap E(R_i)$. By the choice of $\vec{G}_1$ and $\vec{G}_2$, the parity of the number of forward edges of R_1 in $\vec{G}_1$ is equal to the parity of the number of forward edges of R_1 in $\vec{G}_2$. Thus, $|S_1|$ is even. By hypothesis, Q is oddly oriented in each of $\vec{G}_1$ and $\vec{G}_2$. Thus, $|S_2|$ is also even. By Proposition 3.15, each of S_1 and S_2 is a cut of G. But $\{S_1, S_2\}$ is a partition of S, whence S, which is equal to $S_1 \oplus S_2$, is a cut of G. As asserted, $\vec{G}_1 \sim \vec{G}_2$.

We may thus assume that the three orientations $\vec{H}_1$, $\vec{H}_2$ and $\vec{H}_3$ are not identical. Assume that $\vec{H}_1$, $\vec{H}_2$ and $\vec{H}_3$ are similar. Let D denote the cut of $G - R$ that consists of the edges of $G - R$ that have distinct orientations in $\vec{H}_1$ and $\vec{H}_2$. Let X be a shore of D. Let C be the corresponding cut $\partial(X)$ in G. Let $\vec{G}'_2$ denote orientation $\vec{G}_2 \oplus C$. Then,

$\vec{H}_2' := \vec{G}_2' - R$ and $\vec{H}_1$ are equal. Moreover, as Q is oddly oriented in $\vec{G}_2$, it follows that it is also oddly oriented in $\vec{G}_2'$. We may thus replace $\vec{G}_2$ by $\vec{G}_2'$ and assume that $\vec{H}_2 = \vec{H}_1$. A similar reasoning allows us to assume that $\vec{H}_3 = \vec{H}_1$. By the previous case, at least two of $\vec{G}_1$, $\vec{G}_2$ and $\vec{G}_3$ are similar. □

Corollary 3.21
Let G be a Pfaffian matching covered graph, R a removable ear of G. Then, $|\Phi(G)| \leq (\rho(R)+1) \cdot |\Phi(G-R)|$.

Proof: Let $\mathcal{G}$ be a set of $|\Phi(G)|$ nonsimilar Pfaffian orientations of G. For each $\vec{G}$ in $\mathcal{G}$, $\vec{G} - R$ is a Pfaffian orientation of $G - R$, by Proposition 3.16. Assume, to the contrary, that $|\Phi(G)| > (\rho(R)+1) \cdot |\Phi(G-R)|$. Then, there exists a subset $\mathcal{G}_0$ of $\mathcal{G}$ containing $\rho(R)+2$ orientations, such that for any two orientations $\vec{G}$ and $\vec{G}'$ in $\mathcal{G}_0$, orientations $\vec{G} - R$ and $\vec{G}' - R$ are similar. By Corollary 3.18, G has a nice circuit that contains all the edges of R. By Lemmas 3.19 and 3.20, at least two orientations in $\mathcal{G}_0$ are similar. This is a contradiction to the definition of $\mathcal{G}$. □

Theorem 3.22
Let G be a matching covered graph. If G is Pfaffian then $|\Phi(G)| = 2^{b(G)}$.

Proof: By induction on the size of G. Assume that G is Pfaffian. If G is K_2 then the two orientations of G are Pfaffian and similar, and $b(G) = 0$, whence the assertion holds immediately. We may thus assume that G is distinct from K_2.

Consider next the case in which graph G has a nontrivial tight cut, say C. Let G_1 and G_2 denote the two C-contractions of G. By Theorem 3.1, $b(G) = b(G_1) + b(G_2)$. By induction hypothesis, $|\Phi(G_i)| = 2^{b(G_i)}$, for $i = 1, 2$. By Corollary 3.11, $|\Phi(G)| = |\Phi(G_1)| \cdot |\Phi(G_2)|$. Thus,

$$|\Phi(G)| = |\Phi(G_1)| \cdot |\Phi(G_2)| = 2^{b(G_1)}2^{b(G_2)} = 2^{b(G_1)+b(G_2)} = 2^{b(G)}.$$

We may thus assume that G is either a brick or a brace distinct from K_2.

We assert that $|\Phi(G)| \geq 2^{b(G)}$. By hypothesis, G is Pfaffian. Thus, G has at least one Pfaffian orientation. If G is bipartite then $b(G) = 0$, whence the inequality holds. If G is a brick then it is nonbipartite, whence it has at least two Pfaffian orientations, by Lemma 2.8 and Proposition 2.9; moreover, $b(G) = 1$. In both cases, the asserted inequality holds.

In order to complete the proof we must show that for every brick and every brace distinct from K_2, $|\Phi(G)| \leq 2^{b(G)}$. By Theorem 3.14, G has a $(b+p)$-removable ear, R. Then $(b+p)(G-R) = (b+p)(G) - \rho(R)$. By hypothesis, G is Pfaffian. By Proposition 3.16, $G-R$ is also Pfaffian. By Corollary 3.13, $p(G) = 0 = p(G-R)$. Thus, $b(G-R) = b(G) - \rho(R)$. By induction hypothesis, $|\Phi(G-R)| = 2^{b(G-R)}$. By Corollary 3.21, $|\Phi(G)| \leq (\rho(R)+1) \cdot |\Phi(G-R)|$. Thus,

$$|\Phi(G)| \leq (\rho(R)+1) \cdot |\Phi(G-R)| = (\rho(R)+1) \cdot 2^{b(G-R)} = (\rho(R)+1) \cdot 2^{b(G)-\rho(R)}.$$

As $\rho(R)$ lies in $\{0, 1\}$, the asserted inequality holds. The proof of the Theorem is complete. □

Corollary 3.23

Let G be a matching covered graph, R a b-removable ear of G, Q a nice circuit of G that contains all the edges of R, $\vec{G}_0$ an orientation of G such that circuit Q is oddly oriented in $\vec{G}_0$. If $\vec{G}_0 - R$ is a Pfaffian orientation of $G - R$ and G is Pfaffian then $\vec{G}_0$ is a Pfaffian orientation of G.

Proof: Assume that $\vec{G}_0 - R$ is a Pfaffian orientation of $G - R$ and also that G is Pfaffian. To prove that $\vec{G}_0$ is a Pfaffian orientation of G, let $\mathcal{F} := (\vec{G}_1, \vec{G}_2, \cdots, \vec{G}_{\Phi(G)})$ denote a family with $|\Phi(G)|$ nonsimilar Pfaffian orientations of G. Extend $\mathcal{F}$ to a family $\mathcal{G}$ of orientations of G, by adding $\vec{G}_0$ to $\mathcal{F}$. Let $\mathcal{H}$ be the family obtained from $\mathcal{G}$ by replacing each member $\vec{G}_i$ by a new member $\vec{H}_i := \vec{G}_i - R$, for $i = 0, \cdots, |\Phi(G)|$.

By hypothesis, $\vec{G}_0 - R$, that is, $\vec{H}_0$, is a Pfaffian orientation of $G - R$. For $i = 1, \cdots, \Phi(G)$, $\vec{H}_i$ is a Pfaffian orientation of $G - R$, by Proposition 3.16. Thus each of the $|\Phi(G)| + 1$ members of G is a Pfaffian orientation of $G - R$.

Consider first the case in which R is a single ear. Then, $b(G) = b(G - R)$. By Theorem 3.22, $|\Phi(G)| = |\Phi(G - R)|$, whence $|\mathcal{H}| = |\Phi(G - R)| + 1$. As each orientation in $\mathcal{H}$ is a Pfaffian orientation of $G - R$, it follows that, $\mathcal{H}$ has at least two distinct members that are similar. Let i and j be indices such that $0 \le i < j \le |\Phi(G)|$ and $\vec{H}_i \sim \vec{H}_j$. By Lemma 3.19, $\vec{G}_i$ and $\vec{G}_j$ are also similar. By definition of $\mathcal{F}$, it follows that $i = 0$. Thus, $\vec{G}_0$ is similar to Pfaffian orientation $\vec{G}_j$ of G, whence $\vec{G}_0$ is a Pfaffian orientation of G, as asserted.

Consider next the case in which R is a double ear. Then, $\rho(R) = 1$, whence $b(G) = b(G - R) + 1$. By Theorem 3.22, $|\Phi(G)| = 2|\Phi(G - R)|$, whence $|\mathcal{H}| = 2|\Phi(G - R)| + 1$. As each orientation in $\mathcal{H}$ is a Pfaffian orientation of $G - R$, it follows that, $\mathcal{H}$ has at least three distinct members that are similar. Let i, j and k be indices such that $0 \le i < j < k \le |\Phi(G)|$ and $\vec{H}_i \sim \vec{H}_j \sim \vec{H}_k$. By Lemma 3.20, at least two of $\vec{G}_i$, $\vec{G}_j$ and $\vec{G}_k$ are similar. By definition of $\mathcal{F}$, it follows that $i = 0$ and $\vec{G}_0$ is similar to a member of $\mathcal{F}$. Thus, $\vec{G}_0$ is similar to a Pfaffian orientation of G, whence $\vec{G}_0$ is a Pfaffian orientation of G, as asserted. □

3.3 Characteristic Orientations

We now define a recursive procedure for determining an orientation of a matching covered graph. Any orientation determined by this procedure is a *characteristic orientation* of G. The procedure needs to determine (i) a $(b+p)$-removable ear R of G, and (ii) a nice circuit of Q of G that contains all the edges of R.

Theorem 3.14 ensures that if G is not K_2 then it has a $(b + p)$-removable ear. We have seen that it is possible to determine in polynomial time whether or not a graph G is matching covered, and, if G is matching covered, how to determine in polynomial time the invariant $(b+p)(G)$. It is then easy to determine whether or not an ear is $(b+p)$-removable in polynomial time. Moreover, the number of single ears that can possibly be part of a removable ear is linear on the number of edges of G. In fact, the number of single ear candidates is at most the number of edges of G. Therefore, even by brute force there is a polynomial algorithm that determines a $(b + p)$-removable ear of G.

Recall that in order to obtain a perfect matching of a matching covered graph G that contains a specified edge e, it suffices to determine a perfect matching of $G - v - w$ and then add to it edge e, where v and w are the ends of e. The proof of Corollary 3.18 then indicates how to determine in polynomial time a nice circuit of G that contains all the edges of a given removable ear of G. So, let us now describe the polynomial time procedure that produces a characteristic orientation of a matching covered graph G.

1. If G is K_2 then orient the edge of G arbitrarily and return that orientation.

2. Determine a $(b+p)$-removable ear R of G.

3. Determine a nice circuit of G that contains all the edges of R.

4. Recursively determine a characteristic orientation $\vec{H}$ for $G - R$.

5. Extend $\vec{H}$ to an orientation $\vec{G}$ of G by orienting the edges of R so that Q is oddly oriented in $\vec{G}$ and return $\vec{G}$.

Corollary 3.24
Let G be a matching covered graph, $\vec{G}$ a characteristic orientation of G. Then, G is Pfaffian if and only if $\vec{G}$ is a Pfaffian orientation of G.

Proof: Certainly, if $\vec{G}$ is a Pfaffian orientation of G then G is Pfaffian. To prove the converse, assume that G is Pfaffian. We prove that $\vec{G}$ is a Pfaffian orientation of G by induction on the size of G.

If G is K_2 then certainly $\vec{G}$ is a Pfaffian orientation of G. We may thus assume that G is not K_2. The procedure then determines a $(b+p)$-removable ear R of G, a nice circuit Q of G that contains all the edges of R and a characteristic orientation $\vec{H}$ for $G - R$. As G is Pfaffian, then $G - R$ is Pfaffian, by Proposition 3.16. By induction hypothesis, $\vec{H}$ is a Pfaffian orientation of $G - R$. Moreover, $p(G) = 0 = p(G - R)$, by Corollary 3.13. Thus, R is b-removable. As, $\vec{H} = \vec{G} - R$, then $\vec{G} - R$ is a Pfaffian orientation of $G - R$. By Corollary 3.23, $\vec{G}$ is a Pfaffian orientation of G, as asserted. □

4 The Four Problems

In this section, we consider following four problems and show that they are polynomially equivalent. That is, given a polynomial algorithm for solving one of the problems then there are polynomial algorithms for solving the other three problems. The problems are the following:

P1 Given a matching covered graph G, determine whether or not G is Pfaffian.

P2 Given a matching covered graph G, determine the number of Pfaffian similarity classes of G.

P3 Given a matching covered graph G, determine whether or not G is Pfaffian and if G is Pfaffian then give a Pfaffian orientation for G.

P4 Given an orientation $\vec{G}$ of a matching covered graph G, determine whether or not $\vec{G}$ is Pfaffian.

Corollary 4.1
Problems P1 and P2 are polynomially equivalent.

Proof: If there exists a polynomial algorithm for solving problem P2 then certainly there is a polynomial algorithm for solving problem P1. To prove the converse, assume that there is a polynomial algorithm for solving problem P1. In order to determine $\Phi(G)$, for a given matching covered graph G, in polynomial time, determine whether or not G is Pfaffian, using the given algorithm, and then evaluate $|\Phi(G)|$ to be zero if G is not Pfaffian and $2^{b(G)}$ if G is Pfaffian. □

Corollary 4.2
Problems P1 and P3 are polynomially equivalent.

Proof: If there exists a polynomial algorithm for solving problem P3 then certainly there is a polynomial algorithm for solving problem P1. To prove the converse, assume that there is a polynomial algorithm for solving problem P1. In order to solve problem P3 for a given matching covered graph G, first determine whether or not G is Pfaffian, using the given algorithm. If G is Pfaffian, then determine a characteristic orientation of G, and return that Pfaffian orientation of G. □

Corollary 4.3
Problems P1 and P4 are polynomially equivalent.

Proof: Assume that there exists a polynomial algorithm for solving problem P4. To solve problem P1 in polynomial time, let G be a matching covered graph. Determine a characteristic orientation $\vec{G}$ of G and then use the given algorithm to determine whether or not $\vec{G}$ is a Pfaffian orientation of G. Then, G is Pfaffian if and only if $\vec{G}$ is Pfaffian.

Conversely, assume that there exists a polynomial algorithm for solving problem P1. Let us now describe an algorithm for solving problem P4 in polynomial time. Let $\vec{G}$ be an orientation of a matching covered graph G. If G is K_2 then certainly $\vec{G}$ is Pfaffian. Assume thus that G is not K_2, let R be a $(b+p)$-removable ear of G, Q a nice circuit of G that contains all the edges of R. Determine if Q is oddly oriented in G: if the answer is negative then $\vec{G}$ is not Pfaffian. Recursively determine whether or not $\vec{G}-R$ is a Pfaffian orientation of $G-R$. If the answer is negative then $\vec{G}$ is not Pfaffian. Determine whether or not G is Pfaffian, using the given algorithm. If G is Pfaffian, then, by Corollary 3.13, $p(G)=0=p(G-R)$, whence R is b-removable. In that case, $\vec{G}$ is Pfaffian, by Corollary 3.23. □

4.1 Some Comments

There exists a polynomial algorithm for determining whether or not a given matching covered bipartite graph is Pfaffian (see [9, 8]). Consequently, all the above problems are polynomially solvable for bipartite matching covered graph. The status of the four problems for nonbipartite matching covered graphs is open. It is not even known whether problems P1 and P4 lie in NP.

Acknowledgment Some of the ideas presented here are original work by Marcelo H. de Carvalho, U. S. R. Murty and the second-named author. However, the proof of the classical result stated as Theorem 1.2 was entirely created by the first-named author.

References

[1] M. H. de Carvalho, C. L. Lucchesi, and U. S. R. Murty. On a conjecture of Lovász concerning bricks. I. The characteristic of a matching covered graph. *Journal of Combinatorial Theory (B)*, 85:94–136, 2002.

[2] M. H. de Carvalho, C. L. Lucchesi, and U. S. R. Murty. On a conjecture of Lovász concerning bricks. II. Bricks of finite characteristic. *Journal of Combinatorial Theory (B)*, 85:137–180, 2002.

[3] M. H. de Carvalho, C. L. Lucchesi, and U. S. R. Murty. Optimal ear decompositions of matching covered graphs. *Journal of Combinatorial Theory (B)*, 85:59–93, 2002.

[4] J. Edmonds. Paths, trees and flowers. *Canad. J. Math.*, 17:449–467, 1965.

[5] J. Edmonds, L. Lovász, and W. R. Pulleyblank. Brick decomposition and the matching rank of graphs. *Combinatorica*, 2:247–274, 1982.

[6] L. Lovász. Matching structure and the matching lattice. *Journal of Combinatorial Theory (B)*, 43:187–222, 1987.

[7] L. Lovász and M. D. Plummer. *Matching Theory*. Number 29 in Annals of Discrete Mathematics. Elsevier Science, 1986.

[8] W. McCuaig. Brace generation. *J. of Graph Theory*, 38:124–169, 2001.

[9] N. Robertson, P. D. Seymour, and R. Thomas. Permanents, Pfaffian orientations and even directed circuits. *Ann. of Math.*, (2) 150:929–975, 1999.

[10] L. Valiant. The complexity of computing the permanent. *Theoretical Computer Science*, 8:189–201, 1979.