INSTITUTO DE COMPUTAÇÃO
UNIVERSIDADE ESTADUAL DE CAMPINAS

Geometrical Cuts for 0–1 Integer Programming

N. Maculan *E. M. Macambira*
C. C. de Souza

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Geometrical Cuts for 0–1 Integer Programming

Nelson Maculan * Elder M. Macambira † Cid C. de Souza ‡

Abstract: In this technical note we introduce a set of cuts for 0–1 Integer Programming with a strong geometrical flavor. These are the spherical and cylindrical inequalities. We show that the geometrical cuts are in one-to-one correspondence with the canonical cuts introduced by Balas and Jeroslow in [2]. Moreover, we show how the well-known subtour elimination constraints for the Traveling Salesman Problem can be obtained via geometrical cuts. By presenting the subtour elimination constraints in this way, we give another easy and intuitive way to explain the validity of such inequalities. We show that the arguments used to derive the subtour elimination constraint as geometrical cut can be repeated to derive strong valid inequalities that are known for other combinatorial optimization problems.
Keywords: 0–1 Integer Programming, geometrical cuts, canonical cuts, cutting planes.

1 Introduction

In this paper we present some linear inequalities that can be used as cutting planes in the solution of 0–1 integer programs. Such inequalities, like those introduced in [1], [3], [6] and [7], are independent from the combinatorial structure of the problem been solved. Inequalities of this sort are sometimes referred as *general purpose cutting planes*. Probably the most well-known example of such inequalities are the Chvátal-Gomory cuts [4].

Our initial motivation to introduce these inequalities is to separate integer points from the set of feasible ones. That is, we are not interested to cut off a specific fractional solution from the set of feasible points of a given relaxation of the problem. Since the arguments used to derive these inequalities are geometrical, we call them *geometrical cuts*.

The text is organized as follows. Section 2 contains some basic results and definitions for a good understanding of the remaining of the text. Section 3 introduces our first class of geometrical inequalities called *spherical cuts* which is generalized in section 4. The next class of geometrical inequalities is presented in section 5 and an extension of these cuts is given in section 6. In section 7 the geometrical cuts are discussed in all its generality while in section 8 it is shown that this generalization establishes an equivalence between the cuts introduced here and those originally presented by Balas and Jeroslow in [2]. Section 9

*Programa de Engenharia de Sistemas e Computação, Universidade Federal do Rio de Janeiro, Caixa Postal: 68511, 21945-970, Rio de Janeiro, RJ - BRASIL. Supported by FINEP and FUJB.

†Programa de Engenharia de Sistemas e Computação, Universidade Federal do Rio de Janeiro, Caixa Postal: 68511, 21945-970, Rio de Janeiro, RJ - BRASIL. Supported by Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro (grant E-26/152.603/2001).

‡Instituto de Computação, Universidade Estadual de Campinas, Caixa Postal: 6176, 13081-970, Campinas, SP - BRASIL. Supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico do Brasil (grant 300883/94-3).

shows that some well-known inequalities for combinatorial optimization problems are special cases of geometrical cuts. Finally section 10 we draw some conclusions and suggest how our results could be used in computations for 0–1 Integer Programming.

2 Basic results and definitions

This section contains a few definitions and results which are referred in the forthcoming sections. These are basic definitions and results which can be found in textbooks on Integer and Linear Programming (cf,. [5]). The only exception is theorem 2.1 which is stated and proved at the end of this section.

Definition 2.1 *The* Hamming distance *between a pair of points x and y in $\mathbb{B}^n$, is denoted by $d(x, y)$ and corresponds to the number of elements in which they differ, i. e., $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$.*

Definition 2.2 *An $m \times n$ integral matrix A is* totally unimodular *(TU) if the determinant of each square submatrix of A is equal to 0, 1 or -1.*

Proposition 2.1 *If A is TU then $\begin{pmatrix} A \\ I \end{pmatrix}$ is TU.*

Corollary 2.1 *Let A be a $(0, 1, -1)$ matrix with no more than two nonzero elements in each column. Then A is TU if and only if the rows in A can be partitioned into two subsets Q_1 and Q_2 such that if a column contains two nonzero elements, the following statements are true:*

1. *If both nonzero elements have the same sign, then one is in a row contained in Q_1 and the other is in a row contained in Q_2.*

2. *If the two nonzero elements have opposite sign, then both are in rows contained in the same subset.*

Definition 2.3 *A polyhedron $P \subseteq \mathbb{R}^n$ is* integral *if all its extreme points have integer coordinates.*

Proposition 2.2 *Let $P = \{x \in \mathbb{R}^n : Ax = b\}$ be a non empty polyhedron. Suppose that A is an $m \times n$ matrix and $b \in \mathbb{Z}^m$. If A is totally unimodular then P is integral.*

Definition 2.4 *Given a full-dimensional polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, y is said to be an* interior *point of P if $Ay < b$.*

Proposition 2.3 *Every full-dimensional polyhedron has an interior point.*

Proposition 2.4 *$S = \{x_0, x_1, \ldots, x_n\}$ is a set of affinely independent points in $\mathbb{R}^n$ if and only if the points in $S' = \{x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0\}$ are linearly independent.*

Proposition 2.5 *If $S = \{x_1, \ldots, x_n\}$ is a set of linearly independent points in $\mathbb{R}^n$ then, for every constant $\alpha \neq 0$, the points in $S' = \{\alpha x_1, \ldots, x_n\}$ are also linearly independent.*

Theorem 2.1 *Let $P = \{x \in \mathbb{R}^n : \pi x \leq \pi_0, a_i x \leq b_i \text{ for } i = 1, \ldots, m\}$ be a full-dimensional polyhedron and $P' = \{x \in \mathbb{R}^n : a_i x \leq b_i \text{ for } i = 1, \ldots, m\}$. If $P' \setminus P \neq \emptyset$ then $\pi x \leq \pi_0$ defines a facet of P.*

Proof: Let us start by assuming that $\pi_0 \neq 0$ since if this is the case we can always translate the origin to satisfy this condition.

From proposition 2.3, there exists an interior point y in P. Then, there exists a scalar $\epsilon > 0$ such that the ball centered at y with radius ϵ, denoted by $B(y, \epsilon)$, only contains points which are interior to P.

Moreover, since $P' \setminus P \neq \emptyset$, there exists $z \in P'$ such that $\pi z > \pi_0$.

Consider the hyperplane passing through point y and which is orthogonal to $(y - z)$. This hyperplane is given by $H = \{x \in \mathbb{R}^n : (y - z)^T x = (y - z)^T y\}$. Since $H \cap B(y, \epsilon)$ is an $(n-1)$-dimensional sphere (centered at y and with radius ϵ), it contains a set with n affinely independent points, say $\{y^1, \ldots, y^n\}$. Now, because $z \notin H$, the points in $\{z, y^1, \ldots, y^n\}$ are affinely independent. Thus, by proposition 2.4, the points in $\{y^1 - z, \ldots, y^n - z\}$ are linearly independent.

Since for every $i = 1, \ldots, n$, the segment $\overline{zy^i}$ intersects the hyperplane $\{x \in \mathbb{R}^n : \pi x = \pi_0\}$, there exists a scalar $\alpha_i > 0$ such that $u^i = z + \alpha_i (y^i - z)$ satisfies $\pi u^i = \pi_0$. Hence u^i belongs to P and from proposition 2.5 the points in $\{u^1 - z, \ldots, u^n - z\}$ are linearly independent. Now, because $\pi z \neq \pi_0$, according to proposition 2.4 the points in $\{z, u^1, \ldots, u^n\}$ are affinely independent. Thus, the face $F(\pi, \pi_0) = \{x \in P : \pi x = \pi_0\}$ has dimension $n - 1$ and the proof is complete. □

3 1-spherical cuts

In this section we characterize the inequalities that define the convex hull of $X_1(\overline{x}) = \mathbb{B}^n \setminus \{\overline{x}\}$, for $n > 1$ and for some $\overline{x} \in \mathbb{B}^n$. To this end, we first obtain a linear inequality which is valid for all point in $\mathbb{B}^n$ but $\overline{x}$.

Consider the sphere S with radius one and centered at $\overline{x}$. The points in S are those satisfying

$$\sum_{j=1}^{n} (x_j - \overline{x}_j)^2 = 1. \tag{1}$$

If we want to characterize the points whose distance to $\overline{x}$ is at least one than we obtain

$$\sum_{j=1}^{n} (x_j - \overline{x}_j)^2 \geq 1. \tag{2}$$

Now since we are only interested in points belonging to $\mathbb{B}^n$ and, for such points, we have that $x_j^2 = x_j$, inequality (2) can be written as:

$$\begin{aligned}
\sum_{j=1}^{n}(x_j^2 - 2x_j\overline{x}_j + \overline{x}_j^2) &\geq 1 \Longrightarrow \\
\sum_{j=1}^{n}(x_j - 2x_j\overline{x}_j + \overline{x}_j) &\geq 1 \Longrightarrow \\
\sum_{j=1}^{n}(1 - 2\overline{x}_j)x_j &\geq 1 - \sum_{j=1}^{n}\overline{x}_j. \qquad (3)
\end{aligned}$$

We call inequality (3), the *1-spherical inequality* for $\overline{x}$. Notice that this inequality is valid for all points in $\mathbb{B}^n$ whose Hamming distance to $\overline{x}$ is at least one. Moreover, the 0–1 points satisfying it at equality are precisely those points that are adjacent to $\overline{x}$ in the n-dimensional hypercube.

Figure 1 shows an example of the 1-spherical inequality for point $(1, 1, 1)$ in the three dimensional case. The corresponding inequality is given by $x_1 + x_2 + x_3 \leq 2$.

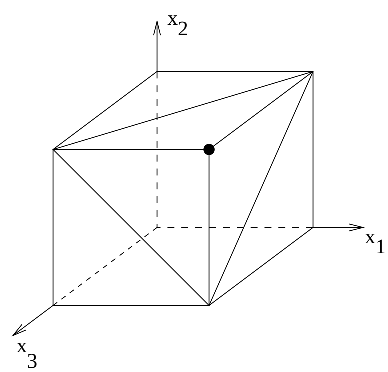

Figure 1: 1–spherical inequality $x_1 + x_2 + x_3 \leq 2$ for $\overline{x} = (1, 1, 1)$.

Thus, consider the linear system given by

$$\mathcal{S}_1(\overline{x}) = \left\{ x \in \mathbb{R}^n : 0 \leq x_j \leq 1, \; j = 1, \ldots, n, \; \sum_{j=1}^{n}(1 - 2\overline{x}_j)x_j \geq 1 - \sum_{j=1}^{n}\overline{x}_j \right\}. \qquad (4)$$

Below we show that $\mathcal{S}_1(\overline{x})$ is an integral polyhedron and that (3) defines a facet of it.

Theorem 3.1 *The polyhedron $\mathcal{S}_1(\overline{x})$ is integral.*

Proof: Consider the linear subsystem of $\mathcal{S}_1(\overline{x})$ formed by the 1–spherical inequality and the nonnegativity constraints on the x variables. The constraint matrix of this subsystem is totally unimodular. To see this, it suffices to observe that it has no more than two nonzero elements in each column and that the rows of this subsystem can be divided according to Corollary 2.1 as follows. The row corresponding to the 1–spherical inequality is assigned to the set Q_1. For each $j \in \{1, \ldots, n\}$, the row corresponding to the nonnegativity constraint $x_j \geq 0$ is assigned to Q_1 if $\overline{x}_j = 1$ and to Q_2 otherwise.

Finally, from Proposition 2.1, we conclude that the linear system $\mathcal{S}_1(\overline{x})$ is totally unimodular and, therefore, the polyhedron is integral. □

Theorem 3.2 *The polyhedron $\mathcal{S}_1(\overline{x})$ is full-dimensional for every $\overline{x}$ in $\mathbb{B}^n$.*

Proof: It suffices to consider the case where $\overline{x} = (1, 1, \ldots, 1)$ (the proof for all other points in $\mathbb{B}^n$ follows from rotation and translation). Thus, for $n > 1$, the points $(0, 0, \ldots, 0)$ and e_i for $i = 1, \ldots, n$ belong to $X_1(\overline{x})$ (where e_i is the n vector with all elements equal zero but the i-th element which is one). Therefore, X_1 has $n + 1$ affinely independent points. This concludes the proof. □

Theorem 3.3 *The 1–spherical inequality defines a facet of $\mathcal{S}_1(\overline{x})$ for every $\overline{x}$ is $\mathbb{B}^n$.*

Proof: Again it is enough to show that the theorem is valid for $\overline{x} = (0, 0, \ldots, 0)$. The proof is immediate from theorems 3.2 and 2.1. □

4 p-spherical cuts

We can generalize the results from section 3 in the following way. Given a point $\overline{x} \in \mathbb{B}^n$, let $W_p(\overline{x})$ be the set of all points in $\mathbb{B}^n$ whose distance to $\overline{x}$ is at most $p - 1$ where $0 < p < n$ and $n > 1$. Moreover, let $X_p(\overline{x}) = \mathbb{B}^n \setminus W_p(\overline{x})$.

The points in $X_p(\overline{x})$ satisfy

$$\sum_{j=1}^{n} (x_j - \overline{x}_j)^2 \geq p. \tag{5}$$

As in the previous section, since $x_j^2 = x_j$, for all $j = 1, \ldots, n$, we can linearize inequality (5) to obtain the p-spherical inequality

$$\sum_{j=1}^{n} (1 - 2\overline{x}_j) x_j \geq p - \sum_{j=1}^{n} \overline{x}_j. \tag{6}$$

As before, we can define the polyhedron below

$$\mathcal{S}_p(\overline{x}) = \left\{ x \in \mathbb{R}^n : 0 \leq x_j \leq 1, \;\; j = 1, \ldots, n, \;\; \sum_{j=1}^{n} (1 - 2\overline{x}_j) x_j \geq p - \sum_{j=1}^{n} \overline{x}_j \right\} \tag{7}$$

Assuming that $n \geq 2$ and $0 < p < n$, similar results to those presented in section 3 can be shown. These results are now stated as follows.

Theorem 4.1 *The polyhedron $\mathcal{S}_p(\overline{x})$ is integral.*

Theorem 4.2 *The polyhedron $\mathcal{S}_p(\overline{x})$ is full-dimensional for every $\overline{x}$ in $\mathbb{B}^n$.*

Theorem 4.3 *The 1–spherical inequality defines a facet of $\mathcal{S}_p(\overline{x})$ for every $\overline{x}$ is $\mathbb{B}^n$.*

The proof of theorem 4.1 follows the same steps as those given in its analogous counterpart in section 3, namely theorem 3.1.

To prove theorems 4.2 and 4.3, we first assume that $\overline{x} = (1, 1, \ldots, 1)$. Then, it suffices to observe that the set of points represented in the columns of the matrix below belongs to $\mathcal{S}_p(\overline{x})$ and are at distance p from $\overline{x}$. Clearly, these points are affinely independent and since 0 is also in $\mathcal{S}_p(\overline{x})$ we have all the arguments to complete the proofs.

$$
\left.\begin{array}{|ccccc|ccccc|}
\hline
0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 1 \\
\hline
0 & 1 & \ldots & 1 & 1 & 0 & 1 & \ldots & 1 & 1 \\
1 & 0 & \ldots & 1 & 1 & 0 & 0 & \ldots & 1 & 1 \\
1 & 1 & \ldots & 1 & 1 & 1 & 0 & \ldots & 1 & 1 \\
1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 0 & 1 \\
1 & 1 & \ldots & 0 & 1 & 1 & 1 & \ldots & 0 & 0 \\
1 & 1 & \ldots & 1 & 0 & 1 & 1 & \ldots & 1 & 0 \\
\hline
\end{array}\right.
\begin{array}{l}
\left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\end{array}}\right\} p-1 \\
\left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\\0\\0\\0\end{array}}\right\} n-p+1
\end{array}
$$

Clearly, inequality (6) generalizes inequality (3) since the latter corresponds to the case where p equals one.

5 Cylindrical cuts

In this section we consider the possibility of deriving a linear inequality which can cut off two points of $\mathbb{B}^n$ simultaneously. The example in Figure 2 shows that, if $\overline{x}^1$ and $\overline{x}^2$ are these two points, the addition of the two corresponding 1–spherical inequalities to the linear system describing the n-dimensional hypercube does not produce an integral polyhedron. Moreover, it is easy to see that if $\overline{x}^1$ and $\overline{x}^2$ are not adjacent, there exists no hyperplane separating $\overline{x}^1$ and $\overline{x}^2$ from the remaining points in $\mathbb{B}^n$.

Our goal is then to derive a linear inequality which is capable to cut off a pair of adjacent points of $\mathbb{B}^n$ simultaneously. Initially, we observe that such points differ in exactly one component, say $k \in \{1, \ldots, n\}$. Thus, for the pair $(\overline{x}^1, \overline{x}^2)$, we have that $d(\overline{x}^1, \overline{x}^2) = 1$.

If we are to separate $\overline{x}^1$ and $\overline{x}^2$ which differ only at the k-th component from the remaining points in $\mathbb{B}^n$, then we are interested in the points of $\mathbb{B}^n$ satisfying

$$\sum_{j=1, j \neq k}^{n} (x_j - \overline{x}_j^1)^2 \geq 1. \tag{8}$$

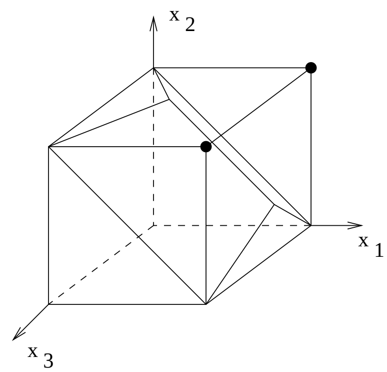

Figure 2: The 1–spherical inequalities $x_1 + x_2 + x_3 \leq 2$ and $x_1 + x_2 - x_3 \leq 1$ corresponding to the points $\overline{x}^1 = (1, 1, 1)$ and $\overline{x}^2 = (1, 1, 0)$ respectively.

Notice that, this inequality ensures that the x point differ either from $\overline{x}^1$ or $\overline{x}^2$ in at least one component or, in other words, the Hamming distance between x and $\overline{x}$ is at least one.

Now, developing the left-hand side of the above inequality and using the fact that both x and $\overline{x}_j^1$ are 0–1 vectors, we have that

$$\begin{aligned}
\sum_{j=1, j \neq k}^{n} (x_j^2 - 2x_j\overline{x}_j^1 + (\overline{x}_j^1)^2) &\geq 1 \Longrightarrow \\
\sum_{j=1, j \neq k}^{n} (x_j - 2x_j\overline{x}_j^1 + \overline{x}_j^1) &\geq 1 \Longrightarrow \\
\sum_{j=1, j \neq k}^{n} [(1 - 2\overline{x}_j^1)x_j + \overline{x}_j^1] &\geq 1 \Longrightarrow \\
\sum_{j=1, j \neq k}^{n} (1 - 2\overline{x}_j^1)x_j &\geq 1 - \sum_{j=1, j \neq k}^{n} \overline{x}_j^1 \qquad (9)
\end{aligned}$$

Inequality (9) is called the *1–cylindrical cut* with respect to points $\overline{x}^1$ and $\overline{x}^2$. Figure 3 below shows an example of such inequalities for the three-dimensional cube. As suggested by this picture, the inequality is used to eliminate from the cube those points of $\mathbb{B}^n$ which lie in the interior of the cylinder whose axis is the straight line going through the points $\overline{x}^1$ and $\overline{x}^2$.

In the next section we generalize the 1–cylindrical inequality and and discuss about the polyhedron obtained when we intersect the n-dimensional hypercube and the half space defined by the a generalization of inequality (9).

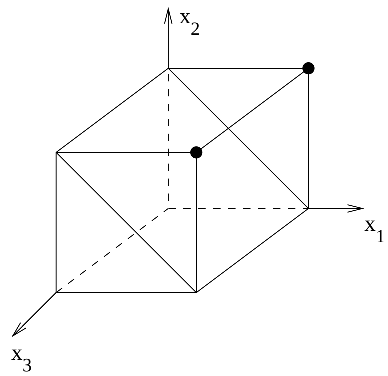

Figure 3: The *1–cylindrical cut* $x_1 + x_2 \leq 1$ for points $\overline{x}^1 = (1,1,0)$ and $\overline{x}^2 = (1,1,1)$.

6 p-cylindrical cuts

Suppose that $n > 2$ and $0 < p < n-1$ and let $\overline{x}^1$ and $\overline{x}^2$ be two adjacent points in $\mathbb{B}^n$. Let $W_p(\overline{x}^1, \overline{x}^2)$ be the set of points in $\mathbb{B}^n$ whose distance to $\overline{x}^1$ or to $\overline{x}^2$ is strictly less than p. In order to generalize inequality (9), we derive a linear inequality which is valid for $\mathbb{B}^n \setminus W_p(\overline{x}^1, \overline{x}^2)$. If k is the only component in which $\overline{x}^1$ and $\overline{x}^2$ differ, then this inequality is written as

$$\sum_{j=1, j\neq k}^{n} (1 - 2\overline{x}_j^1) x_j \geq p - \sum_{j=1, j\neq k}^{n} \overline{x}_j^1. \tag{10}$$

Let us define the polyhedron $\mathcal{C}_p(\overline{x}^1, \overline{x}^2)$ such that

$$\mathcal{C}_p(\overline{x}^1, \overline{x}^2) = \left\{ x \in \mathbb{R}^n : 0 \leq x_j \leq 1, \ \ j = 1, \ldots, n, \ \ \sum_{j=1, j\neq k}^{n} (1 - 2\overline{x}_j^1) x_j \geq p - \sum_{j=1, j\neq k}^{n} \overline{x}_j^1 \right\}. \tag{11}$$

Then the following theorems hold.

Theorem 6.1 *The polyhedron $\mathcal{C}_p(\overline{x}^1, \overline{x}^2)$ is integral.*

Proof: analogous to that of Theorem 3.1. □

Theorem 6.2 *The polyhedron $\mathcal{C}_p(\overline{x}^1, \overline{x}^2)$ is full-dimensional for every pair of adjacent points $\overline{x}^1$ and $\overline{x}^2$ in $\mathbb{B}^n$.*

Proof: The proof is done for $\overline{x}^1 = (1, \ldots, 1, 1)$ and $\overline{x}^2 = (1, \ldots, 1, 0)$ since the arguments hold for any other pair of adjacent point in $\mathbb{B}^n$ as it can be seen by applying rotation and translation.

Notice that we are under the assumption that $n > 2$ and $0 < p < n-1$. Then, the zero vector and the vectors e_i, for all $i = 1, \ldots, n$ are in $\mathcal{C}_p(\overline{x}^1, \overline{x}^2)$. Therefore, we have enough affinely independent points to complete the proof. □

Theorem 6.3 *The p-cylindrical inequality defines a facet of* $\mathcal{C}_p(\overline{x}^1, \overline{x}^2)$ *for every pair of adjacent points* $\overline{x}^1$ *and* $\overline{x}^2$ *in* $\mathbb{B}^n$*.*

Proof: Immediate from theorems 6.2 and 2.1. □

7 General form of a geometrical cut

The geometrical cuts from the two previous sections can be generalized. To see how this generalization is done, assume that V is the set of vertices of the n-dimensional hypercube. Moreover, let X^i denote a set of points in V which have exactly i components that are equal, where $i \in \{1, \ldots, n\}$. Clearly, we have that $|X^i| = 2^{n-i}$.

Notice that the spherical cut in (6) is an inequality that is valid only for the points of V which are at a distance at least p from any point in a set X^n.[1]

On the other hand, the cylindrical cut in (10) is an inequality that is valid only for the points in V which are at a distance at least p from any point in a set X^{n-1}.

Therefore, in general, a geometrical cut is an inequality that is valid only for the 0–1 points which are at a distance at least p from any point in a set X^k, where X is a set of points in V which differ in exactly k components. If $\overline{x}$ is a point in X, then this inequality is given by

$$\sum_{j=1, j \notin K}^{n} (1 - 2\overline{x}_j) x_j \geq p - \sum_{j=1, j \notin K} \overline{x}_j, \tag{12}$$

where K is the set of index in $\{1, \ldots, n\}$ for which the components of the points in X^k are the same.

8 Canonical cuts

At this point it is interesting to establish the relation between the cuts given so far and the canonical cuts introduced by Balas and Jeroslow in [2]. We show that both the p-spherical and p-cylindrical inequalities turn out to be special cases of the canonical cuts. Moreover, from the generalization of the geometrical cuts given in the previous section we show that the canonical and geometrical cuts are indeed equivalent. Thus, the work developed here can be viewed as a reinterpretation of the original results of Balas and Jeroslow.

Consider the n-dimensional hypercube K_n

$$K_n = \{x \in \mathbb{R}^n \mid 0 \leq x_j \leq 1 \,, j \in N\}, \tag{13}$$

[1] Clearly, in this case, X has only one point.

where $N = \{1, \ldots, n\}$. Clearly, every vertex x of K_n is a 0–1 vector satisfying n inequalities of (13) at equality. Let V be the set vertices of K_n.

A face F^k of K_n of dimension k, for $0 \leq k < n$, is a set of points of K_n satisfying exactly $n-k$ inequalities of (13) at equality. Thus all points in $V \cap F^k$ have $n-k$ identical components (those associated to the $n-k$ equations $x_j = 0$ or $x_j = 1$ which define the face F^k).

Given $d \in \{0, 1, \ldots, n-k\}$, the canonical hyperplane of order k associated to a face F^k of K_n denoted by $H(F^k)_d$ is given by

$$\sum_{i \in N(F^k)^+} x_i - \sum_{i \in N(F^k)^-} x_i = |N(F^k)^+| - d, \tag{14}$$

where

$$N(F^k)^+ = \{j \in N \mid x_j = 1 \text{ for all } x \in V \cap F^k\}$$

and

$$N(F^k)^- = \{j \in N \mid x_j = 0 \text{ for all } x \in V \cap F^k\}.$$

Let $H(F^k)_d^+$ be the closed halfspace defined by the canonical hyperplane $H(F^k)_d$ when we replaced the equality in equation (14) by an inequality of the form '$\leq$'. The halfspace $H(F^k)_d^+$ is called a *canonical cut*.

When $k = 0$, we consider zero-dimensional face of K_n, that is, F^0 coincides with a point $\overline{x}$ in V. In this case, we obtain the canonical cut $H(F^0)_d$ of the form

$$\begin{aligned}
\sum_{i|\overline{x}_i=1} x_i - \sum_{i|\overline{x}_i=0} x_i &\leq \sum_{i|\overline{x}_i=1} \overline{x}_i - d \Longrightarrow \\
\sum_{i=1}^{n} (2\overline{x}_i - 1)x_i &\leq \sum_{i=1}^{n} \overline{x}_i - d \Longrightarrow \\
\sum_{i=1}^{n} (1 - 2\overline{x}_i)x_i &\geq d - \sum_{i=1}^{n} \overline{x}_i.
\end{aligned}$$

The last inequality is precisely the d–spherical cut for point $\overline{x}$ described in section 4.

Now suppose that $k = 1$ and consider the 1-dimensional face of K_n, that is, F^1 coincides with the straight line segment of unitary length joining two adjacent vertices $\overline{x}^1$ and $\overline{x}^2$ of V. Suppose that $\overline{x}^1$ and $\overline{x}^2$ differ only at a component k. In this case, we obtain the canonical cut $H(F^1)_d$ of the form

$$\begin{aligned}
\sum_{i|\overline{x}_i^1=\overline{x}_i^2=1} x_i - \sum_{i|\overline{x}_i^1=\overline{x}_i^2=0} x_i &\leq \sum_{i|\overline{x}_i^1=\overline{x}_i^2=1} \overline{x}_i^1 - d \Longrightarrow \\
\sum_{i|\overline{x}_i^1=\overline{x}_i^2} (2\overline{x}_i^1 - 1)x_i &\leq \sum_{i|\overline{x}_i^1=\overline{x}_i^2=1} \overline{x}_i^1 - d \Longrightarrow \\
\sum_{i=1, i\neq k}^{n} (1 - 2\overline{x}_i^1)x_i &\geq d - \sum_{i=1, i\neq k}^{n} \overline{x}_i^1.
\end{aligned}$$

The last inequality is precisely the d–cylindrical cut for points $\overline{x}^1$ and $\overline{x}^2$ described in section 6.

Following the discussion above, one can easily obtain that the canonical cut for a face F^k general geometrical cut with respect to the hyperplane $H(F^k)_d^+$ is equivalent to the geometrical cut of the form of equation (12) computed for some point $\overline{x} \in F^k$ when $d = p$.

9 Subtour elimination inequalities and geometrical cuts

The purpose of this section is to show that the well-known subtour elimination constraint for the Traveling Salesman Problem (TSP) corresponds to a geometrical cut or, according to the previous section, to a canonical cut.

Consider the graph $G = (V, E)$ and let S be a subset of vertices of G satisfying $2 < |S| \leq |V|/2$. The subtour elimination constraint for the TSP is the inequality

$$\sum_{e \in E(S)} x_e \leq |S| - 1, \tag{15}$$

where $E(S)$ is the set of edges in the graph induced by S in G.

Now, consider the set X of all 0–1 vectors in $\mathbb{R}^{|E|}$ whose components associated to edges in $E(S)$ are 1, in other words, X is the set of vertices of the $|E|$-dimensional hypercube which lie on the face defined by the equations $x_e = 1$ for all $e \in E(S)$.

Essentially, equation (15) says that all incidence vectors of hamiltonian cycles in G have to be at distance at least

$$\binom{|S|}{2} - (|S| - 1)$$

from every point in X.

Assume that the edges of $E(S)$ are those corresponding to the first $|E(S)|$ components of the 0–1 vectors. Thus, if $\overline{x}$ is a point in X, the statement above can be written in terms of a geometrical cut as

$$\sum_{e \in E(S)} (1 - 2\overline{x}_e) x_e \geq \binom{|S|}{2} - (|S| - 1) - \sum_{e \in E(S)} \overline{x}_e.$$

However, by the definition of X, $\overline{x}_e = 1$ for all $e \in E(S)$. Therefore, the inequality above can be rewritten as

$$-\sum_{e \in E(S)} x_e \geq \binom{|S|}{2} - (|S| - 1) - \binom{|S|}{2},$$

or, after simplifying terms,

$$\sum_{e \in E(S)} x_e \leq |S| - 1$$

which is precisely the subtour elimination constraint given in (15).

In general, if P is an integral polytope in $\mathbb{B}^n$ and $\sum_{j\in S} x_j \leq k$ is a valid inequality for P for some $S \subseteq \{1,\ldots,n\}$, this inequality can be written as a geometrical cut of the form

$$\sum_{j\in S}(1-2\overline{x}_j)x_j \geq (n-k) - \sum_{j\in S}\overline{x}_j, \tag{16}$$

where $\overline{x}$ is the incidence vector of S.

This includes several well-known inequalities for problems in combinatorial optimization such as the cover inequalities for the knapsack polytope and the clique inequalities for the clique polytope.

10 Conclusions

In this paper we introduce the geometrical cuts for 0–1 Integer Programming. These cuts are shown to be equivalent to the canonical cuts introduced in [2] in the early 70's. Several well-known valid and facet defining inequalities for problems in combinatorial optimization are shown to be special cases of geometrical cuts.

Geometrical cuts can be used computationally in cutting plane methods to eliminate integer points which are infeasible, like it is the case for the subtour elimination constraints for the TSP, or to eliminate feasible solutions whose costs have already been evaluated. In the latter case, the goal is to reduce the feasibility region aiming that the geometrical cuts added to the formulation are strong valid inequalities for the convex hull of the remaining integer points (those not cut off by the geometrical cuts).

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