INSTITUTO DE COMPUTAÇÃO
UNIVERSIDADE ESTADUAL DE CAMPINAS

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M. H. de Carvalho *C. L. Lucchesi*
U. S. R. Murty

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Marcelo H. de Carvalho*
Dept. of Computing and Statistics,
UFMS, Campo Grande, Brasil

Cláudio L. Lucchesi†
Institute of Computing,
UNICAMP, Campinas, Brasil

U. S. R. Murty‡
University of Waterloo, Waterloo, Canada

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Abstract

A graph with at least two vertices is *matching covered* if it is connected and each edge lies in some perfect matching. A matching covered graph G is *extremal* if the number of perfect matchings of G is equal to the dimension of the lattice spanned by the set of incidence vectors of perfect matchings of G. We first establish several basic properties of extremal matching covered graphs. In particular, we show that every extremal brick may be obtained by splicing graphs whose underlying simple graphs are odd wheels. Then, using the main theorem proved in [2] and [3], we find all the extremal cubic matching covered graphs.

1 Introduction

All graphs considered in this paper are loopless. An edge e of a graph G is a *multiple edge* if there are at least two edges of G with the same ends as e. For terminology and notation not defined here, we refer the reader to [1] and [7]. This work may be regarded as an application of the techniques developed in our earlier papers [2] and [3].

Petersen (1890) proved that every 2-connected cubic graph has a perfect matching (see [1]). Tutte (1947) proved the following theorem which characterizes graphs that have a perfect matching.

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THEOREM 1.1 (TUTTE)
A graph G has a perfect matching if and only if,

$$|\mathcal{O}(G-B)| \leq |B|$$

for all $B \subset V$, where $\mathcal{O}(G-B)$ denotes the set of odd components of $G-B$.

Using the above theorem it can be shown that every edge in a 2-connected cubic graph is in some perfect matching of the graph. Thus, every such graph has at least three perfect matchings. Lovász and Plummer have conjectured that there exist constants $c > 0$ and $d > 1$ such that every 2-connected cubic graph on n vertices has at least cd^n perfect matchings (see [7]). This conjecture has been verified for bipartite cubic graphs (see [7]) but, beyond this, not much seems to be known about the number of perfect matchings in 2-connected cubic graphs.

A graph is *matching covered* if it has at least two vertices, is connected and, given any edge of the graph, there is a perfect matching of the graph that contains it. As noted above, using Theorem 1.1 it can be shown that every 2-connected cubic graph is matching covered. Figure 1 shows four cubic graphs which have played important roles in our work.

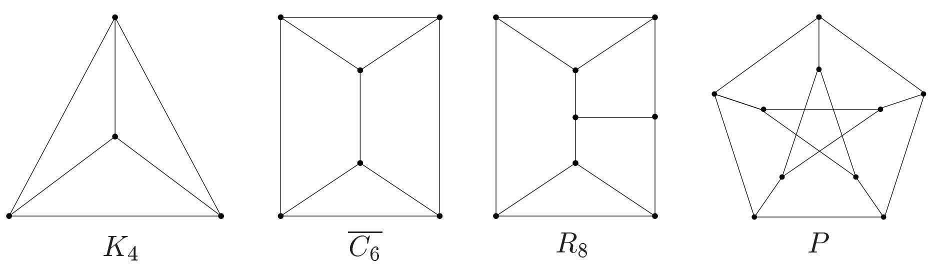

Figure 1: Four important cubic matching covered graphs

For any graph G, we denote by $\mathcal{M}(G)$ the set of all perfect matchings of G and for $M \in \mathcal{M}(G)$, we denote the incidence vector of M by χ^M. The *matching lattice* of a matching covered graph G is the set of all integer linear combinations of vectors in $\{\chi^M : M \in \mathcal{M}(G)\}$. We denote the numbers of vertices, edges and bricks of a matching covered graph G by $m(G)$, $n(G)$ and $b(G)$, respectively. (The invariant $b(G)$ will be explained in greater detail in the next section.) Whenever G is understood, we shall simply write $\mathcal{M}$, m, n and b, instead of $\mathcal{M}(G)$, $m(G)$, $n(G)$ and $b(G)$, respectively.

Edmonds, Lovász and Pulleyblank [5] proved that the dimension $\dim(G)$ of the matching lattice of a matching covered graph G is equal to $m-n+2-b$. Clearly the number of perfect matchings in G is at least $\dim(G)$. A matching covered graph G is *extremal* if $|\mathcal{M}(G)| = dim(G)$. For brevity, we shall refer to an extremal matching covered graph as an extremal graph.

It can be shown that a 2-connected cubic graph on n vertices has at most $n/4$ bricks. Thus, the dimension formula mentioned above implies that every 2-connected cubic graph on n vertices has at least $n/4 + 2$ perfect matchings. It occurred to us that if the Lovász and Plummer conjecture were true, then there can only be a finite number of extremal 2-connected cubic graphs. This does indeed turn out to be true; we show that there are precisely eleven extremal cubic matching covered graphs. The *theta* graph, the cubic bipartite graph with precisely two vertices, is one of them. The four graphs shown in Figure 1 are also extremal. We shall show that every graph in this list, other than the theta graph and the Petersen graph, can be obtained by splicing copies of K_4.

In the next section, we shall briefly recall the necessary terminology from the theory of matching covered graphs. In section 3 we shall establish basic properties of extremal graphs. In section 4 we shall give a characterization of bipartite extremal graphs. In section 5 we shall give a characterization of extremal bricks. Finally, in section 6, we shall show that there are precisely eleven extremal cubic graphs.

2 Splicing and Separation

Let G and H be two disjoint graphs and let u and v be vertices of G and H, respectively, such that the degrees of u in G and of v in H coincide. Let d denote the common degree of u and v. Further, suppose that an enumeration $\pi := (e_1 = u_1u, e_2 = u_2u, ..., e_d = u_du)$ of the edges of G incident with u, and an enumeration $\sigma := (f_1 = v_1v, f_2 = v_2v, ..., f_d = v_dv)$ of the edges of H incident with v are given. Then the graph obtained from $G - u$ and $H - v$ by joining, for $1 \leq i \leq d$, u_i and v_i by a new edge is said to be obtained by *splicing* G and H *at* the specified vertices u and v with respect to the given enumerations π and σ of the sets of edges of G and H incident with u and v, respectively.

In general, the graph resulting from splicing two graphs G and H depends on the choice of u, v, π and σ. For example, there are several ways in which two 5-wheels can be spliced at their hubs; the graphs that can be obtained by choosing various enumerations of edge sets incident with their hubs include the pentagonal prism and the Petersen graph. However, if G is cubic, then, up to isomorphism, the graph obtained by splicing G and $H = K_4$ depends only on the choice of the vertex u in G. In this case we shall denote the graph by $(G \odot K_4)_u$ and say that it is obtained by *splicing G and K_4 at u*. If G is vertex transitive, even the choice of the vertex u is irrelevant; in this case, we shall simply denote the resulting graph by $G \odot K_4$. As examples, let us consider the graphs $\overline{C}_6$ and R_8 shown in Figure 1. It is easy to see that $\overline{C}_6 = K_4 \odot K_4$. The graph $\overline{C}_6$ is clearly vertex transitive; there is only one way of splicing $\overline{C}_6$ and K_4 and $\overline{C}_6 \odot K_4 = R_8$. The automorphism group of R_8 has three orbits and, therefore, it is possible to obtain three different graphs by splicing R_8 and K_4.

The following Proposition is a simple consequence of the definition of a matching covered graph.

Proposition 2.1
Any graph obtained by splicing two matching covered graphs is also matching covered.

Let G be a graph. Then, for any subset X of V, $\nabla(X)$ denotes the cut of G with X and $\overline{X} := V - X$ as its *shores*; in other words, $\nabla(X)$ is the set of all edges of G which have precisely one end in X. A cut $\nabla(X)$ is *trivial* if either X or $\overline{X}$ is a singleton.

Let G be a connected graph and let $C := \nabla(X)$ be a cut of G, where X is a nonnull proper subset of $V(G)$. Then, the graph obtained from G by contracting $\overline{X}$ to a single vertex $\overline{x}$ is denoted by $G\{X;\overline{x}\}$ and the graph obtained from G by contracting X to a single vertex x is denoted by $G\{\overline{X};x\}$. We shall refer to these two graphs $G\{X;\overline{x}\}$ and $G\{\overline{X};x\}$ as the *C-contractions* of G. If the names of the new vertices in the C-contractions are irrelevant, we shall simply denote the two C-contractions of G by $G\{X\}$ and $G\{\overline{X}\}$.

Let G be a matching covered graph and let X be an odd subset of V. Then, the cut $C := \nabla(X)$ is called a *separating cut* of G if the two C-contractions G_1 and G_2 of G are matching covered. The following proposition provides conditions under which a cut of a matching covered graph is a separating cut.

Proposition 2.2 (See [2])
A cut C of a matching covered graph G is a separating cut of G if and only if, given any edge e of G, there exists a perfect matching M_e of G such that $e \in M_e$ and $|C \cap M_e| = 1$. □

Suppose that G is a matching covered graph that is obtained by splicing G_1 and G_2 at vertex u of G_1 and v of G_2. Then, $C := \nabla(V(G_1) - u) = \nabla(V(G_2) - v)$ is a separating cut of G. Conversely, suppose that G is a matching covered graph, C a separating cut of G, and G_1 and G_2 the two C-contractions of G. Then G can be obtained by suitably splicing G_1 and G_2. Thus, the two operations of splicing and separation may be regarded as the opposites of each other.

2.1 Tight Cuts, Braces and Bricks

Let G be a matching covered graph. A cut $C := \nabla(X)$ of G is *tight* if $|C \cap M| = 1$ for every $M \in \mathcal{M}$. It is easy to see that every trivial cut of a matching covered graph G is tight. The following result can be easily deduced from the definitions:

Proposition 2.3
For any tight cut C of a matching covered graph, the two C-contractions of G are also matching covered.

Thus every tight cut of a matching covered graph G is a separating cut of G. But the converse is not true. For example, the graphs $\overline{C}_6$, R_8 and P of Figure 1 have separating cuts that are not tight.

A *barrier* of a graph G is a nonnull set of vertices B such that $|\mathcal{O}(G-B)| = |B|$. A barrier is *trivial* if it consists of just one vertex. A matching covered graph is *bicritical* if it is free of nontrivial barriers. By Theorem 1.1, a matching covered graph G is bicritical if and only if $G - u - v$ has a perfect matching, for each pair u, v of distinct vertices u and v of G.

Lemma 2.4
Any splicing of two bicritical graphs is bicritical.

Proof: Let G be a matching covered graph, C a separating cut of G such that each C-contraction of G is bicritical.

Let $G_1 := G\{X; \overline{x}\}$ and $G_2 := G\{\overline{X}; x\}$ denote the two C-contractions of G. Let u and v denote two distinct vertices of G.

We assert that $G - u - v$ has a perfect matching. For this, adjust notation so that vertex u lies in X. If vertex v also lies in X then $G_1 - u - v$ has a perfect matching that is extendable to a perfect matching of $G - u - v$. Assume thus that v lies in $\overline{X}$. Then, $G_1 - u - \overline{x}$ has a perfect matching M_1 and $G_2 - v - x$ has a perfect matching M_2. The union of M_1 and M_2 is a perfect matching of $G - u - v$.

In all cases, we conclude that $G - u - v$ has a perfect matching. This conclusion holds for each pair u, v of distinct vertices of G. As asserted, G is bicritical. □

The following result plays a fundamental role in the theory of matching covered graphs.

Theorem 2.5 (Edmonds, Lovász and Pulleyblank [5])
A matching covered graph with at least four vertices is free of nontrivial tight cuts if and only if it is 3-connected and bicritical.

A bipartite matching covered graph without nontrivial tight cuts is called a *brace*. A nonbipartite matching covered graph without nontrivial tight cuts is called a *brick*. By Theorem 2.5 it follows that bricks are precisely the nonbipartite matching covered graphs that are 3-connected and bicritical.

2.2 Cubic Bricks

In case of cubic graphs, Theorem 2.5 has the following consequences:

Corollary 2.6
A cubic matching covered graph with at least four vertices is a brick if and only if it is bicritical.

Proof: Let G be a cubic matching covered graph with at least four vertices. If G is a brick then it is bicritical. To prove the converse, suppose that G is bicritical.

Assume, to the contrary, that G is not 3-connected. Graph G, a matching covered graph, is 2-connected. Moreover, G has more than two vertices, by hypothesis.

Let $\{u, v\}$ denote a 2-separation of G. The number of odd components of $G-u-v$ is even. If $G-u-v$ has odd components then it has precisely two, whence $\{u, v\}$ is a nontrivial barrier of G, a contradiction. We may thus assume that each component of $G-u-v$ is even. By hypothesis, G is cubic, whence v is joined to vertices of some component K of $G-u-v$ by at most one edge. If v is not joined to vertices of K then u is a cut vertex, a contradiction. We conclude that v is joined to vertices of K by precisely one edge, e. Then, the end of e in $V(K)$ plus vertex u constitute a nontrivial barrier of G, a contradiction. We conclude that if G is bicritical then it is 3-connected. By Theorem 2.5, G is a brick. □

COROLLARY 2.7
Any splicing of two cubic bricks is a (cubic) brick.

Proof: Let G be the result of the splicing of two cubic bricks. By Lemma 2.4, G is bicritical. By Corollary 2.6, G is a brick. □

We note that, in general, a graph obtained by splicing two bricks is not necessarily a brick. This is so because a splicing of two 3-connected graphs need not necessarily be 3-connected (see Figure 2)

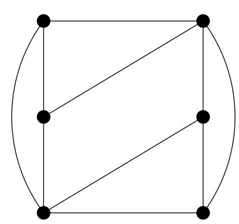

Figure 2: A splicing of two bricks that is not a brick

The following theorem gives conditions under which a splicing of two bricks is a brick.

THEOREM 2.8
Let G be a matching covered graph, $C := \nabla(X)$ a nontrivial separating cut of G such that each C-contraction of G is a brick. Then, G is a brick if and only if no pair of vertices of G, one in X, the other in $\overline{X}$, covers the set of edges of C.

Proof: Assume that G is a brick. Cut C, a nontrivial separating cut of G, is odd but not tight. Therefore G has a perfect matching M that contains at least three edges in C. Thus, no pair of vertices of G covers $M \cap C$.

Conversely, assume that for every vertex v in X and every vertex w in $\overline{X}$ cut C has an edge that is not incident with any of v and w. Each C-contraction of G is a brick, therefore 3-connected and bicritical. By Lemma 2.4, G is bicritical. To prove that G is 3-connected, let v_1 and v_2 be any two vertices in G. Adjust notation so that v_1 lies in X.

Consider first the case in which v_2 also lies in X. Graph $H := G\{X;\overline{x}\} - v_1 - v_2$ is connected because each C-contraction of G is 3-connected. But H is the graph obtained from $G - v_1 - v_2$ by contracting the set $\overline{X}$ to $\overline{x}$. Moreover, $G[\overline{X}]$ is connected. Therefore, $G - v_1 - v_2$ is connected.

Consider now the case in which v_2 lies in $\overline{X}$. Graph $H_1 := G[X] - v_1$ is connected, because each C-contraction of G is 3-connected. Likewise, $H_2 := G[\overline{X}] - v_2$ is connected. By hypothesis, C has an edge e that is not incident with v_1 and v_2. Therefore, e is an edge of $G - v_1 - v_2$ that joins a vertex in H_1 to a vertex in H_2. We conclude that $G - v_1 - v_2$ is connected. This conclusion holds for each pair v_1, v_2 of vertices of G, whence G is 3-connected. □

2.3 Tight cut decomposition

Suppose that G is a matching covered graph and that $C := \nabla(X)$ is a nontrivial tight cut of G. Then, by Proposition 2.3, the two C-contractions $G_1 = G\{X\}$ and $G_2 = G\{\overline{X}\}$ are both matching covered graphs that are smaller than G and we say that we have *decomposed* G into G_1 and G_2. If either G_1 or G_2 has a nontrivial tight cut, that graph can be decomposed into smaller matching covered graphs. This process may be repeated until each of the resulting graphs is either a brick or a brace and is called a *tight cut decomposition* of G. Remarkably, tight cut decompositions are unique up to multiplicities of edges:

Theorem 2.9 (Lovász ([6]))
Any two tight cut decompositions of a matching covered graph yield the same list of bricks and braces (except possibly for multiplicities of edges).

In particular, any two decompositions of a matching covered graph G yield the same number of bricks; this number is denoted by $b(G)$ (or simply b, if G is understood).

A *near-brick* is a matching covered graph G such that $b = 1$. In particular, every brick is a near-brick.

Proposition 2.10
Every bicritical near-brick is a brick.

Proof: Assume, to the contrary, that a bicritical near-brick G has a nontrivial tight cut $C := \nabla(X)$. Let $G_1 := G\{X;\overline{x}\}$ and $G_2 := G\{\overline{X};x\}$ denote the two C-contractions of G. By Theorem 2.9, one of G_1 and G_2 is a near-brick, the other a bipartite matching covered graph. Adjust notation so that G_1 is bipartite. Let $\{A, B\}$ denote the bipartition of G_1 such that $\overline{x}$ lies in A. Then, B is a nontrivial barrier of G. This contradicts the hypothesis that G is bicritical. □

2.4 Removable edges

Let G be a matching covered graph. An edge e of G is *removable* if the subgraph $G-e$ obtained by deleting e is matching covered. A removable edge e of G is *b-invariant* if $b(G-e) = b(G)$. Of the four graphs in Figure 1, K_4 and $\overline{C}_6$ have no removable edges, R_8 has precisely one removable edge which is also b-invariant, whereas every edge of P is removable but is not b-invariant. Confirming a conjecture that had been proposed by Lovász in 1987, we proved the following theorem in [2] and [3]:

THEOREM 2.11
Every brick different from K_4, $\overline{C}_6$, and the Petersen graph has a removable edge that is b-invariant.

The theorem we proved in [2] and [3] is in fact stronger than the above theorem. Using that theorem, we were able to describe a procedure for finding a basis for the matching lattice of matching covered graph G consisting of incidence vectors of perfect matchings of G (see [4]).

2.5 Separating cuts in bricks

By definition, a brick does not have any nontrivial tight cuts. But it is also possible for a brick to have no nontrivial separating cuts at all; any such brick is *solid*. More generally, a matching covered graph G is *solid* if each separating cut of G is tight.

The *odd wheel* W_{2k+1} $(k \geq 1)$ is the graph obtained from an odd circuit $C = (v_0, v_1, ..., v_{2k}, v_0)$ by adding a new vertex h and joining it to each vertex of C. The vertex h is called the *hub* of W_{2k+1} and the circuit C its *rim*. The odd wheel W_3 is isomorphic to K_4; in this case, any vertex of the graph may be regarded as its hub. However, for $k \geq 2$, W_{2k+1} has a unique hub. For every $k \geq 1$, W_{2k+1} is a solid brick. (See [2] for a proof of this and for other examples of solid bricks.)

Clearly any brick whose underlying simple graph is an odd wheel is also solid. We shall refer to such a graph as an *odd wheel up to multiple edges*.

Suppose that G is a brick, C is a nontrivial separating cut of G and that G_1 and G_2 are the two C-contractions of G. Then C is called a *robust cut* of G if $b(G_1) = 1$ and $b(G_2) = 1$. That is, a cut C of a brick G is robust if and only if each C-contraction of G is a near-brick. We were able to show in [3] that every nonsolid brick has a robust cut. This was a crucial step in the inductive proof of Theorem 2.11.

In the current context, we shall see that every solid extremal brick is an odd wheel (up to multiple edges) and that every extremal nonsolid brick has a robust cut C so that one of its C-contractions is an odd wheel (up to multiple edges) and the other C-contraction is an extremal brick. In case of cubic graphs, we prove something stronger, namely that every extremal cubic brick different from K_4 and P has a robust cut C of cardinality three so that one of its C-contractions is K_4, the other C-contraction is an extremal cubic brick. It is this result that enables us to determine the list of all extremal cubic matching covered graphs.

3 Extremal Graphs

As mentioned in the introduction, Edmonds, Lovász and Pulleyblank ([5]) established the following formula for the dimension of the matching lattice of a matching covered graph.

THEOREM 3.1
For any matching covered graph G, $\dim(G) = m - n + 2 - b$.

As an immediate consequence of the above theorem we have:

COROLLARY 3.2
For any matching covered graph G, $|\mathcal{M}| \geq m - n + 2 - b$.

A matching covered graph G is *extremal* if $|\mathcal{M}| = m - n + 2 - b$. If G is a cubic brick on n vertices, then $\dim(G) = 3n/2 - n + 2 - 1 = n/2 + 1$. All the four graphs in Figure 1 are cubic bricks. Furthermore, $|V(K_4)| = 4$ and $|\mathcal{M}(K_4)| = 3$; $|V(\overline{C}_6)| = 6$ and $|\mathcal{M}(\overline{C}_6)| = 4$; $|V(R_8)| = 8$ and $|\mathcal{M}(R_8)| = 5$; and $|V(P)| = 10$ and $|\mathcal{M}(P)| = 6$. Therefore, all these four bricks are extremal. It is also easy to verify that for every $k \geq 1$, the odd wheel W_{2k+1} is an extremal brick. However, not every graph whose underlying simple graph is an odd wheel is extremal, see Proposition 3.6.

3.1 Three Fundamental Properties of Extremal Graphs

An edge of a graph is *solitary* if it lies in precisely one perfect matching of the graph. The following theorem shows the relevance of the notion of b-invariance to the study of extremal graphs.

THEOREM 3.3
Let G be a matching covered graph and let e be a removable edge of G that is b-invariant. Then G is extremal if and only if $G - e$ is extremal and e is solitary in G.

Proof: Since e is a removable edge of G, the graph $G - e$ is matching covered. Moreover, e is b-invariant, whence $b(G - e) = b$. Thus,

$$|\mathcal{M}(G - e)| \geq (m - 1) - n + 2 - b,$$

with equality if and only if $G - e$ is extremal. Also,

$$|\mathcal{M}| \geq |\mathcal{M}(G - e)| + 1,$$

with equality if and only if e is solitary in G. Adding up the two inequalities and simplifying, we have that

$$|\mathcal{M}| \geq m - n + 2 - b,$$

with equality if and only if $G - e$ is extremal and e is solitary in G. As asserted, G is extremal if and only if $G - e$ is extremal and e is solitary in G. □

We now establish simple relations between the number of vertices, the number of edges and the number of perfect matchings of a matching covered graph and the values of these parameters in the cut-contractions of G with respect to a tight cut and also with respect to a separating cut.

THEOREM 3.4
Let G be a matching covered graph, C a tight cut of G, G_1 and G_2 the two C-contractions of G. Graph G is extremal if and only if (i) each of G_1 and G_2 is extremal and (ii) each edge of C is solitary in at least one of G_1 and G_2.

Proof: For $i = 1, 2$, let $m_i := |E(G_i)|$, $n_i := |V(G_i)|$, $b_i := b(G_i)$ and $\mathcal{M}_i := \mathcal{M}(G_i)$. For each edge e in C, let $\mathcal{M}_i(e)$ denote the set of perfect matchings of G_i that contain edge e and $\mathcal{M}(e)$ the set of perfect matchings of G that contain edge e. Then,

$$|\mathcal{M}(e)| = |\mathcal{M}_1(e)| \cdot |\mathcal{M}_2(e)| \geq |\mathcal{M}_1(e)| + |\mathcal{M}_2(e)| - 1, (\forall e \in C) \tag{1}$$

with equality if and only if e is solitary in at least one of G_1 and G_2. Adding up (1) over all edges e in C, we deduce that

$$|\mathcal{M}| \geq |\mathcal{M}_1| + |\mathcal{M}_2| - |C|, \tag{2}$$

with equality if and only if each edge of C is solitary in at least one of G_1 and G_2. On the other hand,

$$|\mathcal{M}_1| \geq m_1 - n_1 + 2 - b_1, \tag{3}$$

with equality if and only if G_1 is extremal. Likewise,

$$|\mathcal{M}_2| \geq m_2 - n_2 + 2 - b_2, \tag{4}$$

with equality if and only if G_2 is extremal. Adding (2), (3) and (4), and taking into account that $m = m_1 + m_2 - |C|$, $n = n_1 + n_2 - 2$ and $b = b_1 + b_2$, we deduce that

$$|\mathcal{M}| \geq m - n + 2 - b, \tag{5}$$

with equality if and only if each of G_1 and G_2 is extremal and each edge of C is solitary in at least one of G_1 and G_2. □

Theorem 3.5
Let G be a brick, C a robust cut of G, G_1 and G_2 the two C-contractions of G. Graph G is extremal if and only if (i) each of G_1 and G_2 is extremal, (ii) each edge of C is solitary in at least one of G_1 and G_2, and (iii) G has precisely one perfect matching that contains more than one edge in C.

Proof: For $i = 1, 2$, let $m_i := |E(G_i)|$, $n_i := |V(G_i)|$, $b_i := b(G_i)$ and $\mathcal{M}_i := \mathcal{M}(G_i)$. By hypothesis, C is robust, whence $b_1 = 1 = b_2$. For each edge e in C, let $\mathcal{M}_i(e)$ denote the set of perfect matchings of G_i that contain edge e and $\mathcal{M}(e)$ the set of perfect matchings M of G such that $M \cap C = \{e\}$. Then,

$$|\mathcal{M}(e)| = |\mathcal{M}_1(e)| \cdot |\mathcal{M}_2(e)| \geq |\mathcal{M}_1(e)| + |\mathcal{M}_2(e)| - 1, (\forall e \in C) \tag{6}$$

with equality if and only if e is solitary in at least one of G_1 and G_2. Graph G is a brick and cut C is robust in G, therefore at least one perfect matching of G contains more than one edge in C. Adding up (6) over all edges e in C, we deduce that

$$|\mathcal{M}| \geq |\mathcal{M}_1| + |\mathcal{M}_2| - |C| + 1, \tag{7}$$

with equality if and only if each edge of C is solitary in at least one of G_1 and G_2, and G has precisely one perfect matching that contains more than one edge in C. On the other hand,

$$|\mathcal{M}_1| \geq m_1 - n_1 + 2 - b_1 = m_1 - n_1 + 1, \tag{8}$$

with equality if and only if G_1 is extremal. Likewise,

$$|\mathcal{M}_2| \geq m_2 - n_2 + 2 - b_2 = m_2 - n_2 + 1, \tag{9}$$

with equality if and only if G_2 is extremal. Adding (7), (8) and (9), and taking into account that $m = m_1 + m_2 - |C|$ and $n = n_1 + n_2 - 2$, we deduce that

$$|\mathcal{M}| \geq m - n + 1 = m - n + 2 - b, \tag{10}$$

with equality if and only if each of G_1 and G_2 is extremal, each edge of C is solitary in at least one of G_1 and G_2, and G has precisely one perfect matching that contains more than one edge in C. □

The Petersen graph P provides an interesting illustration of the above theorem. If X is the vertex set of any pentagon of P, $C := \nabla(X)$ is a robust cut of G. The two C-contractions of P are both 5-wheels and are extremal. Furthermore, there is only one perfect matching of P that has more than one edge in C.

We conclude this subsection with useful observations on multiple edges in extremal graphs. Let e be a b-invariant removable edge in an extremal graph G and let M be a perfect matching containing e. Since e is solitary, it follows that no edge in M other than e can be a multiple edge. In particular, since any multiple edge in an extremal graph is a b-invariant removable edge, it follows that no perfect matching of such a graph can contain more than one multiple edge. It is also easy to see that if e is a b-invariant removable edge in an extremal graph G, then the graph $G + f$ obtained from G by adding an edge f joining the two ends of e is also extremal. Applying these observations to odd wheels, we have:

Proposition 3.6
Let G be a graph whose underlying simple graph is an odd wheel W_{2k+1}. Then G is extremal if and only if either $n = 4$ and any two multiple edges of G are adjacent, or $n > 4$ and all multiple edges of G are incident with the hub of W_{2k+1}.

3.2 Solitary Edges

In this subsection, we shall discuss conditions under which an edge of a graph is solitary. The following proposition is a simple observation.

Proposition 3.7
In any graph G, an edge $e := vw$ is solitary in G if and only if graph $G - v - w$ has precisely one perfect matching. □

Proposition 3.8
Let G be a bipartite graph with a unique perfect matching M, P a maximal M-alternating path in G. Then, P has odd length, the first and last edges of P lie in M, and the origin and terminus of P both have degree one in G.

Proof: Let v denote the origin of P.

Assume, to the contrary, that the edge e of M in $\nabla(v)$ does not lie in P, let w denote the end of e distinct from v. By the maximality of P, w lies in P, whence the subpath of P from v to w, plus edge e, is an M-alternating circuit in G. This implies that M is not unique, a contradiction.

We conclude that the first edge of P lies in M. Likewise, the last edge of P also lies in M. Then, P has odd length. Assume, to the contrary, that the degree of v is at least two. Let f denote any edge of $\nabla(v) - M$, let x denote the end of f distinct from v. By the maximality of P, x lies in P. Thus, the subpath of P from v to x plus edge f is an M-alternating circuit in G. This implies that M is not unique, a contradiction. We conclude that vertex v has degree one in G. Likewise, so too does vertex w, the terminus of P. □

The following result is due to Kötzig (see [7]):

Theorem 3.9
Every graph that has a unique perfect matching has a cut edge.

Proof: Let G be a graph with a unique perfect matching M and let B be a maximal barrier of G. Let $\mathcal{K}$ denote the set of all odd components of $G - B$. We observe that each $K \in \mathcal{K}$ is factor- critical. If not, by Tutte's theorem (Theorem 1.1), there exists a nonempty subset B' of $V(K)$ such that $|\mathcal{O}(K - B')| \geq |B'| + 1$ and it follows that $B \cup B'$ is a barrier of G. This is not possible, by the maximality of B. We also note that $G - B$ has no even components. If not, then, for every even component K of $G - B$ and any vertex v of $V(K)$, $B \cup \{v\}$ is a barrier of G. Again, this not possible, by the maximality of B.

Let H denote the bipartite graph obtained by contracting each odd component K of $G - B$ to a vertex y_K and deleting all edges with both ends in B. Then (B, Y) is a bipartition of H, where $Y := \{y_K : K \in \mathcal{K}\}$. Clearly, $N := M \cap E(H)$ is a perfect matching of H and, since each $K \in \mathcal{K}$ is factor-critical, any perfect matching of H can be extended to a perfect matching of G. Hence N is the only perfect matching of H.

Let P be a maximal N-alternating path in H. By Proposition 3.8, P has odd length and its origin and terminus both have degree one in H. Adjust notation so that the origin of P lies in Y. Then, the first edge of P is a cut edge of G. □

The next property gives a recursive algorithm to decide whether a given perfect matching of a graph is unique.

Corollary 3.10
Let G be a (nonnull) graph with a perfect matching M. Then, M is the only perfect matching of G if and only if (i) G has a cut edge e that lies in M, and (ii) $M - e$ is the only perfect matching of $G - v - w$, where v and w are the ends of e.

3.3 Reduction of 2-Vertices

A *2-vertex* of a graph G is a vertex of degree two in G that is adjacent to precisely two vertices of G. Let G be a matching covered graph, u a 2-vertex of G, v and w the two neighbors of u. Let $X := \{u, v, w\}$. Then, $C := \nabla(X)$ is a tight cut of G. The C-contraction $G_1 := G\{\overline{X}; x\}$ is thus a matching covered graph. We say that *G_1 is obtained from G by the reduction of a 2-vertex.* If G_1 also has a 2-vertex the operation may be repeated until we obtain a graph free of 2-vertices: that graph is said to be *the 2-reduction of G.*

Proposition 3.11
The 2-reduction of a matching covered graph G is unique up to isomorphisms.

Proof: By induction on G. If G is free of 2-vertices then G is the only 2-reduction of G, and the assertion holds trivially.

Assume thus that G has at least one 2-vertex, say u. Let v and w denote the neighbors of u, $X := \{u, v, w\}$, $C := \nabla(X)$, $G_1 := G\{\overline{X}; x\}$. If u is the only 2-vertex of G then G_1 is free of 2-vertices and is thus the only 2-reduction of G. Again, the assertion holds in this case.

Assume thus that G has a 2-vertex u' distinct from u. Let v' and w' denote the two neighbors of u', $X' := \{u', v', w'\}$, $C' := \nabla(X')$ and $G'_1 := G\{\overline{X'}; x'\}$.

We assert that each 2-reduction of G_1 is isomorphic to each 2-reduction of G'_1. For this, consider first the case in which C and C' are not disjoint. In that case, graphs G_1 and G'_1 are isomorphic. By induction hypothesis, all 2-reductions of G_1 are isomorphic, whence so too are all 2-reductions of G_1 and G'_1.

Consider last the case in which C and C' are disjoint. In that case, vertex u' is a 2-vertex of G_1 and vertex u is a 2-vertex of G'_1. By induction hypothesis, each 2-reduction of G_1 is isomorphic to each 2-reduction of $G_1\{\overline{X'}; x'\}$ and each 2-reduction of G'_1 is isomorphic to each 2-reduction of $G'_1\{\overline{X}; x\}$. But $G_1\{\overline{X'}; x'\} = G'_1\{\overline{X}; x\}$.

In both cases, every 2-reduction of G_1 is isomorphic to every 2-reduction of G'_1. This conclusion holds for each pair of distinct 2-vertices u and u' of G. Therefore, all 2-reductions of G are isomorphic. □

Corollary 3.12

A matching covered graph G is extremal if and only if its 2-reduction is extremal.

Proof: By induction on G. If G is free of 2-vertices then the assertion holds immediately. Assume thus that G has a 2-vertex u. Let v and w denote the neighbors of u, $X := \{u, v, w\}$, $C := \nabla(X)$, $G' := G\{X; \overline{x}\}$, $G'' := G\{\overline{X}; x\}$.

Let m', n' and b' denote respectively the number of edges, vertices and bricks of G'. Let $\mathcal{M}'$ denote the set of perfect matchings of G'.

For each perfect matching M of G, $M \cap E(G')$ is a perfect matching of G'; conversely, for each perfect matching M' of G' precisely one of $M' \cup \{uv\}$ and $M' \cup \{uw\}$ is a perfect matching of G. Thus, there exists a one-to-one correspondence relating $\mathcal{M}$ and $\mathcal{M}'$, whence

$$|\mathcal{M}| = |\mathcal{M}'|.$$

Clearly,

$$m - n = m' + 2 - (n' + 2) = m' - n'.$$

Graph G'' is C_4, up to multiple edges in C. Moreover, C is tight. Therefore,

$$b = b'.$$

We conclude that $|\mathcal{M}| - (m - n + 2 - b) = |\mathcal{M}'| - (m' - n' + 2 - b')$. Thus, G is extremal if and only if G' is extremal. By induction hypothesis, G' is extremal if and only if its 2-reduction is extremal. But the 2-reduction of G' is the 2-reduction of G. As asserted, G is extremal if and only if its 2-reduction is extremal. □

4 Bipartite Extremal Graphs

In this section we characterize bipartite extremal matching covered graphs.

Lemma 4.1
Let G be a bipartite matching covered graph free of vertices of degree two. Then, G is extremal if and only if it contains precisely two vertices.

Proof: By hypothesis, G is free of vertices of degree two and matching covered. Every matching covered graph is 2-connected. Therefore, either G has precisely two vertices or each vertex of G is incident with at least three edges.

Assume, to the contrary, that G has at least four vertices. Then, each vertex of G is incident with at least three edges. Let $\{A, B\}$ denote the bipartition of G. Since G has at least four vertices and is free of vertices of degree two, it has a removable edge, say e (see [4] for a proof). Every removable edge of G is b-invariant. In particular, e is b-invariant. By Theorem 3.3, edge e is solitary in G.

Let u and v denote the ends of e. Adjust notation so that u lies in A, whereupon v lies in B. Let M be the perfect matching of G that contains edge e. Let $N := M - e$, $H := G - u - v$. Then, N is a perfect matching of H. For each perfect matching N' of H, set $N' \cup \{e\}$ is a perfect matching of G. Since e is solitary, it follows that N is the only perfect matching of H.

Let P denote a maximal N-alternating path in H, w and x the origin and terminus of P. By Proposition 3.8, path P has odd length, the first and last edges of P lie in M and each of w and x has degree one in H.

The degree of each vertex of G is at least three. Therefore, in G, w is joined by at least two edges to vertex v. Likewise, vertex x is joined to vertex u by at least two edges.

Therefore, P may be extended in G to an M-alternating circuit that contains two edges not in M, each of which is a multiple edge in G. Each such edge is removable, b-invariant, but not solitary in G. By Theorem 3.3, graph G is not extremal, a contradiction. □

Corollary 4.2
A bipartite matching covered graph is extremal if and only if its 2-reduction contains precisely two vertices.

5 Extremal Bricks

In this section we show that every brick is either an odd wheel up to multiple edges or the result of the splicing of an odd wheel with an extremal brick. We accomplish this by showing that (i) every solid extremal brick is an odd wheel and (ii) every nonsolid extremal brick is the result of the splicing of two extremal bricks, at least one of which is solid. In both cases we shall make use of the following result:

Theorem 5.1
Let G be an extremal matching covered graph. If G is free of vertices of degree two then G is bicritical.

Proof: By induction on G. Assume that G is free of vertices of degree two. Assume, to the contrary, that G has a nontrivial barrier B.

If each component of $G - B$ is trivial then G is bipartite. That is, G is a bipartite extremal matching covered graph free of vertices of degree two that contains at least four vertices. That is a contradiction to Lemma 4.1.

We may thus assume that $G - B$ has a nontrivial component, say K. Let $C := \nabla(V(K))$. Then, C is a nontrivial tight cut of G. Let $G_1 := G\{V(K); \overline{v_K}\}$ and $G_2 := G\{\overline{V_K}; v_K\}$ denote the two C-contractions of G.

By Theorem 3.4, each of G_1 and G_2 is extremal. Moreover, each edge of C is solitary in at least one of G_1 and G_2.

Graph G_2 is an extremal matching covered graph and B is a nontrivial barrier of G_2. By induction hypothesis, G_2 is not free of vertices of degree two. It follows that C contains precisely two edges. Moreover, vertex v_K is the only vertex of degree two in G_2 and vertex $\overline{v_K}$ is the only vertex of degree two in G_1. By Proposition 5.2, asserted below, no edge of C is solitary in any of G_1 and G_2. This is a contradiction to Theorem 3.4. As asserted, G is bicritical.

Proposition 5.2
Let G be a matching covered graph, v the only vertex of degree two in G. Then, no edge of $\nabla(v)$ is solitary in G.

Proof: Let e_1 and e_2 denote the two edges of $\nabla(v)$. For $i = 1, 2$, let v_i denote the end of e_i distinct from v. By hypothesis, vertex v_1 has degree greater than two. Each edge of $\nabla(v_1) - e_1$ lies in a perfect matching that necessarily contains edge e_2. Therefore, e_2 lies in at least $d_1 - 1$ perfect matchings of G, where d_1 denotes the degree of v_1. Therefore, e_2 is not solitary in G. Likewise, e_1 is not solitary in G. □

The proof of Proposition 5.2 completes the proof of Theorem 5.1. □

Corollary 5.3
Every extremal cubic matching covered graph with at least four vertices is a brick.

Proof: Let G be an extremal cubic graph with at least four vertices. By Theorem 5.1, G is bicritical. By Corollary 2.6, G is a brick. □

5.1 Extremal Solid Bricks

In this section we characterize the solid extremal bricks: they are all odd wheels. In order to do that we need two lemmas:

Lemma 5.4

Let G be a matching covered graph such that no two 2-vertices are adjacent. Then, an edge e is removable in G if and only if it lies and is removable in the 2-reduction of G.

Proof: By induction on G. If G is free of 2-vertices then G is its 2-reduction and the assertion holds trivially. Assume thus that G has a 2-vertex, u. Let v and w denote the two vertices of G that are adjacent to v. Let $X := \{u, v, w\}$, $C := \nabla(X)$. Let $G_1 := G\{X; \overline{x}\}$ and $G_2 := G\{\overline{X}; x\}$ denote the two C-contractions of G.

Assume that e lies and is removable in G_2. The underlying simple graph of G_1 is C_4. By hypothesis, neither v nor w has degree two (or less) in G, whence each edge of C is a multiple edge in G_1. If e lies in C then it is a multiple edge of G_1, whence $G_1 - e$ is matching covered; if e does not lie in C then $G_1 = G_1 - e$. In both cases, $G_1 - e$ is matching covered. By hypothesis, e is removable in G_2, whence $G_2 - e$ is matching covered. We conclude that both $(C - e)$-contractions of $G - e$ are matching covered, whence so too is $G - e$. Thus, e is removable in G.

Conversely, assume that e is a removable edge of G. Edges $e_1 := uv$ and $e_2 := uw$ are clearly nonremovable in G. Therefore, e is an edge of G_2. Moreover, u is a 2-vertex of $G - e$, whence each $(C - e)$-contraction of $G - e$ is matching covered. In particular, $G_2 - e$ is matching covered. That is, e lies in G_2 and is removable in G_2.

We conclude that an edge of G is removable in G if and only if it lies in G_2 and is removable in G_2. Vertex x of G_2 has degree at least four, because its degree is equal to two less than the sum of the degrees of v and w in G. Therefore, no two 2-vertices of G_2 are adjacent. By induction hypothesis, an edge of G_2 is removable in G_2 if and only if it lies and is removable in the 2-reduction of G_2. But the 2-reduction of G_2 is the 2-reduction of G, by Proposition 3.11. As asserted, an edge of G is removable in G if and only if it lies and is removable in the 2-reduction of G. $\square$

Lemma 5.5 (The Exchange Property)

Let G be a solid matching covered graph, e a removable edge of G. Every edge f that is removable in $G - e$ is also removable in G.

Proof: Assume, to the contrary, that f is not removable in G. By hypothesis, f is removable in $G - e$. Thus, e is the only edge of $G - f$ that does not lie in some perfect matching of $G - f$. By Tutte's Theorem (Theorem 1.1), $G - f$ has a barrier B such that e is the only edge of G that has both ends in B. Moreover, since G is matching covered, B is not a barrier of G, whence f has its ends in distinct (odd) components of $G - f - B$.

Let $\mathcal{K}$ denote the set of (odd) components of $G - f - B$. For each K in $\mathcal{K}$, let $C_K := \nabla_G(V(K))$. Let g denote any edge of G. We assert that g lies in a perfect matching M_g of G that has precisely one edge in C_K, for each K in $\mathcal{K}$. For this, consider first the case in which g is distinct from both e and f. Graph $G - e - f$ is matching covered. Let M_g be a perfect matching of $G - e - f$ that contains edge g.

A simple counting argument shows that M_g contains precisely one edge in C_K, for each K in $\mathcal{K}$. Consider next the case in which G is one of e and f. Let $M_e := M_f$ be a perfect matching of G that contains edge e. A straightforward counting argument shows that M_e contains edge f and precisely one edge in C_K, for each K in $\mathcal{K}$. As asserted, each edge g of G lies in a perfect matching M_g of G that contains precisely one edge in C_K, for each K in $\mathcal{K}$. It follows that C_K is a separating cut, for each component K in $\mathcal{K}$.

Let M be a perfect matching of $G - e$ that contains edge f. Thus, M is a perfect matching of G that contains edge f but does not contain edge e. A counting argument shows that M contains precisely three edges in C_K, for some component K in $\mathcal{K}$. Thus, C_K is separating but not tight in G. This contradicts the hypothesis that G is solid. As asserted, f is removable in G. □

Theorem 5.6
Let G be a brick on n vertices. If $n = 4$ then G is extremal and solid if and only if all multiple edges of G are pairwise adjacent. If $n > 4$ then G is extremal and solid if and only if it is an odd wheel up to multiple spokes.

Proof: We have already noted that every odd wheel is an extremal solid brick. The sufficiency of the asserted conditions thus follows from Proposition 3.6. Let us now proceed to prove their necessity. Thus, assume that G is an extremal solid brick. We shall prove that either (i) $n \geq 6$ and G is an odd wheel up to multiple spokes, or (ii) $n = 4$ and all multiple edges of G are pairwise adjacent. This shall be done by induction on $|V(G)|$.

Consider first the case in which no removable edge of G is b-invariant. By Theorem 2.11, G is one of K_4, $\overline{C_6}$ and P. Graphs $\overline{C_6}$ and P are not solid. Thus, $G = K_4$, an odd wheel. The assertion holds in this case.

We may thus assume that G has a removable edge e that is b-invariant. By Theorem 3.3, edge e is solitary and graph $G - e$ is extremal. Let M be the perfect matching of G that contains edge e.

Let W denote the 2-reduction of $G - e$. Every vertex of G, a brick, is adjacent to at least three vertices. Let us now examine in some detail the relation involving $G - e$ and W. In [2], we proved a result (Theorem 2.28) that implies the following assertion:

Theorem 5.7
If a matching covered graph G is solid and e is a removable edge in G then $G - e$ is also solid. □

Proposition 5.8
Graph W is a solid extremal brick.

Proof: Graph $G-e$ is extremal. Since G is solid, $G-e$ is also solid, by Theorem 5.7. By definition, e is b-invariant, whence $G-e$ is a near-brick. Thus, $G-e$ is an extremal solid near-brick.

The operation of reduction of a 2-vertex, when applied to a graph G_0, corresponds to the replacement of G_0 by G_1, one of its C-contractions, where C is a tight cut of G_0. Moreover, the C-contraction of G distinct from G_1 is bipartite. Every separating nontight cut of G_1 is also a separating nontight cut of G_0. We conclude that the 2-reduction of a solid graph is solid. We also conclude that the operation of 2-reduction preserves the number of bricks of the graph. In our case, the 2-reduction W of $G-e$ is a solid near-brick. Moreover, W is extremal, by Corollary 3.12. Thus, W is an extremal solid near-brick. By Theorem 5.1, W is bicritical. By Proposition 2.10, W is a brick. As asserted, W is an extremal solid brick. □

By induction hypothesis, either (i) W has at least six vertices and is an odd wheel up to multiple spokes, or (ii) W has precisely four vertices and all multiple edges are pairwise adjacent.

Proposition 5.9
No edge of $M-e$ is removable in W.

Proof: Assume, to the contrary, that some edge f of $M-e$ is removable in W. By Lemma 5.4, f is removable in $G-e$. By Lemma 5.5, f is removable in G. Then, $G-f$ has a perfect matching M' that contains edge e. Thus, M' is a perfect matching of G that contains edge e but not f, whence it is distinct from M. We conclude that e is not solitary in G, a contradiction. □

Lemma 5.10
At most one end of e has degree three in G.

Proof: Let v and v' denote the ends of e in G. Every vertex of G is adjacent to at least three vertices in G. Assume, to the contrary, that both v and v' have degree three in G. Let b_1 and b_2 denote the vertices of G that, in addition to v', are adjacent to v in G. Likewise, let b'_1 and b'_2 denote the vertices of G that, in addition to v, are adjacent to v' in G. We now consider three cases, depending on the number of vertices that $\{b_1, b_2\}$ and $\{b'_1, b'_2\}$ have in common. In each case we deduce a contradiction.

Case 1 *Sets $\{b_1, b_2\}$ and $\{b'_1, b'_2\}$ are identical.*

Adjust notation so that $b_1 = b'_1$, whereupon $b_2 = b'_2$ (Figure 3(a)). Consider first the case in which $n = 4$. In that case, the underlying simple graph of G is K_4, whence b_1 and b_2 are adjacent. No perfect matching of $G-e$ contains an edge that joins b_1 and b_2. Therefore, edge e is not removable in G, a contradiction. Consider next the case in which $n > 4$. Then, $G-b_1-b_2$ is not connected, whence G is not 3-connected. This is a contradiction to Theorem 2.5. In both alternatives we derived a contradiction.

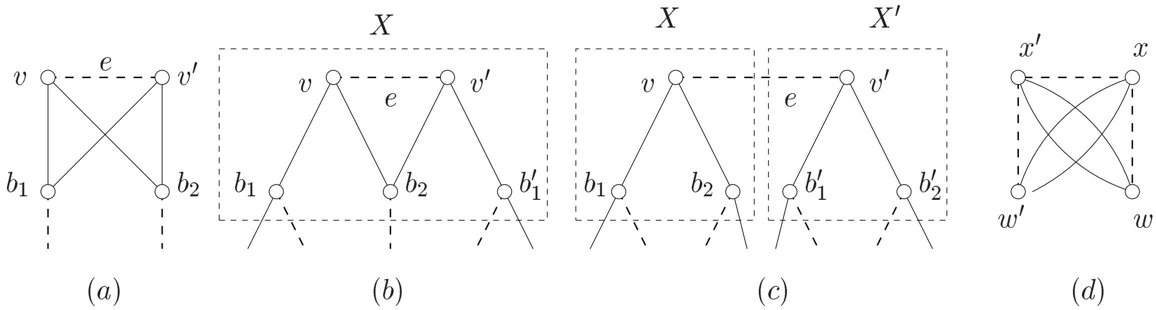

Figure 3: Illustrations for the proof of Lemma 5.10. Dashed lines indicate edges that lie in the perfect matching M of G that contains edge e.

Case 2 *Sets* $\{b_1, b_2\}$ $\{b'_1, b'_2\}$ *have precisely one vertex in common.*

Adjust notation so that $b_2 = b'_2$ (Figure 3(b)). Let $B := \{b_1, b_2, b'_1\}$, $I := \{v, v'\}$, $X := B \cup I$. Then, W is isomorphic to $G\{\overline{X}; x\}$. No edge of G joins any two vertices in B, because e is removable. Thus, $|M \cap \nabla(X)| = 3$ and $|\nabla(X)| \geq 5$. Moreover, the ends of the three edges of $M \cap \nabla(X)$ in $\overline{X}$ are distinct.

Consider first the case in which x is adjacent in W to precisely three vertices. Then, at least one edge of $M \cap \nabla(X)$ is multiple in W, whence removable in W. This is a contradiction to Proposition 5.9. Consider next the case in which x is adjacent in W to more than three vertices. Then, W is an odd wheel with at least six vertices and x is its hub. Each spoke of W is removable in W. In particular, each of the three edges of $M \cap \nabla(X)$ is removable in W. In both alternatives we get a contradiction to Proposition 5.9.

Case 3 *Sets* $\{b_1, b_2\}$ $\{b'_1, b'_2\}$ *are disjoint.*

Let $B := \{b_1, b_2\}$, $X := B \cup \{v\}$. Likewise, let $B' := \{b'_1, b'_2\}$, $X' := B' \cup \{v'\}$. Then, W is isomorphic to the graph obtained from G by contracting X to a single vertex x and X' to a single vertex x' (Figure 3(c)).

No edge of G has both ends in B, otherwise e would not be removable. Likewise, no edge of G has both ends in B'. Let L be the subgraph of W spanned by $M - e$. Then, each of x and x' has degree two in L, all the other vertices have degree one in L. Consider first the case in which W is an odd wheel with more than four vertices. The hub of W is incident with an edge of $M - e$. Moreover, each spoke of W is removable. This is a contradiction to Proposition 5.9. Consider next the case in which W has precisely four vertices. The degrees of the vertices of L are $1, 1, 2, 2$. Moreover, L is free of multiple edges, by Proposition 5.9. Therefore, L is a path $Q := (w, x, x', w')$ of length three (Figure 3(d)). The degree of x in $W - M$ is at least two and the two edges of $M \cap \nabla_W(x)$ are not multiple edges in W. Therefore, at least two edges join

x to w' in W. Likewise, at least two edges join x' to w in W. This is a contradiction to the fact that the multiple edges of W are pairwise adjacent. In both alternatives we derived a contradiction.

In all cases considered we derived a contradiction based on the hypothesis that both ends of e have degree three in G. As asserted, at most one end of e has degree three in G. □

We now complete the proof of the Theorem considering two cases: in the first case, no end of e has degree three in G; in the second case, precisely one end of e has degree three in G.

Case 1 *No end of e has degree three in G.*

Therefore, $W = G - e$. Consider first the case in which $n = 4$. Then, the multiple edges of $G - e$ are pairwise adjacent in $G - e$. Moreover, since e is solitary, at most one edge of G is not adjacent to edge e. We conclude that the multiple edges of G are pairwise adjacent and the assertion holds.

Consider now the case in which $n > 4$. Then, $G - e$ is an odd wheel, up to multiple spokes. Let h denote the hub of $G - e$. Every spoke is removable in $G - e$. By Proposition 5.9, no edge of $M - e$ is removable in $G - e$. Therefore, no edge of $M - e$ is incident with h. We conclude that edge e is the edge of M incident with h, whence G is an odd wheel, up to multiple spokes.

Case 2 *Precisely one end of e has degree three in G.*

Let v denote the end of e of degree three. Let h denote the other end of e. Let b_1 and b_2 denote the two vertices that are adjacent to v in $G-e$, let $X := \{b_1, b_2, v\}$ (Figure 4). For $i = 1, 2$, let $f_i := b_i v_i$ denote the edge of M incident with b_i, let$g_i := b_i w_i$ denote any edge in $\nabla(b_i) - (M \cup \nabla(v))$. No edge of G joins vertices b_1 and b_2, because e is removable. Then, $\{e, f_1, g_1, f_2, g_2\} \subseteq \nabla(X)$ (Figure 4(a)). Graph W is isomorphic to $G\{\overline{X}; x\}$.

We assert that $w_1 = h$. For this, assume the contrary. Consider first the case in which w_1 does not lie in $\{v_1, v_2, h\}$. Then, in W, x is adjacent to at least four vertices. In that case, W is an odd wheel with at least six vertices and x is its hub. Each spoke of W is thus removable in W. In particular, f_1 is removable in W, a contradiction to Proposition 5.9. Consider next the case in which w_1 lies in $\{v_1, v_2\}$. If $w_1 = v_1$ then f_1 is a multiple edge in W. If $w_1 = v_2$ then f_2 is a multiple edge in W. In both alternatives, an edge of $M - e$ is removable in W, a contradiction to Proposition 5.9. As asserted, $w_1 = h$. This equality holds for each edge $g_1 = b_1 w_1$ of $\nabla(X) - M$ that is incident with b_1. Therefore, b_1 is adjacent to precisely three vertices: v, v_1 and h. Likewise, b_2 is adjacent to precisely three vertices: v, v_2 and h (Figure 4(b)). Moreover, edges g_1 and g_2 are multiple in W.

If $V(W) = \{x, h, v_1, v_2\}$ then the underlying simple graph of W is K_4; moreover, precisely one edge joins v_1 and v_2, because g_1 and g_2 are multiple edges in W: in

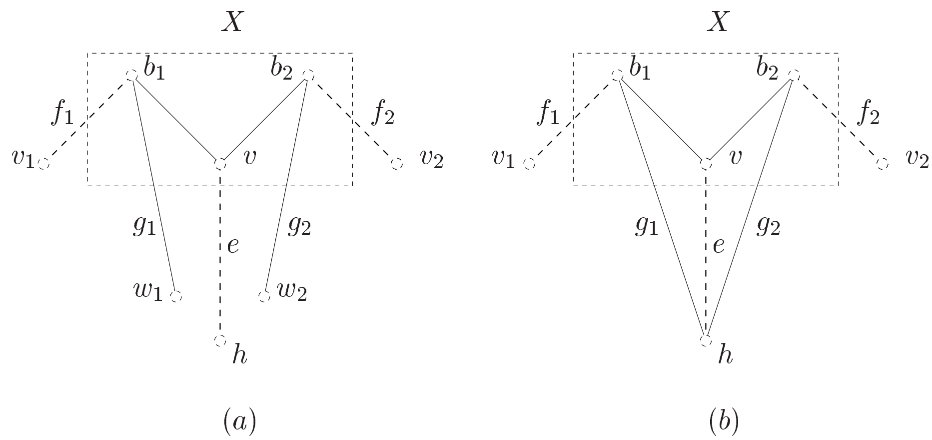

Figure 4: Illustration of the proof of Theorem 5.6, in the case in which edge e has precisely one end of degree three. Dashed lines indicate the edges of the perfect matching M of G that contains edge e.

that case, $W - h$ is the triangle $Q := (x, x_1 = v_1, x_2 = v_2)$. Alternatively, if W has more than four vertices then it is an odd wheel up to multiple spokes; the only vertex of W not incident with edges in M is h; every spoke of W is removable, whence, by Proposition 5.9, vertex h is the hub of W: in that case, $W - h$ is the polygon $Q := (x, x_1 = v_1, x_2, \cdots, x_{n-4} = v_2)$. In both alternatives, replacement of x in Q by (b_2, v, b_1) yields an odd polygon R with precisely $n - 1$ vertices, which is equal to $G - h$. Moreover, h is adjacent to each vertex of R. We conclude that G is an odd wheel up to multiple spokes, with at least six vertices, with hub h and rim R.

The analysis of the two possible cases completes the proof of Theorem 5.6. □

5.2 Extremal Nonsolid Bricks

We are now in position to show that every nonsolid extremal brick is the result of splicing an odd wheel (up to multiple edges) with an extremal brick.

Let G be a matching covered graph. A cut C *precedes* cut D in G, written $C \preceq D$, if $|M \cap C| \leq |M \cap D|$ for each perfect matching M of G. If equality holds for each perfect matching M of G then C and D are *matching equivalent.* If strictly inequality holds for at least one perfect matching M then C *strictly precedes* D in G, written $C \prec D$. The following result was proved in [2] as Corollary 2.4:

Theorem 5.11
Let G be a brick, M_0 a perfect matching of G, C a nontrivial separating cut of G such that $|M_0 \cap C| > 1$. If C is minimal with respect to the relation $\preceq$ of precedence

among all nontrivial separating cuts D of G such that $D \preceq C$ and $|M_0 \cap D| > 1$, then C is robust.

Corollary 5.12
Every nonsolid brick has a robust cut.

Proof: Assume that G is a nonsolid brick. Then, G has a separating nontight cut C. Let M_0 be a perfect matching of G that contains more than one edge in C. Let $\mathcal{C}$ denote the collection of those nontrivial separating cuts D of G such that $D \preceq C$ and $|M_0 \cap D| > 1$. Cut C certainly lies in $\mathcal{C}$, whence $\mathcal{C}$ is nonnull. Let D be a cut in $\mathcal{C}$ that is minimal with respect to the relation of precedence. By Theorem 5.11, D is robust. □

Theorem 5.13
Every nonsolid brick G has a robust cut C such that one of the C-contractions of G is solid.

Proof: By Corollary 5.12, G has robust cuts. Let $C := \nabla(X)$ be a robust cut in G, with $|X|$ as small as possible. Let $G_2 := G\{\overline{X}; x\}$, and let M_0 denote a perfect matching of G such that $|M_0 \cap C| > 1$. We assert that the C-contraction $G_1 := G\{X; \overline{x}\}$ of G is solid.

Assume, to the contrary, that G_1 has a separating nontight cut D_1. Let M_1' denote any perfect matching of G_1 containing more than one edge in D_1. Let M_1'' denote a perfect matching of G_2 that contains the edge of M_1' in C. Let $M_1 := M_1' \cup M_1''$. Then M_1 is a perfect matching of G that contains more than one edge in D_1 and just one edge in C.

We first observe that C and D_1 are "coherent" separating cuts of G, in the following sense: for every edge e of G there exists a perfect matching M_e of G that contains edge e and just one edge in each of C and D_1. To see this, consider first the case in which e lies in G_1; in that case, G_1 has a perfect matching, M_e', that contains e and just one edge in D_1; let M_e'' denote a perfect matching of G_2 that contains the edge of M_e' in C. Then $M_e' \cup M_e''$ is a perfect matching of G that contains edge e and just one edge in each of C and D_1. Now consider the case in which edge e does not lie in G_1. Let M_e'' denote a perfect matching of G_2 that contains edge e. Let M_e' denote a perfect matching of G_1 that contains the edge of M_e'' in C and just one edge in D_1. Again, $M_e' \cup M_e''$ is a perfect matching of G that contains edge e and just one edge in each of C and D_1.

Let $\mathcal{C}$ denote the collection of those nontrivial separating cuts D of G such that $D \preceq D_1$ and $|M_1 \cap D| > 1$. Collection $\mathcal{C}$ is nonnull, because cut D_1 lies in $\mathcal{C}$. Let D denote a cut in $\mathcal{C}$ that is minimum with respect to the relation of precedence. By Theorem 5.11, D is robust. Since M_1 has more than one edge in D, but has only one edge in C, it follows that D is not matching-equivalent to C. In particular, $D \neq C$.

Let Y denote the shore of D such that $|X \cap Y|$ is odd. Then, $I := \nabla(X \cap Y)$ and $U := \nabla(\overline{X} \cap \overline{Y})$ are odd cuts. A simple counting argument then shows that for every set M of edges of G,

$$|M \cap C| + |M \cap D| = |M \cap I| + |M \cap U| + 2\,|M \cap \nabla(X - Y, Y - X)|\,, \tag{11}$$

where $\nabla(X - Y, Y - X)$ denotes the set of edges of G that have one end in $X - Y$, the other in $Y - X$.

For any edge e of G, matching M_e, defined above, contains edge e and just one edge in each of C and D_1. Since $D \preceq D_1$, it follows that M_e also has just one edge in D. Thus, $|M_e \cap C| + |M_e \cap D| = 2$. From (11), we deduce that M_e has just one edge in each of I and U. Moreover, M_e and $\nabla(X - Y, Y - X)$ are disjoint. These conclusions hold for each edge e of G, and edge e lies in M_e. Therefore, each of I and U is separating. Moreover, no edge of G joins a vertex of $X - Y$ to a vertex of $Y - X$. Thus modularity applies to the pair C, D, that is, for every set M of edges of G,

$$|M \cap C| + |M \cap D| = |M \cap I| + |M \cap U|\,. \tag{12}$$

We assert that U is a tight cut in G_2. For this, assume, to the contrary, that there exists a perfect matching M_2'' of G_2 that contains more than one edge in U. Let M_2' denote a perfect matching of G_1 that contains the edge of M_2'' in C and just one edge in D_1. Then $M_2 := M_2' \cup M_2''$ constitutes a perfect matching of G that contains one edge in each of C and D_1, but more than one edge in U. From (12), we deduce that M_2 contains more than one edge in D. Thus M_2 has just one edge in D_1 but more than one edge in D. This contradicts the relation $D \preceq D_1$. As asserted, U is tight in G_2.

We assert that cuts C and U are not matching equivalent. For this, assume the contrary. From (12), we deduce that D and I are also matching equivalent. Therefore, I also lies in $\mathcal{C}$ and is minimal with respect to the relation $\preceq$ of precedence. Therefore, I is robust. Moreover, M_1 has one edge in C and more than one edge in D, whence it has more than one edge in I. Thus, C and I are distinct. We conclude that $X \cap Y$ is a proper subset of X, a contradiction to the minimality of $|X|$. As asserted, C and U are not matching equivalent.

We now show that cut U is trivial in G_2. For this, assume the contrary. We have seen that cut U is tight in G_2. Thus, cut U is a nontrivial tight cut of G_2. Cut C is robust in G, whence G_2 is a near-brick. By the uniqueness of the tight cut decomposition, one of the U-contractions of G_2 is bipartite, the other is a near-brick. Let Z denote the shore of U in G_2 such that $H := G_2\{Z; \overline{z}\}$ is bipartite. Let $\{A, B\}$ denote the bipartition of H. Adjust notation so that $\overline{z}$ lies in A, whereupon B does not contain vertex $\overline{z}$. Then, B is a nontrivial barrier of G_2.

If the contraction vertex x of G_2 does not lie in B then B is a nontrivial barrier of G, a contradiction. Therefore, x lies in B. We conclude that $\overline{z}$ lies in A and x lies in B. These are the only vertices of H that are not original vertices of G. Moreover, H is matching covered. It follows that every perfect matching of G contains the same number of edges in C and in U. That is, C and U are matching equivalent, a contradiction. As asserted, U is trivial in G_2.

Finally, U is trivial in G, otherwise $U = C$, whence C and U are matching equivalent, a contradiction. We conclude that $\overline{X} \cap \overline{Y}$ is a singleton. Thus, U is tight in G. In that case, I is not tight in G, otherwise both C and D would be tight cuts, by (12). Thus, $X \cap Y$ is not a singleton. We conclude that $|\overline{Y}| = |X| - |X \cap Y| + |\overline{X} \cap \overline{Y}| < |X|$. This contradicts the minimality of $|X|$.

In all cases, we obtained a contradiction, based on the hypothesis that G_1 is not solid. As asserted, G_1 is solid. □

Figure 5 illustrates the necessity of the minimality of $|X|$ in the proof of Theorem 5.13. It shows that minimality of X is not sufficient: it depicts a brick G and a robust cut $C := \nabla(X)$. The C-contraction $G\{X\}$ of G is $\overline{C_6} + e$, the other C-contraction of G is $\overline{C_6}$ plus a multiple edge. Thus, neither C-contraction of G is solid. Cut D is also a robust cut of G, one of its C-contractions is K_4, up to multiple edges. The triangle in $G[X]$ spans a cut I that is robust in $G\{X\}$ and naturally a separating cut of G. However, I is not robust in G. To see this, observe that for every perfect matching M of G,

$$|M \cap C| + |M \cap D| = |M \cap I| + 1.$$

In particular, if M has just one edge in I then it has precisely one edge in each of C and D. Therefore, one of the I-contractions of G is not a near-brick, it has a 2-separation, the cuts C and D are the associated tight cuts. Finally, we note that G is not extremal.

Corollary 5.14
Every extremal nonsolid brick G has a robust cut $C := \nabla(X)$ such that C-contraction $G\{X\}$ is an odd wheel up to multiple edges, and C-contraction $G\{\overline{X}\}$ is an extremal brick.

Proof: By Theorem 5.13, G has a robust cut $C := \nabla(X)$ such that C-contraction $G_1 := G\{X\}$ is solid. Let $G_2 := G\{\overline{X}\}$.

By definition of robust cut each of G_1 and G_2 is a near-brick. By Theorem 3.5, each of G_1 and G_2 is extremal. Cut C, a nontrivial separating cut of G, has at least three edges in some perfect matching of G. Thus, both G_1 and G_2 are free of 2-vertices. By Theorem 5.1, each of G_1 and G_2 is bicritical. By Proposition 2.10 each of G_1 and G_2 is a brick. Thus, each of G_1 and G_2 is an extremal brick. Moreover, G_1 is solid. By Theorem 5.6, G_1 is an odd wheel, up to multiple edges. □

Figure 6 illustrates Corollary 5.14. It shows a graph that is the result of the splicing of two 5-wheels, up to multiple edges, at their hubs. We remark that this graph is also the result of the splicing of an extremal brick and a K_4. We shall see that every extremal nonsolid cubic brick other than the Petersen graph may be obtained by splicing a smaller extremal cubic brick and a K_4. It seems quite plausible that an analogous result (conjecture 5.15) holds for all extremal nonsolid bricks.

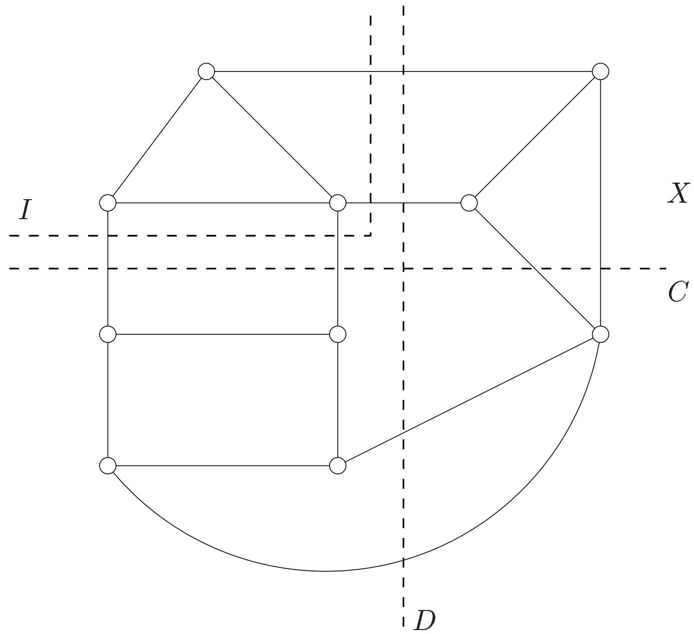

Figure 5: An example that illustrates the need for a robust cut of minimum shore in the proof of Theorem 5.13.

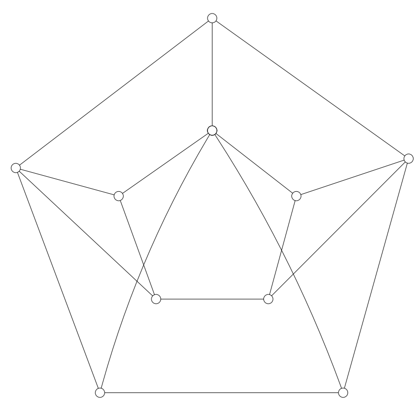

Figure 6: An example of a nonsolid extremal brick.

Conjecture 5.15
Every nonsolid extremal brick distinct from the Petersen graph is the result of the splicing of an extremal brick and K_4, up to multiple edges.

6 Extremal Cubic Graphs

In this section we characterize extremal cubic graphs and show that there are precisely eleven such graphs.

By Corollary 4.2, the theta-graph, the bipartite graph with just two vertices and three edges, is the only cubic extremal bipartite matching covered graph. By Corollary 5.3, every extremal cubic graph with more than two vertices is a brick. Thus, our task now consists of determining the extremal cubic bricks.

As noted in the introduction, every 2-connected cubic graph is matching covered. Furthermore, if G is any 2-connected cubic graph and C is a nontrivial cut of G with $|C| = 3$, then both C-contractions of G are cubic and 2-connected. We therefore have:

Proposition 6.1
In a 2-connected cubic graph, every 3-cut is separating.

We shall see that every extremal cubic brick other than K_4 and the Petersen graph has a nontrivial 3-cut.

6.1 Essentially 4-Connected Extremal Cubic Bricks

A 3-connected cubic graph G is said to be *essentially 4-connected* if $|\nabla(X)| > 3$, for all subsets X of $V(G)$ such that $1 < |X| < |V(G)| - 1$. In this section we shall show that there are precisely two essentially 4-connected extremal cubic bricks: K_4 and P, the Petersen graph.

Theorem 6.2
Let G be an essentially 4-connected cubic brick distinct from K_4 and the Petersen graph. Then G is not extremal.

Proof: Since $\overline{C}_6$ is not essentially 4-connected, it follows that $G \neq K_4, \overline{C}_6, P$. Therefore, by Theorem 2.11, there exists an edge $e = uv$ in G such that e is removable and b-invariant.

Assume, to the contrary, that G is extremal. By Theorem 3.3, edge e is solitary. Thus, graph $H := G - u - v$ has precisely one perfect matching. By Theorem 3.9, graph H has a cut edge, say $f = xy$.

Graph G, a brick, is 3-connected, whence H is connected. Therefore, $H - f$ has precisely two connected components. Let X and Y be the sets of vertices of the two components of $H - f$, where $x \in X$, and $y \in Y$. Since $G \neq K_4$, at least one of X and Y has cardinality greater than one. Suppose that $|X| > 1$. Then, the cut $C := \nabla(X)$ is a nontrivial cut of G. Thus, since G is essentially 4-connected, $|C| \geq 4$. And so there must be at least three edges of G from $\{u, v\}$ to X. However, the total number of edges in $\nabla(\{u, v\})$ is four. This means that there is at most one edge from $\{u, v\}$ to Y. It follows that $|\nabla(Y)| \leq 2$. This is impossible because G is 3-connected. $\square$

6.2 Nonessentially 4-Connected Extremal Cubic Bricks

In this section we show that every nonessentially 4-connected extremal cubic brick is the result of splicing K_4 and an extremal cubic brick distinct from the Petersen graph. For this, we need the following result:

LEMMA 6.3
Let G be an extremal cubic brick, C a nontrivial 3-cut of G. Then, each C-contraction of G is an extremal cubic brick.

Proof: By induction. Assume as the primary inductive hypothesis that the assertion holds for each nontrivial 3-cut of each extremal cubic brick G' having fewer vertices than G. Assume as the secondary inductive hypothesis that the assertion holds for each nontrivial 3-cut C' of G that strictly precedes C. (See the beginning of Section 5.2 for the definition of the relation of strict precedence.)

The main part of the proof is to show that cut C is robust in G. That is, each C-contraction of G is a near-brick. For this, let X be a shore of C, $G_1 := G\{X; \overline{x}\}$. We must show that G_1 is a near-brick.

Cut C is a 3-cut of G, in turn a 3-connected graph. By Proposition 6.1, C is separating in G. Thus, G_1 is a cubic matching covered graph. Every brick is a near-brick. We may thus assume that G_1 is not a brick.

Cut C is nontrivial, whence G_1 has at least four vertices. By Corollary 2.6, graph G_1 is not bicritical. Let B denote a maximal nontrivial barrier of G_1. Graph G, a brick, is bicritical. Each barrier of G_1 that does not contain vertex $\overline{x}$ is a barrier of G. We conclude that $\overline{x}$ lies in B.

Let $\mathcal{K}$ denote the set of (odd) components of $G_1 - B$. For each K in $\mathcal{K}$, let $C_K := \nabla(V(K))$, let $G_K := G\{V(K); \overline{v_K}\}$.

Graph G_1, a contraction of G, in turn a brick, is 3-edge-connected. Therefore, $|C_K| \geq 3$ for each K in $\mathcal{K}$. On the other hand, since G_1 is cubic, $\sum_{K\in\mathcal{K}} |C_K| = 3|B|$. It follows that C_K is a 3-cut, for each K in $\mathcal{K}$. Thus, G_K is cubic.

Let B_K denote any barrier of G_K. If B_K does not contain $\overline{v_K}$ then it is a barrier of G, whence it is trivial. If B_K contains $\overline{v_K}$ then $B \cup (B_K - \overline{v_K})$ is a barrier of G_1, whence $B_K = \{\overline{v_K}\}$, by the maximality of B. In both cases, B_K is trivial. This conclusion holds for each barrier B_K of G_K. Thus, G_K is a cubic bicritical matching covered graph. If K is nontrivial then, by Corollary 2.6, graph G_K is a brick. We conclude that $b(G_1)$ is equal to the number of nontrivial components of $G_1 - B$.

If every component of $G_1 - B$ is trivial then G_1 is bipartite. Moreover, B is one of the parts of the bipartition of G_1. In that case, the other part is a nontrivial barrier of G, a contradiction. We conclude that at least one component of $G_1 - B$ is nontrivial.

Assume, to the contrary, that $G_1 - B$ has at least two nontrivial components, say J and L. For every perfect matching M of G,

$$|B| - 2 + |M \cap C_J| + |M \cap C_L| \leq \sum_{K\in\mathcal{K}} |M \cap C_K| = |B| - 1 + |M \cap C|,$$

whence $|M \cap C_J| + |M \cap C_L| \leq 1 + |M \cap C|$. Thus, each of C_J and C_L precedes C. Moreover, neither C_J nor C_L is tight in G, therefore each of C_J and C_L strictly precedes C. In particular, C_J is a nontrivial 3-cut of G that strictly precedes C. By induction, each C_J-contraction of G is an extremal cubic brick.

In particular, $H := G\{\overline{V(J)}; v_J\}$ is an extremal cubic brick. Moreover, C is a nontrivial 3-cut of H. By induction, each C-contraction of H is an extremal cubic brick. But B is a nontrivial barrier of the C-contraction of H that contains vertex v_J, a contradiction.

Thus, we have shown that if G_1 is not a brick then it has a nontrivial barrier B such that precisely one component of $G_1 - B$, say J, is nontrivial. Moreover, G_J is a brick. This implies that $b(G_1) = 1$. That is, G_1 is a near-brick. This conclusion holds for each C-contraction G_1 of G. As asserted, C is robust. This completes the main part of the proof.

By Theorem 3.5, each C-contraction of G is an extremal cubic near-brick. By Theorem 5.1, each C-contraction of G is a brick. As asserted, each C-contraction of G is an extremal cubic brick. □

Theorem 6.4

Let G be a cubic brick that is not essentially 4-connected. If G is extremal then it is the result of splicing K_4 and an extremal cubic brick distinct from the Petersen graph.

Proof:By hypothesis, graph G is not essentially 4-connected, therefore it has at least one nontrivial 3-cut. Choose a nontrivial 3-cut $C := \nabla(X)$ such that X is minimal and let $G_1 := G\{X; \overline{x}\}$ and $G_2 := G\{\overline{X}; x\}$ denote the C-contractions of G.

By Lemma 6.3, each of G_1 and G_2 is an extremal cubic brick. By Theorem 3.5, G has precisely one perfect matching that contains more than one edge in C.

Let us first show that G_1 must be essentially 4-connected. Suppose that this is not the case. Then G_1 has a nontrivial 3-cut $C_1 := \nabla(Y)$, where Y is a proper subset of X. This is a contradiction to the minimality of X. Indeed, G_1 is essentially 4-connected.

Let us now show that G_1 is K_4. Graph G_1 is an extremal cubic brick that is essentially 4-connected. By Theorem 6.2, G_1 is one of K_4 and P, the Petersen graph. It can be easily seen that if $G_1 = P$, then G would have at least two perfect matchings that include cut C. This is precluded by Theorem 3.5. Therefore, graph G_1 is K_4. Likewise, G_2 cannot be the Petersen graph, otherwise G would have at least two perfect matchings that include cut C. The assertion holds. □

The following Theorem is an immediate consequence of Theorem 3.5 and Theorem 6.4. A vertex v of a cubic graph H is said to satisfy the *unique matching condition* if the graph obtained from H by deleting v and all the three neighbours of v has a unique perfect matching.

THEOREM 6.5
Let G be an extremal cubic brick different from K_4 and P. Then

$$G = (H \odot K_4)_v,$$

where H is an an extremal cubic brick on $|V(G)| - 2$ vertices distinct from P and v is a vertex of H that satisfies the unique matching condition.

Theorem 6.5 suggests how all extremal bricks distinct from the Petersen graph may be generated. We start with K_4. Clearly it is the only extremal cubic brick on four vertices. For $n \geq 4$, suppose that the set $\mathcal{G}_n$ of all extremal cubic bricks on n vertices is known. Then each graph in the set $\mathcal{G}_{n+2}$, the set of cubic bricks on $n+2$ vertices, is of the form $(H \odot K_4)_v$, where H is a member of $\mathcal{G}_n$ and v is a vertex of H that satisfies the unique matching condition. All graphs (up to isomorphism) that can be generated in this way are shown in Figure 7.

Let $G_1 := K_4$, $G_2 := \overline{C}_6$ and $G_3 := R_8$. Since G_1 and G_2 are vertex-transitive, it follows that $\mathcal{G}_4 = \{G_1\}$, $\mathcal{G}_6 = \{G_2\}$ and $\mathcal{G}_8 = \{G_3\}$. The automorphism group of G_3 has three orbits. Vertices of G_3 marked u and v in Figure 7 belong to different orbits and satisfy the unique matching condition. The vertices of the third orbit of G_3 do not satisfy the unique matching condition. Thus we obtain graphs $G_4 := (G_3 \odot K_4)_u$ and $G_5 := (G_3 \odot K_4)_v$ that constitute the set $\mathcal{G}_{10}$ of extremal cubic bricks on ten vertices. It can be seen that $\mathcal{G}_{12} = \{G_6, G_7\}$ where $G_6 = (G_4 \odot K_4)_u \cong (G_5 \odot K_4)_v$ and $G_7 = (G_5 \odot K_4)_v$, $\mathcal{G}_{14} = \{G_8\}$ where $G_8 = (G_6 \odot K_4)_u \cong (G_7 \odot K_4)_u$ and, finally, $\mathcal{G}_{16} = \{G_9\}$ where $G_9 = (G_8 \odot K_4)_u$. To confirm that the nine graphs $G_1, G_2, ..., G_9$ are the only extremal cubic graphs, other than the Petersen graph and the theta graph, the following facts need to be verified for each G_i:

1. Every labelled vertex of G_i has the unique matching property and different labelled vertices of G_i belong to different orbits under the automorphism group of G_i, and

2. Every unlabelled vertex of G_i belongs to the same orbit as a labelled vertex or it does not have the unique matching property.

There is only one extremal cubic brick on sixteen vertices, namely G_9 (Figure 7) and no vertex of this graph satisfies the unique matching condition. Therefore, there are no extremal cubic bricks on eighteen vertices. It is quite interesting that this procedure cannot be carried on forever.

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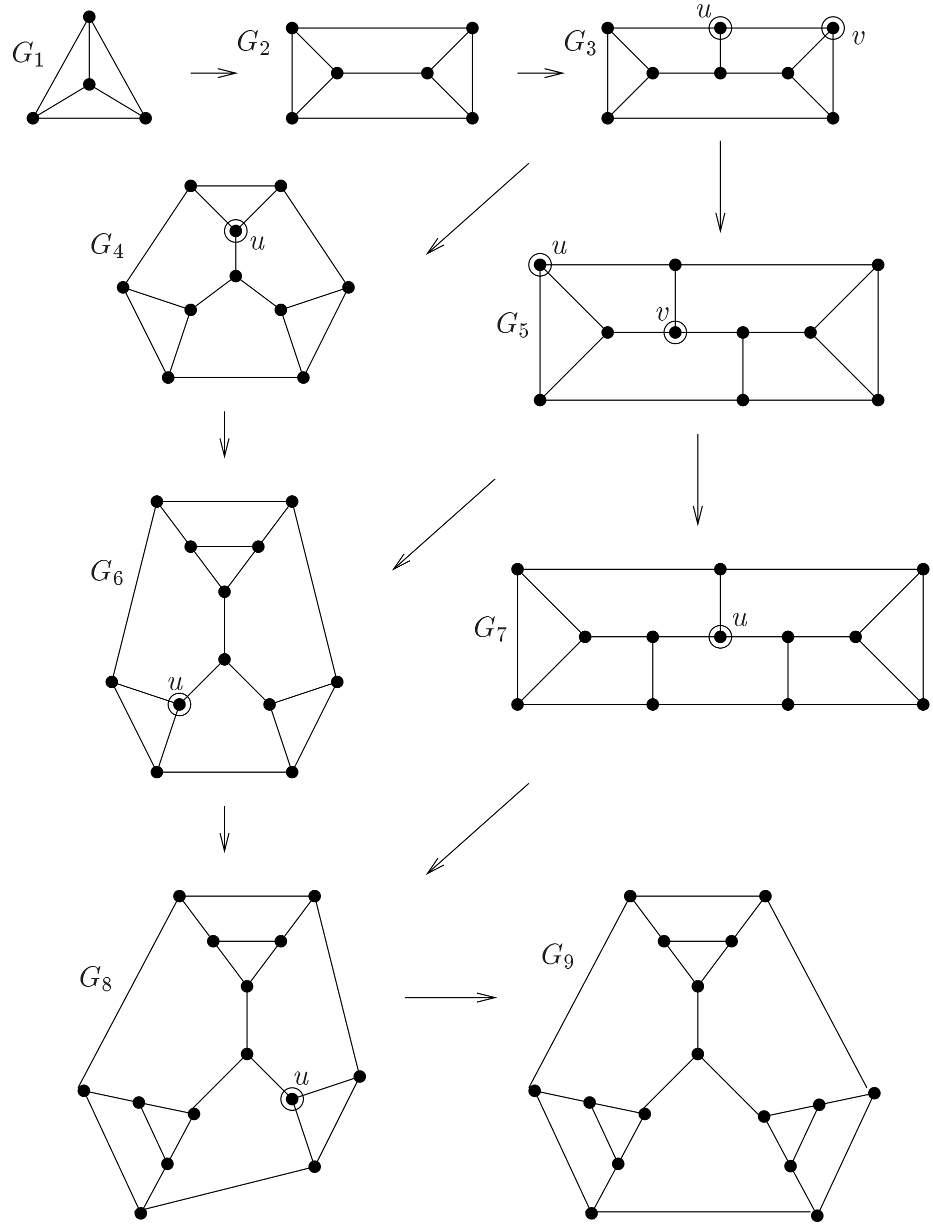

Figure 7: Extremal cubic bricks

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