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Tutte's 3-flow Conjecture and Matchings in Bipartite Graphs

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Abstract

Tutte's 3-flow conjecture is restated as the problem of finding an orientation of the edges of a 4-edge-connected, 5-regular graph G, for which the out-flow at each vertex is $+3$ or -3. The induced equipartition of the vertices of G is called mod 3-orientable. We give necessary and sufficient conditions for the existence of mod 3-orientable equipartitions in general 5-regular graphs, in terms of (i) a perfect matching of a bipartite graph derived from the equipartition and (ii) the size of cuts in G. Also, we give a polynomial time algorithm for testing whether an equipartition is mod 3-orientable.

1 Introduction

A *(nowhere-zero) k-flow* for an undirected graph $G = (V, E)$ is an assignment of directions and integer weights to the edges in E such that (i) the weights are restricted to the values in the range $1, \ldots, k-1$ and (ii) the sum of weights over edges leaving any vertex v in V minus the sum over those entering v, the *out-flow* at v, is equal to zero. A *(nowhere-zero) mod k-flow* is defined similarly, differing from a k-flow only in the restriction on the out-flow, which is allowed to be zero mod k at every vertex. Since any mod k-flow can be converted into a k-flow (see [15] for a proof), a graph admits a k-flow if and only if it admits a mod k-flow.

The theory of k-flows was introduced by Tutte as a generalization of face k-colorings for planar graphs. In particular, Tutte has proposed three well known conjectures, the 5-, 4- and 3-flow Conjectures, stated below, which generalize three famous theorems related to face k-colorings for planar graphs: The Five Color Theorem [7], the Four Color Theorem [8] and Grötzsch's Theorem [6].

5-Flow Conjecture *Every 2-edge-connected graph admits a 5-flow.*

4-Flow Conjecture *Every 2-edge-connected graph without a Petersen minor admits a 4-flow.*

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3-Flow Conjecture *Every 4-edge-connected graph admits a 3-flow.*

These conjectures are still open. Partial results on them can be found in [4, 13, 16]. The best result for graphs in general is the 6-flow Theorem, proved by Seymour [13, 15]:

6-Flow Theorem *Every 2-edge-connected graph admits a 6-flow.*

Recently, Robertson, Sanders, Seymour and Thomas [9, 10, 11, 12] proved the 4-flow Conjecture for cubic graphs.

In this paper we are interested in the 3-flow Conjecture. The best result known so far regarding this Conjecture is due to Dahab and Younger [3] who proved that every 2-edge-connected planar or projective planar graph with at most three 3-cuts admits a 3-flow. This is an extension of a proof due to Steinberg and Younger [14] that every 2-edge-connected planar graph with at most three 3-cuts and every 2-edge-connected projective planar graph with at most one 3-cut admits a 3-flow.

A graph admits a 3-flow if and only if it admits a *mod 3-orientation*, i.e., an assignment of an orientation and weight 1 to every edge, such that at every vertex the out-flow equals zero modulo 3. Observe that any mod 3-flow can be converted into a mod 3-orientation by reversing the orientation of every edge with weight 2 and complementing mod 3 its weight. If $G = (V, E)$ is a 5-regular graph, then a mod 3-orientation induces a natural *equipartition* (V^+, V^-), i.e. a partition in two sets of equal size, of V where V^+ contains the vertices with out-flow equal to $+3$, the *sources*, and V^- contains the vertices with out-flow equal to -3, the *sinks*. Correspondingly, equipartitions of V which are thus induced by some mod 3-orientation of G are called *mod 3-orientable*. Now, in [2, 4], it is shown that any 4-edge-connected graph G can be converted into a 5-regular 4-edge-connected graph G' in such a way that if G' admits a 3-flow, so does G. Therefore Tutte's 3-flow Conjecture is restated as:

3-Flow Conjecture *Every 5-regular 4-edge-connected graph has a mod 3-orientable equipartition of its vertex set.*

This paper presents two characterizations of mod 3-orientable equipartitions for 5-regular graphs. These characterizations lead us to a polynomial time algorithm for testing whether an equipartition is mod 3-orientable. Both characterizations and the algorithm are shown in Section 3. In Section 2 we give the definitions and terminology used in the subsequent proofs.

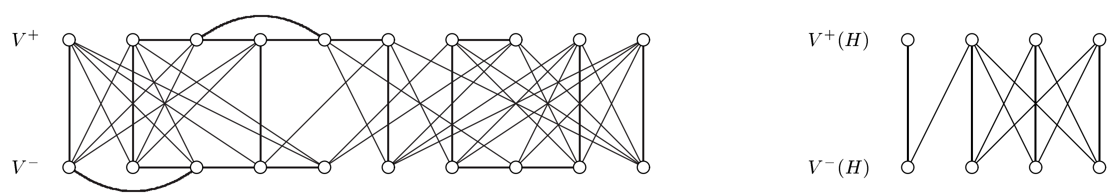
2 Definitions and Terminology

In this paper we take G to be an undirected graph with vertices $V(G)$ and edges $E(G)$, or simply V and E when the context permits. Some basic definitions in graph theory are not presented here, but they can be found in [1].

Given $X \subseteq V$, $G[X]$ denotes the subgraph of G induced by X. For sets $X, Y \subseteq V$, $\delta(X, Y)$ denotes the subset of edges in E having one end in X and the other in Y. When $Y = V \setminus X$, we say that $\delta(X, Y)$ is a *cut*, denoted by δX. Given disjoint subsets X and Y of V, $G[X]$ and $G[Y]$ are *adjacent subgraphs* if $G[X]$ and $G[Y]$ are connected and $\delta(X, Y) \neq \emptyset$. The set of all connected components of G is denoted by $K(G)$. For $X \subseteq V$ and $H \subseteq K(G[X])$, the set of *neighbor components* of H, denoted N_H, is the subset of components in $K(G[V \setminus X])$ which are adjacent to some component of H.

A *tree* T is a connected graph having $|E(T)| = |V(T)| - 1$. A *crown* C is a connected graph having $|E(C)| = |V(C)|$, i.e, it is a tree plus one edge, thus containing exactly one cycle. We denote by $t(G)$ the number of connected components of graph G which are trees.

Let G be a 5-regular graph and (V^+, V^-) an equipartition of its vertices. Then (V^+, V^-) is *mod 3-promising* if for every component S in $K(G[V^+])$ and $K(G[V^-])$ we have $|E(S)| \leq |V(S)|$, i.e. S is a tree or a crown. Figure 1(a) shows an example of a mod 3-promising equipartition. Given a mod 3-promising equipartition (V^+, V^-) of G, we define the *shrink* operation with respect to (V^+, V^-) as the deletion of the crowns of $K(G[V^+])$ and $K(G[V^-])$ followed by the contraction of the trees in $K(G[V^+])$ and $K(G[V^-])$ into single vertices. We also remove multiple edges possibly generated by these contractions. The bipartite graph H resulting from G after this shrink operation has bipartition $(V^+(H), V^-(H))$ corresponding to the trees in $K(G[V^+])$ and $K(G[V^-])$, respectively. Figure 1 illustrates the shrink operation.



(a) A mod 3-promising equipartition of G (b) Shrunken graph H

Figure 1: The shrink operation

3 Characterizations

As we have mentioned in Section 1, the 5-regular graphs which admit a 3-flow are those which admit a mod 3-orientable equipartition. In the case of a bipartite 5-regular graph G with bipartition (V^+, V^-), there is always such an equipartition, namely the bipartition (V^+, V^-) itself. To see it, remember that G admits a perfect matching, by Hall's Theorem (see [1]). Thus, given a perfect matching M of G, we obtain a mod 3-orientation D for G directing the edges of M with tail in V^- and head in V^+ and all other edges with tail in V^+ and head in V^-, as shown in Figure 2.

This technique for the bipartite case inspired, for general 5-regular graphs, two characterizations of mod 3-orientable equipartitions, stated as Theorems 1 and 2:

Theorem 1 *Let $G = (V, E)$ be a 5-regular graph and (V^+, V^-) an equipartition of V. Then (V^+, V^-) is mod 3-orientable if and only if it is mod 3-promising and the graph H obtained by shrinking G with respect to (V^+, V^-) has a perfect matching.*

(a) G with perfect matching M (b) A mod 3-orientation D for G

Figure 2: A mod 3-orientation for a 5-regular bipartite graph

Theorem 2 *Let $G = (V, E)$ be a 5-regular graph and (V^+, V^-) an equipartition of V. Then (V^+, V^-) is mod 3-orientable if and only if for all $Z \subseteq V$ with $Z^+ = Z \cap V^+$ and $Z^- = Z \cap V^-$, the following holds:*

$$|\delta Z| \geq 3|\,|Z^+| - |Z^-|\,|. \tag{1}$$

The necessity of the condition in Theorem 2 can be easily understood. We define the *out-flow at a vertex set Z* as the sum of the weights of the edges of δZ directed with tail in a vertex of Z and head in a vertex of $\overline{Z}$ minus the sum of the weights of the edges of δZ directed with head in a vertex of Z and tail in a vertex of $\overline{Z}$. It is not hard to see that the out-flow at a vertex set Z equals the sum of the out-flow at every vertex in Z. Now, let D be a mod 3-orientation for which every vertex in V^+ is a source and every vertex in V^- is a sink. For every $Z \subseteq V$, the out-flow at Z equals $3(|Z^+| - |Z^-|)$. Then, since every edge has weight one, δZ must have at least $3|\,|Z^+| - |Z^-|\,|$ edges. In particular, this relation must hold for the subsets X of V^+ or V^-, which implies, by Lemma 1 below, that a mod 3-orientable equipartition must be mod 3-promising, as required by Theorem 1.

Lemma 1 *Let $G = (V, E)$ be a 5-regular graph and (V^+, V^-) an equipartition of V. Then (V^+, V^-) is mod 3-promising if and only if for all $X \subseteq V^+$ and for all $X \subseteq V^-$ the following holds:*

$$|\delta X| \geq 3|X|.$$

Proof. (Necessity) Take $X \subseteq V^+$. By hypothesis, every connected component in $K(G[V^+])$ is a tree or a crown. In particular, the connected components of $K(G[X])$ are only trees or crowns. Hence, we have $|E(G[X])| \leq |X|$, with equality holding only when every component in $K(G[X])$ is a crown. By the 5-regularity of G, we have:

$$|\delta X| = 5|X| - 2|E(G[X])| \geq 5|X| - 2|X| = 3|X|.$$

The same argument can be used to prove the result for $X \subseteq V^-$.

(Sufficiency) Take $G_i \in K(G[V^+])$. By hypothesis $|\delta V(G_i)| \geq 3|V(G_i)|$. By the 5-regularity of G, we have:

$$2|E(G_i)| = 5|V(G_i)| - |\delta V(G_i)| \leq 5|V(G_i)| - 3|V(G_i)| = 2|V(G_i)|.$$

Thus, $|E(G_i)| \leq |V(G_i)|$.

On the other hand, since G_i is connected, we have $|E(G_i)| \geq |V(G_i)| - 1$. Hence, either G_i is a tree and $|E(G_i)| = |V(G_i)| - 1$ or G_i is a crown and $|E(G_i)| = |V(G_i)|$. The proof for $G_i \in K(G[V^-])$ is analogous. □

Nevertheless, there are mod 3-promising equipartitions such as that shown in Figure 3 which are not mod 3-orientable. The set $Z = \{1, 7, 8, 9, 10, 11\}$ violates the necessary condition (1) since $|\delta Z| = 10$ and $||Z^+| - |Z^-|| = 4$; so it certifies that this equipartition actually cannot be mod 3-orientable.

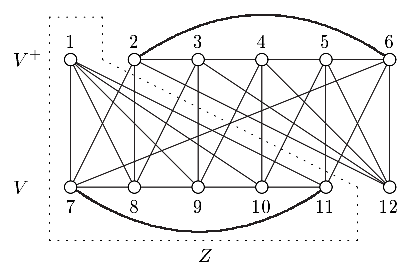


Figure 3: A mod 3-promising equipartition which is not mod 3-orientable

A mod 3-promising equipartition (V^+, V^-) has other interesting properties which follow from the 5-regularity of the graph. The first, which results from a simple counting argument, is that for every tree T and crown C in $K(G[V^+]) \cup K(G[V^-])$, we have $|\delta V(T)| = 3|V(T)| + 2$ and $|\delta V(C)| = 3|V(C)|$, respectively. Furthermore, as shown by Lemma 2, the number of trees in $K(G[V^+])$ equals the number of trees in $K(G[V^+])$.

Lemma 2 *Let G be a 5-regular graph with a mod 3-promising equipartition (V^+, V^-). Then the number of trees in $K(G[V^+])$ and in $K(G[V^-])$ is the same. That is,*

$$t(G[V^+]) = t(G[V^-]).$$

Proof. First,

$$|E(G[V^+])| = \sum_{G_i \in K(G[V^+])} |E(G_i)|.$$

By hypothesis, any component $G_i \in K(G[V^+])$ is a tree or a crown. Moreover,

$$\sum_{\substack{G_i \in K(G[V^+]) \\ G_i\ crown}} |E(G_i)| = \sum_{\substack{G_i \in K(G[V^+]) \\ G_i\ crown}} |V(G_i)|, \text{ and}$$

$$\sum_{\substack{G_i \in K(G[V^+]) \\ G_i\ tree}} |E(G_i)| = \sum_{\substack{G_i \in K(G[V^+]) \\ G_i\ tree}} |V(G_i)| - t(G[V^+]).$$

Hence,

$$|E(G[V^+])| = |V^+| - t(G[V^+]).$$

Analogously,

$$|E(G[V^-])| = |V^-| - t(G[V^-]).$$

By hypothesis, $|V^+| = |V^-|$ and as G is 5-regular, we have $|E(G[V^+])| = |E(G[V^-])|$. Hence, we conclude that $t(G[V^+]) = t(G[V^-])$. □

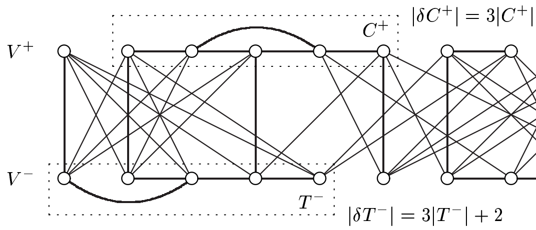
Now, we are ready to prove Theorem 1.

Proof of Theorem 1. (Necessity) Let (V^+, V^-) be a mod 3-orientable equipartition and D an associated mod 3-orientation of G. We already know that (V^+, V^-) must be mod 3-promising. So, let $\mathcal{T}^+$, $\mathcal{C}^+$, $\mathcal{T}^-$, $\mathcal{C}^-$ be the collections of trees and crowns in $K(G[V^+])$ and $K(G[V^-])$. For every tree $T^+ \in \mathcal{T}^+$, $|\delta V(T^+)| = 3|V(T^+)| + 2$; thus D must direct exactly one edge e of $\delta V(T^+)$ with head in $V(T^+)$; this is called a *minority edge*. For every crown $C^+ \in \mathcal{C}^+$, no edges of $\delta V(C^+)$ have heads in $V(C^+)$. Similar statements hold for the trees and crowns of $G[V^-]$. Thus a minority edge of a tree $T^+ \in \mathcal{T}^+$ is also a minority edge of some tree $T^- \in \mathcal{T}^-$ and vice-versa (see Figure 4). The set of all such minority edges form a perfect matching in H, the graph resulting from shrinking G with respect to (V^+, V^-).

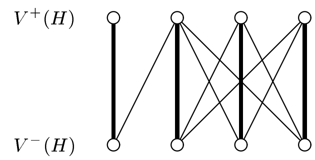
(Sufficiency) Before proceeding we need a definition: A *branching* is a directed tree B for which the indegree at every vertex is exactly one, except at one vertex r, called the *root* of B, which has indegree zero.

Let (V^+, V^-) be a mod 3-promising equipartition such that the graph H obtained after shrinking G with respect to (V^+, V^-) has a perfect matching M. Direct the edges of δV^+ which are images of edges of M so that their tails are in V^- and their heads in V^+. Direct the remaining edges of δV^+ in the opposite direction.

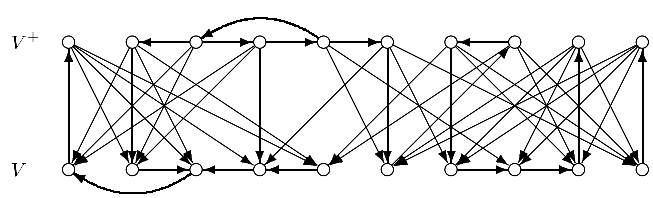
Now, extend this orientation to each tree T^+ in $\mathcal{T}^+$ directing the edges of T so that the resulting (oriented) tree is a branching having as root the vertex incident to the minority edge of $\delta V(T^+)$. This makes every vertex in T^+ a source. For a tree T^- in $\mathcal{T}^-$ the procedure is analogous but with orientations reversed; every vertex in the oriented T^- is thus a sink. For a crown C^+ in $\mathcal{C}^+$, contracting its unique cycle produces a tree L, which can be made into a branching with root the contraction vertex of the cycle. Restoring the cycle back to C^+ and directing its edges so as to form an oriented cycle, produces an orientation for C^+ in which every vertex is a source. The procedure for a crown C^- in $\mathcal{C}^-$ is analogous. This proves that (V^+, V^-) is mod 3-orientable. □



(a) A mod 3-promising equipartition of G



(b) Perfect matching of H



(c) A mod 3-orientation of G

Figure 4: An illustration of the proof of Theorem 1

Proof of Theorem 2. (Sufficiency) We have already proved necessity above. Let F be a subset of the trees in $K(G[V^+])$ and N_F its corresponding set of neighbor components. For each crown C in $K(G[V^+])$, $\delta(C)$ contributes $3|V(C)|$ edges to δV^+; similarly, each tree T contributes $3|V(T)|+2$ edges to δV^+. The same holds for the crowns and trees of $K(G[V^-])$. Therefore we have $|\delta V(F)| = 3|V(F)| + 2t(F)$ and $|\delta V(N_F)| = 3|V(N_F)| + 2t(N_F)$.

Let $Z = V(F) \cup V(N_F)$. By the definition of N_F we have $\delta V(F) \subseteq \delta V(N_F)$; so $\delta Z = \delta V(N_F) \setminus \delta V(F)$. Thus

$$|\delta Z| = |\delta V(N_F)| - |\delta V(F)| = 3(|V(N_F)| - |V(F)|) + 2(t(N_F) - t(F)).$$

Rewriting this equation, we have

$$2(t(N_F) - t(F)) = |\delta Z| - 3(|V(N_F)| - |V(F)|).$$

By hypothesis, $|\delta Z| \geq 3|\,|V(N_F)| - |V(F)|\,|$, so

$$2(t(N_F) - t(F)) \geq 3|\,|V(N_F)| - |V(F)|\,| - 3(|V(N_F)| - |V(F)|) \geq 0.$$

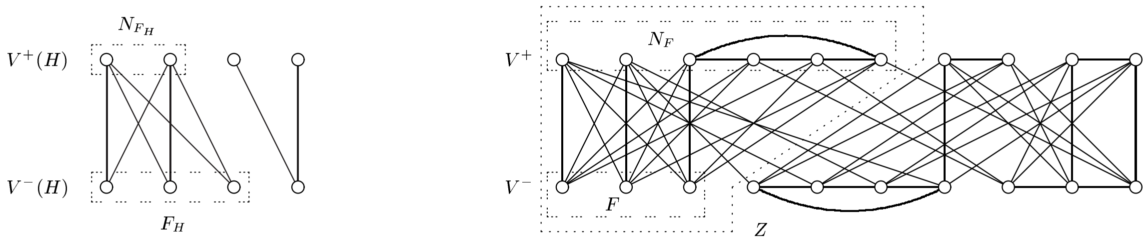
Therefore, $t(N_F) \geq t(F)$. Consider now the bipartite graph H obtained by shrinking G with respect to (V^+, V^-). Let $F_H \subseteq V^+(H)$ be the vertices representing the trees in F; clearly $t(F) = |F_H|$. For every tree in N_F there is a tree in N_{F_H} representing it; thus $t(N_F) = |N_{F_H}|$. Hence $|N_{F_H}| \geq |F_H|$ and by Hall's Theorem there is a matching M in H which covers $V^+(H)$. Since $|V^+(H)| = |V^-(H)|$, M is a perfect matching. By Theorem 1, (V^+, V^-) is mod 3-orientable. □

3.1 Testing the mod 3-orientability of an Equipartition in Polynomial Time

Theorem 1 suggests a simple algorithm for testing whether a given equipartition (V^+, V^-) of a 5-regular graph G is mod 3-orientable: we test if (V^+, V^-) is mod 3-promising and, in the affirmative case, generate the shrunken graph H and test whether H has a perfect matching. Furthermore, the proof of Theorem 1 produces a mod 3-orientation when (V^+, V^-) is mod 3-orientable.

Observe that this test fails in two situations: when (V^+, V^-) is not mod 3-promising or when the shrunken graph H does not have a perfect matching. In the former case, there must be a connected component S either in $K(G[V^+])$ or in $K(G[V^-])$ which is not a tree or crown, i. e. $|\delta V(S)| < 3|V(S)|$. Thus, by Theorem 2, the set $Z = V(S)$ is a certificate that (V^+, V^-) is not mod 3-orientable. In the latter case, there must be a subset F_H either of $V^+(H)$ or of $V^-(H)$, for which $|F_H| < |N_{F_H}|$. Let F be the set of trees represented by F_H, N_F its corresponding set of neighbor components and take $Z = V(F) \cup V(N_F)$, as shown in Figure 5. Then, the sufficiency of Theorem 2 shows that Z is a certificate of non-mod 3-orientability of (V^+, V^-), i. e.,

$$|\delta Z| < 3|\,|V(N_F)| - |V(F)|\,|.$$



(a) $|N_{F_H}| < |F_H|$

(b) $|\delta Z| < 3|\,|V(N_F)| - |V(F)|\,|$

Figure 5: Finding a certificate of non-mod 3-orientability for an equipartition

Therefore, we have an algorithm for testing the mod 3-orientability of an equipartition (V^+, V^-) of a 5-regular graph which either finds a mod 3-orientation in the affirmative case or exhibits a set Z that violates the condition of Theorem 2 otherwise. A pseudo-code for the algorithm is shown in Figure 6.

We will now argue that the algorithm shown in Figure 6 takes polynomial time with respect to the size of the graph G. To test if the equipartition is mod 3-promising it is enough to traverse subgraphs $G[V^+]$ and $G[V^-]$ to determine their connected components and check whether all of them are trees and crowns. This step takes linear time. Having identified all the trees in $K(G[V^+])$ and $K(G[V^-])$, constructing the shrunken graph H takes another linear time traversal of graph G.

Now, to find a perfect matching M for H or a set F_H certifying that such a matching does not exist, we can use an augmenting path algorithm such as that described in Chapter 5 of [1]. This algorithm takes polynomial time with respect to the size of H, and thus to the size of G. Hence, the algorithm is actually polynomial.

4 Final Comments

We gave two necessary and sufficient conditions for the existence of mod 3-orientable equipartitions in general 5-regular graphs. We also derived from these conditions a polynomial time algorithm for testing the mod 3-orientability of an equipartition of a 5-regular graph.

It is known (see GT4 of [5]) that deciding whether an arbitrary graph admits a 3-flow is an NP-complete problem. It is possible to prove (see [2]) that the problem remains NP-complete even if restricted to the class of the 5-regular graphs. Thus there is little hope that our results will lead to polynomial algorithms for this decision problem. However, they may help in settling Tutte's conjecture which is restricted to 4-edge-connected 5-regular graphs.

Input: A 5-regular graph $G = (V, E)$ and an equipartition (V^+, V^-) of V.
Output: A mod 3-orientation for G or a subset $Z \subseteq V$ such that $|\delta Z| < 3\big|\,|Z^+| - |Z^-|\,\big|$.

(1) if (V^+, V^-) is mod 3-promising, then
/* obtain shrunken graph H */
(2) shrink G with respect to (V^+, V^-) obtaining H;
(3) if H has a perfect matching M then
(3.1) direct the edges of M with tail V^- and head in V^+;
(3.2) direct the remaining edges of δV^+ in the opposite direction;
(3.3) extend orientation to trees and crowns in $K(G[V^+]) \cup K(G[V^-])$
(4) else
/* there is F_H for which $|F_H| < |N_{F_H}|$ */
(4.1) take F as the subset of the trees represented by F_H;
(4.2) return $Z = V(F) \cup V(N_F)$.
(5) else
/* there is $S \in K(G[V^+]) \cup K(G[V^-])$ such that $|V(S)| < |E(S)|$ */
(5.1) return $Z = V(S)$.

Figure 6: An algorithm for testing whether an equipartition is mod 3-orientable

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