On the Relation between the Petersen Graph and the Characteristic of Separating Cuts of Matching Covered Graphs

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Abstract

A *matching covered graph* is a connected graph each edge of which lies in some perfect matching. A cut of a matching covered graph is *separating* if each of its two contractions yields a matching covered graph. A cut is *tight* if each perfect matching of the graph contains just one edge in the cut. Every tight cut of a matching covered graph is separating. The *characteristic* of a nontight separating cut is the smallest number of edges greater than one that some perfect matching of the graph has in the cut. The characteristic of a tight cut is defined to be equal to ∞.

We show that the characteristic of every separating cut C of a matching covered graph lies in $\{3, 5, \infty\}$. Moreover, if C has characteristic equal to 5 then graph G has the Petersen graph as a minor, in a very strict sense. In particular, if G is free of nontrivial tight cuts then G is the Petersen graph, up to multiple edges.

1 Introduction

Matching theory has had a fast development after Hall and Tutte's Theorems. Hall's theorem establishes necessary and sufficient conditions for a bipartite graph to have a perfect matching and Tutte's theorem establishes necessary and sufficient conditions for a general graph to have a perfect matching. We refer the reader to Lovász and Plummer [5], Murty [6] or Lovász [4].

We shall use $V(G)$ and $E(G)$, respectively, for the set of vertices and edges of a graph G.

A *matching* of a graph G is a set of edges that do not have any end in common. We say that a matching M of G is *perfect* if every vertex of G is an end of some edge of M. An edge of a graph G is *admissible* in G if it lies in some perfect matching of G. A graph G is *matching covered* if it is connected and each edge is admissible in G.

Let G be a matching covered graph. For subset X of $V(G)$, $\nabla(X)$ denotes the edge-cut associated with X, that is, the set of edges of G having one end in X and the other in $\overline{X}$; we say that X is a *shore* of $\nabla(X)$. Since G is connected, sets X and $\overline{X}$ are the only shores of $\nabla(X)$. Since G has perfect matchings, the size of sets X and $\overline{X}$ have the same parity. Cut $\nabla(X)$ is *odd* or *even*, depending on the parity of $|X|$. We reserve the word *cut* to mean an edge-cut. Cut $\nabla(X)$ is *trivial* if one of X and $\overline{X}$ has at most one vertex.

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Two cuts D_1 and D_2 of a matching covered graph G are *matching-equivalent* if, for every perfect matching M of G, the number of edges of M in D_1 and in D_2 coincide.

Let C denote $\nabla(X)$. The graph obtained from G by contracting set $\overline{X}$ to a single new vertex $\overline{x}$ is a C-*contraction* of G and is denoted by $G\{X;\overline{x}\}$. If the name of the new vertex is irrelevant we then simply denote the contraction by $G\{X\}$. Observe that this notation is inspired in the traditional notation $G[X]$, used to denote the subgraph of G spanned by set X.

Cut C is *tight* in G if every perfect matching of G has precisely one edge in C. If G is free of nontrivial tight cuts then it is a *brace* if it is bipartite, a *brick* otherwise. If cut C is tight, then every C-contraction of G is matching covered (the converse is not necessarily true). This property led Lovász to define a *tight cut decomposition* of G to be a collection of matching covered graphs obtained from the initial collection $\{G\}$ by repeatedly replacing each member H of the collection by the two C-contractions of H, for any nontrivial tight cut C of H, until every member of the collection is free of nontrivial tight cuts. A remarkable result, shown by Lovász in [4], states that any two tight cut decompositions of G yield the same family of graphs, up to multiple edges. Thus, the number of bricks of any tight cut decomposition of G is an invariant of the graph and is denoted $b(G)$. Graph G is a *near-brick* if $b(G) = 1$.

Any graph H obtained during the application of a tight cut decomposition procedure to G is a tight cut minor of G. More formally, a graph H is *tight cut minor* of a matching covered graph G if, and only if, either (i) graph H is graph G, or (ii) graph G has a nontrivial tight cut C such that graph H is a tight cut minor of a C-contraction of G.

A *barrier* B of a matching covered graph is a nonempty set of vertices such that the the number of odd components of $G - B$ is equal to the cardinality of set B. If B is a barrier of a matching covered graph then $G - B$ has no even components. A barrier B is *trivial* if it has at most one vertex.

Let B be a nontrivial barrier of G and K be a nontrivial odd component of $G - B$, thus $\nabla(V(K))$ is a nontrivial tight cut, namely *barrier cut.* Let $\{u, v\}$ be a 2-separation of G that is not a barrier. Let K be an even component of $G - \{u, v\}$. Then $\nabla(K \cup \{u\})$ and $\nabla(K \cup \{v\})$ are both tight cuts of G, namely *2-separation cuts.* Figure 1 shows examples of these cuts. These two cuts are important because of a remarkable result, due to Lovász, Edmonds and Pulleyblank, that states that if a matching covered graph has a nontrivial tight cut then it has a nontrivial tight cut that is, either a barrier cut, or a 2-separation cut.

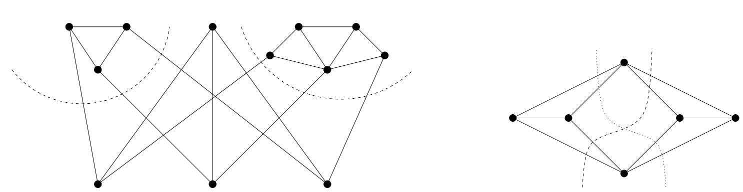


Figure 1: Two special types of tight cuts: barrier cut and 2-separation-cut

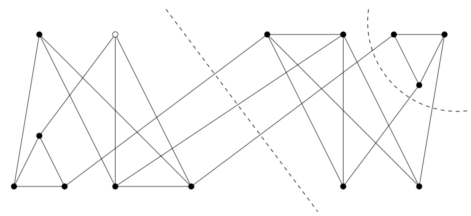


Figure 2: Two separating cuts that do not form a cohesive collection

Cut C of G is *separating* in G if each of its C-contractions is matching covered. Thus, every tight cut of G is separating in G. A separating cut of G is *strictly separating* if both C-contractions of G are non-bipartite.

The *characteristic* of a separating cut C of G, denoted $\lambda_G(C)$, is the minimum number of edges that perfect matchings of G have in C, among all perfect matchings that have more than one edge in C. Thus, $\lambda_G(C) \geq 3$. We extend the definition to tight cuts by defining the characteristic of a tight cut to be infinite. The *characteristic* of G is the minimum of the characteristic of its separating cuts.

Carvalho et al. [2] have shown that the characteristic of every matching covered graph lies in $\{3, 5, \infty\}$. They have also shown that the only brick of characteristic 5 is the Petersen graph. We prove herein two generalizations of their result. We state now the first of the two generalizations:

THEOREM 1.1
The characteristic $\lambda_G(C)$ of any separating cut C of any near-brick G lies in $\{3, 5, \infty\}$. Moreover, if $\lambda_G(C) = 5$ then graph G has a tight cut minor P that is the Petersen graph, up to multiple edges, and cut C is strictly separating in P.

We now prepare the statement of the second generalization of the Theorem of Carvalho et al.. The following characterization of separating cuts is not difficult to prove:

LEMMA 4.1
A cut C of a matching covered graph G is separating if, and only if, every edge of G lies in a perfect matching of G that contains precisely one edge in C.

This result motivates the following definition, which will play a central role in this paper. A collection $\mathcal{C}$ of cuts of G is *cohesive* in G if for each edge of G, there exists a perfect matching M of G such that matching M contains precisely one edge in each cut of $\mathcal{C}$. Note that any collection of tight cuts of G is cohesive. Note also that every member of a cohesive collection of G is separating in G. The converse, however is not necessarily true: Figure 2 shows an example, due to Carvalho, of two separating cuts of a matching covered graph that do not constitute a cohesive collection. We may now state our second main result:

Theorem 1.2
The characteristic of any separating cut $C := \nabla(X)$ of any matching covered graph G lies in $\{3, 5, \infty\}$. Moreover, $\lambda_G(C) = 5$ if, and only if, graph G has a tight cut minor H, in which cut C is strictly separating, such that one of the following two alternatives holds:

(i) *Either graph H is the Petersen graph, up to multiple edges, or*

(ii) *graph H is not a near-brick and there exist two cuts in H, $D_1 := \nabla_H(Y_1)$ and $D_2 := \nabla_H(Y_2)$, each of which has characteristic 5 in G, set Y_1 is a subset of X and set Y_2 is a subset of $\overline{X}$, collection $\{D_1, D_2, C\}$ is cohesive, cuts D_1 and D_2 are not disjoint, and the graph obtained by the contraction of Y_1 and Y_2 is the Petersen graph, up to multiple edges.*

Figure 3 shows an example of alternative (ii) in the statement of Theorem 1.2.

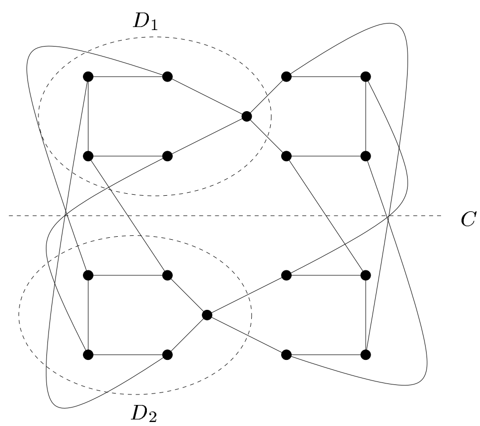


Figure 3: An illustration of alternative (ii) in (1.2)

Section 2 contains most of the basic material that is required, but which may be skipped by the reader that is quite familiar with the subject. Section 3 contains results concerning robust cuts, which are essential to the proof of main results of this paper. Section 4 introduces important results concerning cohesive collections. Section 5 contains an important result, which is called the Theorem on Odd Wheels. The proof of Theorem 1.1 is presented in Section 6. Finally, Section 7 contains a proof of Theorem 1.2.

2 Basics

In this section, we list some elementary or well-known results about matching covered graphs.

Lemma 2.1
Let $C := \nabla(X)$ be a separating cut of a matching covered graph G. Then, the C-contraction $G\{X\}$ is bipartite if, and only if, induced subgraph $G[X]$ of G is also bipartite. Moreover, if $G[X]$ is bipartite, the cardinality of two parts of the bipartition differ by one unit.

<u>Proof:</u> Assume that $G\{X\}$ is bipartite. Graph $G[X]$ is a subgraph of $G\{X\}$. Thus $G[X]$ is bipartite and the assertion holds.

Suppose that $G[X]$ is bipartite. Let (A, B) be a bipartition of $G[X]$. Cut C is a separating cut of G. Thus, for each edge of G there exists a perfect matching of G that has only one edge in C. The restriction of this matching to edges of $G[X]$ is a matching with exactly one single vertex. Therefore, $||A| - |B|| = 1$. Thus, the moreover part follows. Adjust the notation so that $|B| = |A| + 1$.

Let $A' := A \cup \{\overline{x}\}$, whence $|A'| = |B|$. Let $E_A := C \cap \nabla(A)$. Assume that $G\{X\}$ is not bipartite. Thus, the contracted vertex is adjacent to vertices of A, that is $E_A \neq \emptyset$. Let $e \in E_A$. Graph $G\{X\}$ is matching covered because C is a separating cut. Thus, there exists a perfect matching of $G\{X\}$ that includes e. Both ends of e are in A', therefore there exist $|A'| - 2$ vertices of A' to match with $|B|$ vertices of B. But $|A'| = |B|$ and B is an independent set, thus such matching, that uses e, does not exist. This contradicts the admissibility of edge e. Therefore, $G\{X\}$ is bipartite. □

Corollary 2.2

Let C be a separating cut of a matching covered graph G. If one of the C-contractions of G is bipartite then cut C is tight in G.

<u>Proof:</u> Let X be a shore of C. Adjust the notation so that $G\{X\}$ be bipartite. By (2.1) graph $G[X]$ is bipartite. Let (A, B) be a bipartition of $G[X]$, by (2.1) $||A| - |B|| = 1$. Adjust the notation so that $|A| = |B| - 1$.

Graph $G\{X\}$ is bipartite and the ends of edges of C, different from the contracted vertex, lie in B. Therefore, B is a barrier to graph G with $G[\overline{X}]$ as an odd component of $G - B$, thus cut C is a barrier cut. Therefore, cut C is a tight cut of G. □

Corollary 2.3

In a bipartite graph G, a cut is tight if, and only if, it is separating in G.

<u>Proof:</u> Let $C := \nabla(X)$ be an odd cut of G. If C is a tight cut then, as we have already seen, it is separating. So, we can assume that C is a separating cut of G. Graph $G[X]$ is bipartite, thus, by (2.1), graph $G\{X\}$ is also bipartite. By (2.2), cut C is tight in G. □

Lemma 2.4

A matching covered graph G is bipartite if, and only if, it has $b(G) = 0$.

<u>Proof:</u> Let G be a matching covered bipartite graph. Let $C := \nabla(X)$ be a separating cut. Graphs $G[X]$ and $G[\overline{X}]$ are both bipartite. Thus, by (2.1), $G\{X\}$ and $G\{\overline{X}\}$ are both bipartite. By (2.2), cut C is a tight cut of G. Thus, $b(G) = b(G\{X\}) + b(G\{\overline{X}\})$. By induction hypothesis, $b(G\{X\}) = 0$ and $b(G\{\overline{X}\}) = 0$. Therefore $b(G) = 0$.

Assume now, that $b(G) = 0$. Any tight cut decomposition yields the same list of bricks and braces and $b(G)$ is the sum of $b(G_i)$ for each G_i in the list. We conclude that any tight cut decomposition of G has only braces. Let $\mathcal{L}$ be a tight cut decomposition of G, up to multiple edges.

The proof will be by induction on the size of $\mathcal{L}$. If $\mathcal{L} = \{G\}$ then G itself is a brace and the assertion holds. Thus, we can assume that $|\mathcal{L}| \geq 2$.

Let C be a tight cut of G. Let $G\{X\}$ and $G\{\overline{X}\}$ be the C-contractions of G. Let $\mathcal{L}_X$ and $\mathcal{L}_{\overline{X}}$ be the list of bricks and braces, up to multiple edges, of any tight cut decomposition of $G\{X\}$ and $G\{\overline{X}\}$, respectively. List $\mathcal{L} = \mathcal{L}_X \cup \mathcal{L}_{\overline{X}}$, therefore $\mathcal{L}_X$ and $\mathcal{L}_{\overline{X}}$ are composed only braces, whence $b(G\{X\}) = b(G\{\overline{X}\}) = 0$.

By induction hypothesis, $G\{X\}$ and $G\{\overline{X}\}$ are bipartite. By (2.1), graphs $G[X]$ and $G[\overline{X}]$ are bipartite. Let (X_A, X_B) a bipartition of $G[X]$ and $(\overline{X}_A, \overline{X}_B)$ a bipartition of $G[\overline{X}]$. By (2.1), $||X_A| - |X_B|| = 1$ and $\left|\left|\overline{X}_A\right| - \left|\overline{X}_B\right|\right| = 1$. Adjust the notation so that $|X_A| = |X_B| - 1$ and $\left|\overline{X}_A\right| = \left|\overline{X}_B\right| + 1$ (see Figure 4).

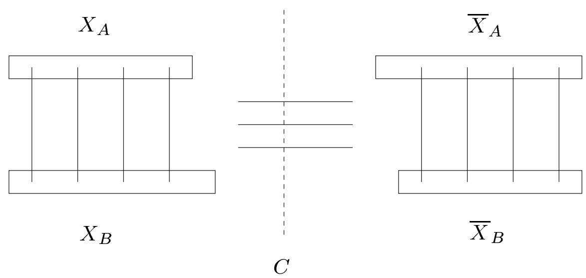


Figure 4: Graph G and cut C.

Graph $G\{X\}$ is bipartite. Thus each edge of C has one end in the contracted vertex and the other in X_B. Therefore, there are no edges of C in $\nabla(X_A)$. By symmetry, there are no edges of C in $\nabla(\overline{X}_B)$. Therefore, the edges of C have one end in X_B and the other n $\overline{X}_A$ and G is bipartite. □

Lemma 2.5

A non-bipartite matching covered graph G is a near-brick if, and only if, graph G is free of strictly separating tight cuts.

Proof: Suppose that G is a near-brick. Let C be a nontrivial tight cut of G. Let G_1 and G_2 be the two C-contractions of G. Thus, $b(G) = b(G_1) + b(G_2)$. By hypothesis, $b(G) = 1$, therefore, either $b(G_1) = 1$ and $b(G_2) = 0$, or the contrary. Adjust the notation so that $b(G_1) = 1$. By (2.4) G_2 is a bipartite graph and then C is not a strictly separating tight cut. This result holds for any tight cut of G. We conclude that G is free of strictly separating tight cuts.

Now, assume that G is free of strictly separating tight cuts. Thus, for any tight cut of G one of the C-contractions is bipartite. If G has no nontrivial tight cuts then it is a brick, whence a near-brick. So, we can assume that G has a nontrivial tight cut C. Let G_1 and G_2 be the C-contractions. Adjust the notation so that G_2 is bipartite. Thus,

$$b(G) = b(G_1) + b(G_2) = b(G_1) + 0 = b(G_1).$$

Graph G_1 is free of nontrivial strictly separating tight cuts. Thus, by induction hypothesis $b(G_1) = 1$, whence $b(G) = 1$ and the assertion holds. □

Corollary 2.6

Graph G is a near brick if, and only if, for each nontrivial tight cut of G one of the C-contractions is a bipartite graph and the other is a near-brick. □

The Petersen graph has a special role in theory of matching covered graphs. If $C := \nabla(X)$ is a separating cut of the Petersen graph, then graphs $G\{X\}$ and $G\{\overline{X}\}$ are odd wheels and $G[X]$ and $G[\overline{X}]$ are pentagons. Moreover, these two pentagons are joined by that special way (see Figure 5). Any separating cut of the Petersen graph has this structure because of the automorphisms of this graph.

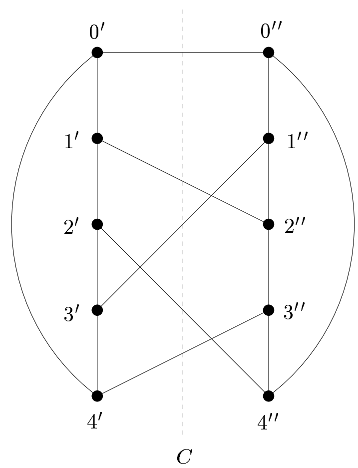


Figure 5: A separating cut in the Petersen graph.

Lemma 2.7

Let G be a matching covered graph, D be a non-tight cut of G. If a D-contraction H of G is the Petersen graph, up to multiple edges, then every nontrivial separating cut of H is a separating cut of G with characteristic three in G.

Proof: Let Y be a shore of D. Adjust the notation so that $H := G\{Y;\overline{y}\}$. Graph H is the Petersen graph, up to multiple edges. Therefore, the subjacent graph of H is cubic, whence, $\overline{y}$ has three adjacent in H. We conclude that any perfect matching of G has at most three edges in D.

Let $C := \nabla(X)$ be a separating cut of H. Cut C separates two pentagons in H. Adjust the notation so that the $C := \{0', 1', 2', 3', 4'\}$ and the contracted vertex of H be vertex $0'$ (see Figure 5). In order to prove that C is a separating cut of G we must find for each edge of G a perfect matching with one edge in C.

Let H' be the other D-contraction of G and let e be an edge of H'. There exists a perfect matching of G that uses e. Let M_e be this matching. This matching has at most three edges in D. If $|M_e \cap D| = 3$ then

$$(M_e \cap E(H')) \cup \{(2', 3'), (1'', 2''), (3'', 4'')\}$$

is a perfect matching of G that uses e and has only one edge in C.

Suppose now, that $|M_e \cap D| = 1$. Let f be the edge of M_e in D. There exists two possibilities to edge f, up to automorphisms, and in each case we have found a perfect matching with one edge in C.

$$\begin{array}{rcl} lcl\, f = (0', 0'') & \Rightarrow & (M_e \cap E(H')) \cup \{(1', 2'), (3', 4'), (1'', 2''), (3'', 4'')\} \\ f = (0', 1') & \Rightarrow & (M_e \cap E(H')) \cup \{(2', 4''), (3', 4'), (0'', 1''), (2'', 3'')\} \end{array}$$

Let $f_1 := (0', 0'')$, $f_2 := (0', 1')$ and $f_3 := (0', 4')$. Each of these edges have adjacent edges in H that are admissible edges in G. Let e be an edge that is adjacent to some f_i. There exists a perfect matching M_e of G that uses e. Cut D is an odd cut, therefore, each perfect matching has one or three edges in D. Matching M_e can not have three edges in C because one of these edges would be adjacent to e, contradicting the fact of M_e being a matching. Therefore, we conclude that $|M_e \cap D| = 1$. So, for each f_i there exist a perfect matching of G that uses f_i and only f_i. Let M_i be the restriction of such matching to edges of $E(H')$.

Let e be an edge of H. Cut C is a separating cut of H. Therefore, there exists a perfect matching of H that uses e and has only one edge in C. By construction, this matching has only one edge in D. Let f_i be this edge. Thus, $M_i \cup M_e$ is a perfect matching of G that uses e and has only one edge in C. We conclude that C is a separating cut of G.

Now, to complete the proof we need to find a perfect matching of G that has three edges in C. By hypothesis, D is a nontight cut, therefore, there exists a perfect matching M of G with three edges in D. Thus,

$$(M \cap E(H')) \cup \{(2', 4''), (3', 1''), (2'', 3'')\}$$

is a perfect matching of G with three edges in C. Therefore, $\lambda(C) = 3$. □

LEMMA 2.8
Let G be the simple graph obtained from the Petersen graph P by adding an edge e. Let C be a nontrivial separating cut of G such that $C - e$ is separating in $G - e$. Then, the characteristic of C in G is equal to three.

Proof: In order to show that $\lambda(C) = 3$ it is enough to find a perfect matching of G with three edges in C. Each separating cut of $G - e$ separates two pentagons. Let $C - e := \nabla(X)$ be a separating cut of $G - e$. Adjust the notation so that the $X := \{0', 1', 2', 3', 4'\}$ and the contracted vertex of D be vertex $0'$ (see Figure 5).

Edge e is not multiple, thus it joins two non-consecutive vertices of G. Suppose first that $e \notin C$. By automorphisms of Petersen graph, we can adjust the notation so that $e = (1', 4')$. Thus,

$$M := \{(0', 0''), (1', 4'), (2', 4''), (3', 1''), (2'', 3'')\}$$

is a perfect matching of G with three edges in C.

Now, we can assume that $e \in C$. Again by the automorphisms of Petersen graph, we can consider $e = (0', 2'')$. Thus,

$$M := \{(0', 2''), (1', 2'), (3', 1''), (4', 3''), (0'', 4'')\}$$

is a perfect matching of G with three edges in C. Therefore, the assertion follows. □

LEMMA 2.9
For any maximal barrier B of a matching covered graph G, every bipartite (odd) component of $G - B$ is trivial.

Proof: Let B be a maximal barrier of G. Let $K_1, \ldots, K_r$ be the odd components of $G - B$. Assume, to the contrary, that there exists K_i such that $G[K_i]$ is bipartite. Let (A_i, B_i) a bipartition of K_i. Cut $\nabla(K_i)$ is a separating cut, thus, by (2.1), $||A_i| - |B_i|| = 1$. Adjust the notation so that $|A_i| = |B_i| + 1$. Thus, $\nabla(A_i) = \nabla(B_i) \cup \nabla(K_i)$. Therefore, $B \cup B_i$ is a barrier to G that contradicts the maximality of B □

Let G be a matching covered graph and let e and f be any two edges of G. Then we say e *depends* on f, or e *implies* f, if every perfect matching that contains e also contains f. We write $e \Rightarrow f$ to indicate that e depends on f. Relation $\Rightarrow$ is reflexive and transitive.

Two edges e and f are *mutually dependent* if $e \Rightarrow f$, and $f \Rightarrow e$. In this case we write $e \Leftrightarrow f$. Clearly $\Leftrightarrow$ is an equivalence relation on $E(G)$. In general, an equivalence class can be arbitrarily large. However, in a 3-edge-connected near-brick, equivalence classes have cardinality at most two as shown in the following lemma.

LEMMA 2.10
For every 3-edge-connected near-brick G, every equivalence class Q with respect to the dependence relation contains at most two edges, with equality only if graph $G - Q$ is bipartite.

Proof: Graph G is non-bipartite matching covered graph, thus if $|Q| = 1$, graph $G - Q$ is non-bipartite then assertion holds. Assume that $|Q| > 2$. Let e, f and g be three edges of Q. Each of these three edges imply the other two. Let B be a maximal barrier of $G - f$ such that both ends of e are in B and the two ends of f are in different components of $G - f - B$.

Suppose that there exists another edge e' which has both its ends in B. Let M be a perfect matching in G that contains edge e'. By counting, we conclude that $f \in M$ and $e \notin M$. But this contradicts the hypothesis that $e \Leftrightarrow f$. Therefore, e is the only edge spanned by B.

Suppose now that $G-f-B$ has at least one nontrivial component. Let K be a nontrivial component of $G-f-B$. Barrier B is maximal, thus by (2.9), K is non bipartite. Thus, cut $C := \nabla(K)$ is a separating cut of G with both its C_K-contractions non-bipartite. Therefore, by (2.6), C_K can not be a tight cut. Let M be a perfect matching of G that has more than one edge in C_K. By counting,

$$|M \cap C_K| = 3, \qquad f \in M \qquad \text{and} \qquad e \notin M.$$

Again contradicts the hypothesis of $e \Leftrightarrow f$.

Therefore, $G-e-f$ is bipartite and $E \setminus \{e,f\}$ is a cut of G. By analogy, $E \setminus \{e,g\}$ is also a cut of G. The symmetric difference of these two cuts is $\{f,g\}$. Moreover, the symmetric difference of any two cuts is also a cut. Therefore, $\{f,g\}$ is a cut of G. But this is a contradiction because G is 3-edge-connected. Therefore, $|Q| \leq 2$. Moreover, if $|Q| = 2$ then $G-Q$ is bipartite. □

Lemma 2.11

Let G be a matching covered graph, $C := \nabla(X)$ a separating cut of G. If each C-contraction of G is bicritical then graph G is bicritical. Moreover, if each C-contraction of G is a brick then, G is a brick if, and only if, subgraph $G[C]$ of G spanned by C has a matching with at least three edges.

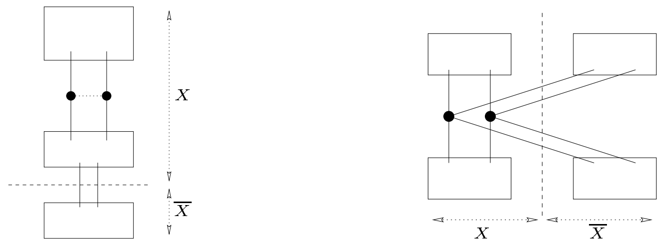
Proof: Let $G_1 := G\{X;\overline{x}\}$ and $G_2 := \{\overline{X};x\}$ be the two C-contractions of G. We will show that G is bicritical by removal of any two vertices, say u and v, of G and finding a perfect matching to $G-u-v$. Consider first the case in which both u and v are vertices of X. By hypothesis, graph G_1 is bicritical, therefore, $G_1 - \{u,v\}$ has a perfect matching M_1. Let e be the edge of $M_1 \cap C$. Graph G_2 is matching covered, thus there exists a matching M_2 in G_2 that uses e. Therefore, $M_1 \cup M_2$ is a perfect matching of G.

Consider now the case in which $u \in X$ and $v \in \overline{X}$. Graph $G_1 - u - \overline{x}$ has perfect matchings. Let M_1 be a matching of $G_1 - u - \overline{x}$. This matching has no edges in C. Analoguely, there is a perfect matching, M_2, to $G_2 - v - x$ without edges in C. Thus, $M_1 \cup M_2$ is a perfect matching to $G-u-v$. Therefore, G is bicritical.

Now, we will show the second part of the lemma. Suppose that G is a brick and cut C is a separating cut of G, thus cut C can not be tight. Therefore, there exist at least one perfect matching of G with at least three edges in C. In particular, the subgraph $G[C]$ spanned by C has a matching with at least three edges.

Now, suppose that $G[C]$ has a matching with at least three edges. Assume, to the contrary, that G has a nontrivial tight cut. Thus, graph G has a nontrivial tight cut that is either a barrier cut or a 2-separation cut. Graph G is bicritical, thus this tight cut must be a 2-separation cut. Let $\{u,v\}$ be this 2-separation and let K_1 and K_2 be the (even) components of $G - \{u,v\}$.

Suppose first that u and v lie in X. If $K_1 \subset X$ then $\{u,v\}$ is a 2-separation of G_1 (see Figure 6(a)) contradicting the hypothesis that G_1 is a brick. Otherwise, that is, if $K_i \cap X \neq \emptyset$ and $K_i \cap \overline{X} \neq \emptyset$ for $1 \leq i \leq 2$ then graph G_2 has a vertex cut (see Figure 6(b)), a contradiction with the hypothesis of G_2 is matching covered.



(a) (b)

Figure 6: Vertices of the 2-separation lie in X.

Now, we may assume that $u \in X$ and $v \in \overline{X}$. Suppose that $K_i \cap X \neq \emptyset$ for $1 \leq i \leq 2$. In this case, $\{u, \overline{x}\}$ is a 2-separation of G_1 (see Figure 7(a)) a contradiction because, by hypothesis, this graph is a brick. Assume then that $K_2 \cap X = \emptyset$ and $K_1 \cap \overline{X} = \emptyset$. In this case cut C is a tight cut of this 2-separation (see figure 7(b)). By hypothesis, there exists a perfect matching with more than one edge in C and, again we have a contradiction. Therefore, as we have asserted graph G is a brick.

(a) (b)

Figure 7: One vertex of the 2-separation lies in X and the other lies in $\overline{X}$.

□

The first part of the next Lemma was proved in [1]. The last part follows trivially of the fact that G, in that case, is isomorphic to C_4 with multiple edges.

Lemma 2.12
Let G be a brace with at least four vertices. If G has at least six vertices then every edge is removable in G. If G has just four vertices and is free of vertices of degree two then, for every vertex v of G, at most one edge of $\nabla(v)$ is not removable in G.

3 Robust Cuts and b-removable edges

Robust cuts have been defined and used in [2] and in [3]. We cite here the fundamental results we need involving robust cuts. We remark that most of these results were proved in those two papers.

Let G be a matching covered graph. A cut C of G is *robust* in G if cut C is not tight in G and each C-contraction of G is a near-brick.

LEMMA 3.1
Let G be a matching covered graph, $D := \nabla(Y)$ a separating cut of G that is either tight or robust in G, $H := G\{Y;\overline{y}\}$ a D-contraction of G, C a tight cut of H. Then, either (i) cut C is tight in G or (ii) cuts C and D are matching-equivalent in G, cut C is robust in G and the C-contraction of H that contains vertex $\overline{y}$ is bipartite.

Proof: Let X be the shore of cut C in H that contains vertex $\overline{y}$. Let $\overline{X}$ denote the other shore of C in H. Let $H_1 := H\{X;\overline{x}\}$, $H_2 := H\{\overline{X};x\}$. (See Figure 8).

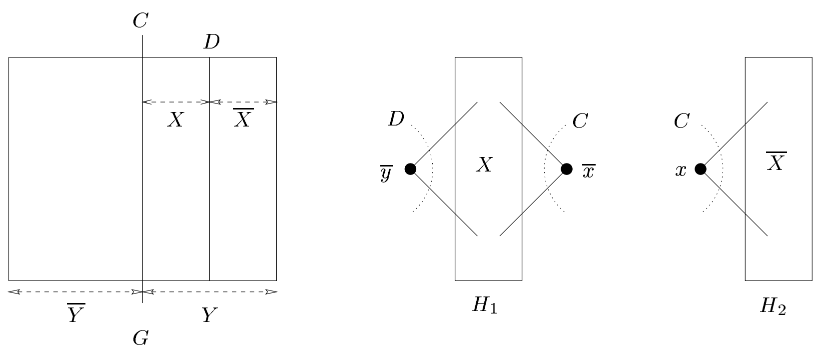


Figure 8: An illustration for Lemma 3.1

Consider first the case in which graph H is not a near-brick. By definition of robust cut, it follows that cut D is not robust. By hypothesis, cut D is either tight or robust in G. We deduce that cut D is tight in G. By hypothesis, cut C is tight in H, therefore it is also tight in G. The assertion thus holds in this case.

We may thus assume that graph H is a near-brick. By hypothesis, cut C is tight in H. Therefore, one of H_1 and H_2 is bipartite, the other is a near-brick. Observe that H_2 is a C-contraction of G. If H_2 is bipartite then cut C is tight in G, and the assertion holds in this case.

We may thus assume that graph H_1 is bipartite. Cut C is tight in H, thus $b(H_2) = b(H) = 1$, we conclude that H_2 is a near-brick. If vertices $\overline{y}$ and $\overline{x}$ lie in the same part of H_1 then the other part of H_1 is a barrier of G, whence cuts C and D are both tight in G, the assertion holds in this case. Alternatively, if vertices $\overline{y}$ and $\overline{x}$ lie in distinct parts of H_1 then cuts C and D are matching-equivalent in G. If D is tight in G then so too is C. So we can assume that D is robust. Therefore, the two D-contractions of G are near-bricks. Thus, the other C-contraction of G that includes cut D is a near-brick too. So, cut C is a non-tight cut and both its C-contractions are near-bricks, whence C is a robust cut. In both alternatives the assertion holds. □

Lemma 3.2 (See [2], Theorem 4.3)
If a matching covered graph has a robust cut then it is a near-brick. □

Let G be a matching covered graph, let C and D denote two (not necessarily distinct) cuts of G such that

$$|M \cap C| \leq |M \cap D|, \text{ for every perfect matching } M \text{ of } G. \tag{1}$$

We then say that cut C *precedes* cut D, and denote this fact by writing $C \preceq D$. In addition, if equality holds in (1) for each perfect matching M of G then we say that cuts C and D are *matching-equivalent*. If equality does not hold in (1) for some perfect matching M of G then we say that cut C *strictly precedes* cut D and denote this fact by writing $C \prec D$. For any collection $\mathcal{C}$ of cuts of G, a cut C of $\mathcal{C}$ is *minimal with respect to* $\preceq$ *in* $\mathcal{C}$ if no cut D in $\mathcal{C}$ strictly precedes C in G.

Lemma 3.3 (See [3], Corollary 2.4)
Let C be a separating cut of a brick G, let M_0 be a perfect matching of G that contains more than one edge in C. Let $\mathcal{C}$ be the collection of separating cuts D of G such that $|M_0 \cap D| > 1$ and $D \preceq C$. Then, every cut of $\mathcal{C}$ that is minimal with respect to the relation of precedence is robust in G. □

A matching covered graph G is *solid* if it is free of strictly separating cuts. An edge e of a matching covered graph G is *removable* in G if graph $G - e$ is also matching covered. An edge e of G is b-removable in G if it is removable in G and $b(G - e) = b(G)$.

Lemma 3.4 (See the proof of Theorem 2.23 in [2])
Let e be a removable edge of a matching covered graph G, let C be a cut of G such that $C - e$ is strictly separating in $G - e$. Let $\mathcal{C}$ be the collection of those cuts D of G such that $D - e$ is strictly separating in $G - e$ and $D \preceq C$. Then, every cut of $\mathcal{C}$ that is minimal with respect to the relation of precedence is strictly separating in G.

Proof: Certainly cut C lies in collection $\mathcal{C}$. Let D be a cut in $\mathcal{C}$ that is minimal with respect to the relation of precedence. Each $(D - e)$-contraction of $G - e$ is non-bipartite. Therefore, each D-contraction of G is non-bipartite. If cut D is separating in G then it is strictly separating.

Assume, to the contrary, that cut D is not separating in G. Then, at least one of the D-contractions of G is not matching covered. Let X be a shore of D and $H := G\{X; \overline{x}\}$ be a D-contraction of G that is not matching covered.

By hypothesis, cut $D - e$ is separating in $G - e$. Graph $H - e$, a $(D - e)$-contraction of $G - e$, is thus matching covered. We conclude that edge e lies in H but is not admissible in H. Let B denote a maximal barrier of graph H that contains both ends of edge e. If vertex $\overline{x}$ does not lie in B then B is a barrier of G that spans edge e, whence e is not admissible in G, a contradiction. We conclude that vertex $\overline{x}$ lies in B.

Let $\mathcal{K}$ denote the set of (odd) components of $H - e - B$. For each component K in $\mathcal{K}$, let C_K denote cut $\nabla(V(K))$ of G.

By hypothesis, cut $D - e$, a member of collection $\mathcal{C}$, is strictly separating in $G - e$. This observation has two important implications. The first is that graph $H - e$ is non-bipartite, whence at least one component in $\mathcal{K}$, say L, is nontrivial, therefore the C_L-contraction $G\{V(L)\}$ is non-bipartite, by (2.9). The other implication is that the C_L-contraction $(G - e)\{\overline{V(L)}\}$ of $G - e$ is non-bipartite, by (2.1). We conclude that both C_L-contractions of $G - e$ are non-bipartite.

We now show that cut C_L is separating in $G - e$ and also that $C_L \prec D$, thereby contradicting the definition of D. For this, observe that for every perfect matching M of G, the number of edges of M in cut $\nabla_H(B)$ may be expressed in two ways:

$$|\,M \cap D\,| + |B| - 1 - 2\,|\,M \cap \{e\}\,| = \sum_{K \in \mathcal{K}} |\,M \cap C_K\,| \geq |\,M \cap C_L\,| + |\mathcal{K}| - 1.$$

Since $|\mathcal{K}| = |B|$, it follows that

$$|\,M \cap D\,| - 2\,|\,M \cap \{e\}\,| \geq |\,M \cap C_L\,|\,.$$

From the equation above, it follows that for every perfect matching M of G, $|\,M \cap C_L\,| \leq |\,M \cap D\,|$, with equality only if edge e does not lie in M. Since edge e is admissible in G, it follows that $C_L \prec D$.

Let f be any edge of $G - e$. Cut $D - e$ is separating in $G - e$, therefore there exists a perfect matching M_f of $G - e$ that contains edge f and just one edge in D. From the equation above it then follows that matching M_f contains just one edge in C_L. This conclusion holds for each edge f of $G - e$, therefore cut C_L is separating in $G - e$. Since both C_L-contraction of $G - e$ are non-bipartite, cut C_L is strictly separating in $G - e$.

In sum, cut C_L strictly precedes cut D in G, cut C_L is strictly separating in $G - e$. This conclusion contradicts the minimality of cut D in $\mathcal{C}$. As asserted, cut D is strictly separating in G. $\square$

Corollary 3.5

If a near-brick G is solid then every removable edge of G is b-removable in G.

Proof: Let e denote a removable edge of G.

We observe first that graph $G - e$ is not bipartite. For if $G - e$ is bipartite, then either edge e has both ends in the same part of $G - e$ or graph G itself is bipartite. If edge e has both ends in the same part of $G - e$ then it is not admissible in G; if graph G is bipartite then it is not a near-brick. In both alternatives we derive a contradiction. Indeed, graph $G - e$ is non-bipartite.

Assume, to the contrary, that edge e is not b-removable in G. Then, graph $G - e$ is neither bipartite nor a near-brick, whence it has a strictly separating tight cut, by (2.5). Therefore by previous lemma, graph G also has a strictly separating tight cut. This contradicts the hypothesis that G is solid. $\square$

4 Cohesive Collections of Cuts

In this section we list some important properties of cohesive collections of cuts of a matching covered graph. As we have defined, a collection of cuts $\mathcal{C}$ is *cohesive* if every edge of G lies in a perfect matching of G that contains precisely one edge in each cut in $\mathcal{C}$. The following result characterizes separating cuts.

LEMMA 4.1
A cut C of a matching covered graph G is separating if, and only if, every edge of G lies in a perfect matching of G that contains precisely one edge in C.

COROLLARY 4.2
Every tight cut of a matching covered graph is separating.

COROLLARY 4.3
A cut C of a matching covered graph G is separating, if, and only if, collection $\{C\}$ is cohesive.

COROLLARY 4.4
For each cohesive collection $\mathcal{C}$ of a matching covered graph G and every tight cut C of G, collection $\{C\} \cup \mathcal{C}$ is also cohesive.

Two cuts $\nabla(X)$ and $\nabla(Y)$ of a graph G *cross* if each of $X \cap Y$, $X \cap \overline{Y}$, $\overline{X} \cap Y$ and $\overline{X} \cap \overline{Y}$ is non-null. A collection of cuts is *laminar* if no two of its cuts cross.

LEMMA 4.5
For any cohesive laminar collection $\{C, D\}$ of cuts of a matching covered graph G, let H denote the D-contraction of G that contains cut C. Then, cut C is separating in H. Moreover, $\lambda_G(C) \leq \lambda_H(C)$, with equality if cut D is tight in G.

Proof: Collection $\{C, D\}$ is cohesive. Thus, for each edge of G there exists a perfect matching with one edge in C and with one edge in D. The restriction of this matching to edges of H is a perfect matching in H with one edge in C. In particular, for each edge of H there exists a perfect matching with one edge in C. Therefore, C is a separating cut in H. Moreover, each perfect matching of H has one edge in D and can be expanded to a perfect matching of G. Therefore $\lambda_G(C) \leq \lambda_H(C)$.

If D is a tight cut then the restriction of each perfect matching of G to edges of H is a perfect matching of H. Therefore, the set of perfect matchings of H is exactly the set of the perfect matching of G restricting the edges of these matchings to edges of H. Therefore, $\lambda_G(C) = \lambda_H(C)$. □

LEMMA 4.6
Let $C := \nabla(X)$ and $D := \nabla(Y)$ be two crossing cuts of a matching covered graph G. Adjust notation so that $|X \cap Y|$ be odd. Let $I := \nabla(X \cap Y)$, let $U := \nabla(\overline{X} \cap \overline{Y})$. If collection $\{C, D\}$ is cohesive, then the following properties hold:

(i) *For every set F of edges of G, the following relation of modularity holds:*

$$|F \cap C| + |F \cap D| = |F \cap I| + |F \cap U|.$$

(ii) *Collection $\{C, D, I, U\}$ is cohesive.*

(iii) *Let λ_I denote the characteristic of cut I in $G\{Y\}$ and λ_U denote the characteristic of cut U in $G\{\overline{Y}\}$. Then, $\lambda_C(G) \leq \min\{\lambda_I, \lambda_U\}$, with equality if cut D is tight in G.*

Proof: Let S be the set of edges that have one end in $\overline{X} \cap Y$ and the other in $X \cap \overline{Y}$. For any set of edges the following relation is true:

$$|F \cap C| + |F \cap D| = |F \cap I| + |F \cap Y| + 2|F \cap S|$$

Suppose that $S \neq \emptyset$. Collection $\{C, D\}$ is cohesive. Therefore, for each edge of G there exists a perfect matching with exactly one edge in C and one edge in D. Let M be a perfect matching of G that uses $e \in S$ and has one edges in C and one edge in D. Thus,

$$2 = |M \cap C| + |M \cap D| = |M \cap I| + |M \cap Y| + 2|M \cap S| > 2$$

Therefore, S must be empty and the modularity property holds.

Collection $\{C, D\}$ is cohesive and modularity holds. Thus, $\{C, D, I, U\}$ is a cohesive collection. If $\{C, D, I, U\}$ is a cohesive collection, so too is $\{D, I\}$ and $\{D, U\}$. By (4.5), we conclude that I is separating in $G\{Y\}$ and U is separating in $G\{\overline{Y}\}$.

Let M be a perfect matching with λ_I edges in I and one edge in D and U. Thus, by modularity, $|M \cap C| = \lambda_I$. Let M be a perfect matching with λ_U edges in U and one edge in D and I. Thus, by modularity, $|M \cap C| = \lambda_U$. Therefore, $\lambda(C) \leq \min\{\lambda_I, \lambda_U\}$.

Suppose now that D is a tight cut. Let M be a perfect matching of G with λ_C edges in C. Cut D is tight and the modularity property holds. Therefore,

$$|M \cap I| + |M \cap U| = \lambda_C + 1. \tag{2}$$

Moreover,

$$|M \cap I| + |M \cap U| \geq 2\min\{\lambda_I, \lambda_U\}. \tag{3}$$

By 2 and 3 we conclude that

$$\lambda_C \geq \min\{\lambda_I, \lambda_U\}$$

We know that $\lambda_C \leq \min\{\lambda_I, \lambda_U\}$. Therefore the equality holds and the proof is complete. □

5 The Theorem on Odd Wheels

In this section we present a result that establishes, under certain conditions, that a near-brick is an odd wheel. That result plays a fundamental role in the proof of the Main Theorem.

For any vertex v of a graph G, a *v-matching* of G is a set M of edges of G such that every vertex of G distinct from v is incident with precisely one edge of M. A trivial but important remark: if M is a v-matching of a graph that has an even number of vertices then $|M \cap \nabla(v)|$ is odd.

THEOREM 5.1 (THEOREM ON ODD WHEELS)
For any vertex v of a 3-edge-connected near-brick G and any v-matching M of G, one of the following properties holds:

(i) *Either graph G is an odd wheel of hub v, up to multiple edges incident with vertex v,*

(ii) *or graph G is not solid,*

(iii) *or graph G has a removable singleton or doubleton that is disjoint with $M \cup \nabla(v)$.*

Proof: By induction on the size of G.

CASE 1 *Graph G is not a brick.*

By hypothesis, graph G is a near-brick. Thus, for every tight cut C of G, one of the C-contractions of G is bipartite, the other is a near-brick. By hypothesis of the case, graph G is not a brick. Let thus C be a nontrivial tight cut of G such that the set of edges of the bipartite C-contraction of G is minimal. Then, that C-contraction is a brace.

Let X be a shore of C, $G_1 := G\{X; \overline{x}\}$, $G_2 := G\{\overline{X}; x\}$ the two C-contractions of G. Adjust notation so that G_1 is bipartite. Let (A, B) denote the bipartition of G_1 such that vertex $\overline{x}$ lies in A.

Consider first the case in which vertex v does not lie in $V(G_1)$. In that case, the restriction of M to G_1 is a perfect matching of G_1. Let w be any vertex of $A - \overline{x}$. This vertex is incident with exactly one edge of M. By (2.12), at most one edge of G_1 incident with vertex w is not removable in G_1. By hypothesis, graph G is 3-edge-connected, whence it is free of vertices of degree two. We conclude that $\nabla(w) - M$ contains an edge, say e, that is removable in G_1. Moreover, vertices w and $\overline{x}$ lie on the same part A of G_1, therefore edge e does not lie in C. Finally, vertex v does not lie in $V(G_1)$, therefore edge e does not lie in $\nabla(v)$. We conclude that edge e is removable in G and does not lie in $M \cup \nabla(v)$. The assertion holds in this case.

Consider next the case in which vertex v lies in A. Then, it is distinct from vertex $\overline{x}$. Moreover, each vertex of $V(G_1) - \{v, \overline{x}\}$ is incident with precisely one edge of M, and $|M \cap \nabla(v)|$ is odd, therefore M is a perfect matching of G, and cut C is tight in G. By (2.12), cut C contains an edge, e, that does not lie in M and is removable in G_1. Moreover, edge e does not lie in $\nabla(v)$. Let R be a minimal class of G_2 induced by edge e. Observe that the restriction of M to G_2 is a perfect matching of G_2, and edge e does not lie in M, therefore R and M are disjoint. Moreover, graph G_2, a C-contraction of G, is 3-edge-connected, therefore R contains at most two edges. If R and C are disjoint then R is removable in G and is disjoint with $M \cup \nabla(v)$. If R and C are not disjoint then e is the only edge of R in C. In both cases, R is a removable singleton or doubleton of G that is disjoint with $M \cup \nabla(v)$. The assertion holds in this case.

Consider last the case in which vertex v lies in B. In this case, cut C has as many edges in M as does cut $\nabla(v)$. Therefore, the restriction of M to G_2 is an x-matching of G_2. By induction hypothesis, with x playing the role of v, $M \cap E(G_2)$ the role of M and G_2 the role of G, we have one of the alternatives of the assertion. We consider each one of them separately. If G_2 has a removable singleton or doubleton R that is disjoint with $M \cup C$, then R is also a removable doubleton or singleton in G that is disjoint with $M \cup \nabla(v)$. For every separating cut D of G_2, its characteristic in G_2 equals its characteristic in G: if G_2 is not solid, neither is G. Finally, assume that G_2 is an odd wheel of hub x, up to multiple edges in C, let w be any vertex of $B - v$. By (2.12), $\nabla(w)$ has an edge, say e, that is removable in G_1 and does not lie in $M \cup \nabla(v)$. If edge e does not lie in C then it is removable in G; if edge e lies in C and either the order of the wheel G_2 is greater than three or edge e is a multiple edge in C, then edge e is removable in G; if edge e is not a multiple edge in G_2 and the order of G_2 is three, then there exists a doubleton in G_2 that is disjoint with M and removable in both G_2 and G. In all cases considered, either graph G has a removable singleton or doubleton that is disjoint with $M \cup \nabla(v)$, or graph G is not solid. The analysis of this case is complete.

We may thus assume that graph G is a brick. To proceed with the proof, we need a theorem due to Lovász (Theorem 5.5.1, page 196, [5]):

Theorem 5.2
Every critical graph G can be represented as

$$G = P_0 + P_1 + \cdots + P_r, \tag{4}$$

where P_0 is K_1 and each P_i $(1 \leq i \leq r)$ is either an odd path or an odd circuit having precisely its origin and terminus in common with $P_0 + P_1 + \cdots + P_{i-1}$.

Sequence $\mathcal{P} := (P_0, P_1, \cdots, P_r)$ is an *ear decomposition* of critical graph G, and for each i $(1 \leq i \leq r)$, P_i is an *ear* of $\mathcal{P}$. The following assertion is the converse of (5.2), and is easily proved by induction.

Proposition 5.3
If a graph G has an ear decomposition as in (4), *then it is critical.* □

Graph G is bicritical, therefore graph $G - v$ is critical. By (5.2), graph $G - v$ has ear decompositions. Set M is a v-matching of G therefore $M - \nabla(v)$ is a matching of $G - v$, and $E(G - v)$ is not a subset of M. Thus, for each ear decomposition $\mathcal{P} = (P_0, P_1, \cdots, P_r)$ of $G - v$, there exists at least one integer i such that $0 < i \leq r$ and $E(P_i)$ is not a subset of M; we define the *index* of $\mathcal{P}$ to be the largest positive integer $q \leq r$ such that $E(P_q) - M$ is nonnull.

Let $\mathcal{P} := (P_0, P_1, \cdots, P_r)$ be an ear decomposition of $G - v$ of highest index. Let q denote the index of $\mathcal{P}$.

PROPOSITION 5.4
For each integer i, $(q < i \leq r)$, ear P_i has length one, its only edge, p_i, lies in M. Therefore, for any permutation $(P'_{q+1}, \cdots, P'_r)$ of $(P_{q+1}, \cdots, P_r)$, sequence $(P_0, P_1, \cdots, P_q, P'_{q+1}, \cdots, P'_r)$ is an ear decomposition of $G - v$ of index q.

Proof: Let i be any integer such that $q < i \leq r$. No edge of P_i is incident with vertex v. Therefore, every vertex of $V(P_i)$ is incident with at most one edge of M. By definition of index q, $E(P_i) \subset M$. Thus, P_i must have length one. This conclusion holds for each index i such that $q < i \leq r$. Therefore, graph $P_0 + P_1 + \cdots + P_q$ is a spanning subgraph of $G - v$. We conclude that we may permute the ears of $\mathcal{P}$ as indicated in the assertion, to obtain another ear decomposition of $G - v$ of index q. □

CASE 2 $q = 1$.

We assert that $r = 1$. To see this, assume the contrary. Then, the edge p_2 of P_2 has both ends in $V(P_1)$. But P_1 is an odd circuit, whence the ends of p_2 split P_1 in two paths, P' and P'', of odd and even length, respectively. If P' has length one then its edge e and p_2 constitute a pair of multiple edges of G; moreover, p_2 lies in M, therefore edge e does not lie in M. Alternatively, if the length of P' is greater than one then $E(P')$ is not a subset of M. In both cases, $E(P')$ is not a subset of M. Let P'_0 be the vertex graph of any vertex of P''. Replacement of P_0, P_1, P_2 in $\mathcal{P}$ by $P'_0, P'' + p_2, P'$, respectively, yields an ear decomposition of $G - v$ of index two, a contradiction. As asserted, $r = 1$. We conclude that $G - v$ is an odd circuit.

Graph G is bicritical, therefore every vertex of G is adjacent to at least three vertices. Thus, each vertex of $G - v$ is adjacent to v. We conclude that G is an odd wheel of hub v, up to multiple edges in $\nabla(v)$. The analysis of the case is complete.

CASE 3 $q > 1$ *and* $|E(P_q)| = 1$.

We assert that $q = r$ in this case. For $i = q + 1, \cdots, r$, path P_i has length one, by (5.4). Therefore, graph $P_0 + P_1 + \cdots + P_{q-1}$ is a spanning subgraph of $G - v$. Therefore, we may replace $(P_q, \cdots, P_r)$ in $\mathcal{P}$ by $(P_{q+1}, \cdots, P_r, P_q)$, thereby obtaining an ear decomposition of index r. By definition of $\mathcal{P}$, it follows that $q = r$, as asserted. Let e be the edge of P_q. Let S be the set of edges of G that depend on edge e.

PROPOSITION 5.5
No edge f of S lies in $M \cup \nabla(v)$.

Proof: Edge e, an edge of graph $G - v$ that does not lie in M, does not lie in $M \cup \nabla(v)$. The assertion holds trivially if $f = e$. We may thus assume that f is an edge of $S - e$. Then, graph $G - e$ has a barrier B that contains both ends of f.

We assert that vertex v does not lie in B. For this, assume the contrary. Let w be any vertex of $B - v$. Let $B' := B - \{v, w\}$. Let $G' := G - v - e - w$. Then, $G' - B' = G - e - B$, whence the number of (odd) components of $G' - B'$ is strictly greater than the number of vertices of B'. Thus, graph G' has no perfect matching, whence graph $G - v - e$ is not critical. But $(P_0, P_1, \cdots, P_{r-1})$ is an ear decomposition of graph $G - v - e$, therefore graph $G - v - e$ is critical, by (5.3). This is a contradiction. As asserted, vertex v does not lie in B.

Set M, a v-matching of G, has precisely one edge incident with each vertex of G distinct from v, therefore it has an odd number of edges incident with each vertex of G. For each (odd) component K of $G - e - B$, set M has thus an odd number of edges in $\nabla(V(K))$. Edge e does not lie in M, therefore set M has at least $|B|$ edges in $\nabla(B)$. Vertex v does not lie in B, therefore each vertex of B is incident with precisely one edge of M. Moreover, that edge lies in $\nabla(B)$. Therefore, edge f does not lie in M. In sum, edge f has both ends in B and does not lie in M, and vertex v does not lie in B. We conclude that edge f does not lie in $M \cup \nabla(v)$, as asserted. □

Let R be any minimal class of G induced by e. Then, set R is disjoint with $M \cup \nabla(v)$. Graph G is a brick, thus R contains at most two edges, by (2.10). We conclude that graph G has a removable singleton or doubleton R that is disjoint with $M \cup \nabla(v)$, as asserted. The analysis of this case is complete.

CASE 4 *$q > 1$ and $|E(P_q)| > 1$.*

Let

$$P_q = (v_0, e_1, v_1, \cdots, e_{2n+1}, v_{2n+1})$$

$X := (V(P_q) - \{v_0, v_{2n+1}\}) \cup \{v\}$, $C := \nabla(X)$. Let $G_1 := G\{\overline{X}; x\}$ and $G_2 := G\{X; \overline{x}\}$ denote the two C-contractions of G.

LEMMA 5.6
Each of $G_1 - x$ and $G_2 - \overline{x}$ is critical.

Proof: Sequence $(P_0, P_1 \cdots, P_{q-1})$ is an ear decomposition of graph $G_1 - x$. Therefore, graph $G_1 - x$ is critical, by (5.3).

The proof that graph $G_2 - \overline{x}$ is critical is more elaborate. For any two integers j and k such that $0 < j, k < 2n + 1$, let $S[j, k]$ denote the subpath of P_q extending from vertex v_j to vertex v_k, if $j \leq k$, otherwise let $S[j, k]$ denote the reversal of $S[k, j]$. For each integer i such that $q < i \leq r$, edge p_i of P_i is an *upper edge* of $\mathcal{P}$.

PROPOSITION 5.7
Let e be any upper edge of $\mathcal{P}$. Then, both ends of edge e are internal vertices of $V(P_q)$. Moreover, if v_j and v_k denote the two ends of e in $V(P_q)$, path $S[j, k]$ has even length.

Proof: We may permute the upper edges of $\mathcal{P}$ so that e is the edge of P_{q+1}. Assume, to the contrary, that no end of edge e is an internal vertex of P_q. Then, we may clearly interchange P_q and P_{q+1} in $\mathcal{P}$, thereby obtaining an ear decomposition of index $q + 1$, a contradiction. We conclude that at least one end of e is an internal vertex of P_q, say v_j, where $0 < j < 2n + 1$.

Assume, to the contrary, that the other end of e is not an internal vertex of P_q. One of $S[0,j]$ and $S[j,2n+1]$ has odd length, the other has even length. Adjust notation, by replacing P_q in $\mathcal{P}$ by its reversal, if necessary, so that the length of $S[0,j]$ is even. Replace, in $\mathcal{P}$, P_q by $S[0,j]\cdot P_{q+1}$ and P_{q+1} by $S[j,2n+1]$. Edge e lies in M, therefore $E(S[j,2n+1])$ is not a subset of M, whence $\mathcal{P}$ has not maximum index, a contradiction. As asserted, both ends of e are internal vertices of P_q.

Let v_k be the other end of edge e. Assume, to the contrary, that $S[j,k]$ has odd length. Adjust notation so that $j<k$. Replace, in $\mathcal{P}$, P_q by $S[0,j]\cdot P_{q+1}\cdot S[k,2n+1]$, and P_{q+1} by $S[j,k]$, thereby obtaining an ear decomposition of $G-v$ of index $q+1$, a contradiction. As asserted, $S[j,k]$ has even length. □

Two upper edges e and f of $\mathcal{P}$ *cross* if the ends v_i and v_j of edge e and the ends v_k and v_l of edge f, with $i<j$ and $k<l$, satisfy the inequality $i<k<j<l$.

Proposition 5.8

Let v_i, v_j be the ends of upper edge e and v_k, v_l the ends of upper edge f such that $i<k<j<l$. Then, each of $S[i,k]$, $S[k,j]$ and $S[j,l]$ has even length.

Proof: We may assume, by permuting upper edges, that e is the edge of P_{q+1} and f the edge of P_{q+2}. Path $S[i,j]$ has even length, by (5.7). Therefore, the lengths of paths $S[i,k]$ and $S[k,j]$ have the same parity. Likewise, the lengths of paths $S[k,j]$ and $S[j,l]$ also have the same parity. Thus, the three paths have lengths of the same parity. Assume, to the contrary, that the common parity is odd. Replace, in $\mathcal{P}$, P_q by $S[0,i]\cdot P_{q+1}\cdot S[k,j]\cdot P_{q+2}\cdot S[l,2n+1]$, P_{q+1} by $S[i,k]$ and P_{q+2} by $S[j,l]$. This replacement yields an ear decomposition of $G-v$ of index $q+2$, a contradiction. As asserted, the common parity is even. □

Proposition 5.9

Let e be an upper edge of $\mathcal{P}$, v_i and v_j its ends in $V(P_q)$. Then, at least one internal vertex of $S[i,j]$ is adjacent to vertex v.

Proof: By induction on $|j-i|$. Adjust notation so that $i<j$. Vertex v_{i+1} has degree at least three in G and is distinct from vertex v_j, because $j-i$ is even. If v_{i+1} is adjacent to vertex v then the assertion holds. Assume thus that vertex v_{i+1} is adjacent to an upper edge f of $\mathcal{P}$. Path $S[i,i+1]$ has odd length, therefore edge f cannot cross edge e, by (5.8). We conclude that edge f has ends v_{i+1} and v_k such that $i+1<k<j$. By induction, path $S[i+1,k]$ has at least one internal vertex adjacent to vertex v. □

Proposition 5.10

Graph $G_2-\overline{x}$ is critical.

Proof: We assert that a spanning subgraph of $G_2-\overline{x}$ has an ear decomposition $\mathcal{Q}:=Q_0+Q_1+\cdots+Q_s$, where $s\leq 3$. For this, recall first that graph G is a brick, therefore $\{v_0, v_{2n+1}\}$ is not a 2-separation of G. No edge of G joins an internal vertex of P_q to vertices of G_1-v. Therefore, at least one internal vertex of P_q is adjacent to vertex v. Let i be the smallest positive integer such that $i < 2n+1$ and vertex v_i is adjacent to vertex v. Likewise, let j be the largest positive integer such that $j < 2n+1$ and v_j is adjacent to vertex v. Thus, $0 < i \leq j < 2n+1$. Let e_i and e_j be edges of $\nabla(v)$ incident with vertices v_i and v_j, respectively.

Consider first the case in which $i = 1$ and $j = 2n$ (Figure 9(a)). In this case, the assertion holds, with $s = 1$ and $Q_1 := S[1, 2n] \cdot (v_{2n}, e_{2n}, v, e_1, v_1)$. Consider next the case in

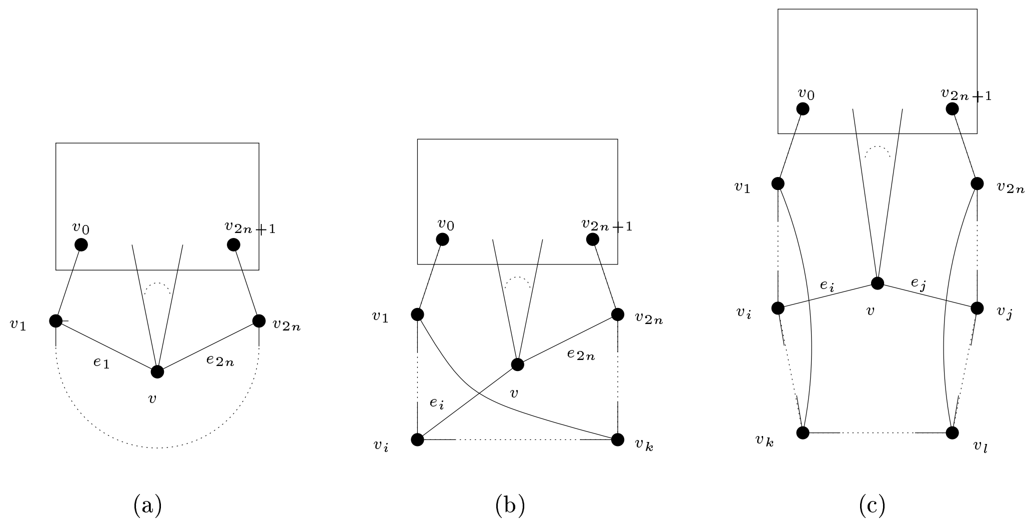


Figure 9: An illustration for the proof of (5.10)

which $i = 1$ and $j < 2n$, or $i > 1$ and $j = 2n$. Adjust notation, by reversing P_q if necessary, so that $j = 2n$ (Figure 9(b)). Then, $i > 1$. Graph G is bicritical, therefore vertex v_1 is adjacent to at least three vertices of G. No edge of G joins vertex v_1 to either v or any vertex of X. Therefore, there is an upper edge of $\mathcal{P}$ incident with vertex v_1. Let e denote that upper edge. Let v_k denote the other end of e. By (5.7), path $S[1, k]$ has even length, whence path $S[k, 2n]$ has odd length. Moreover, by (5.9) and by definition of i, $i < k$. The assertion holds, with $s = 2$, $Q_1 := S[1, k] \cdot (v_k, e, v_1)$ and $Q_2 := S[k, 2n] \cdot (v_{2n}, e_{2n}, v, e_i, v_i)$.

We may thus assume that $1 < i \leq j < 2n$. Graph G is bicritical, therefore both vertices v_1 and v_{2n} are incident with upper edges of $\mathcal{P}$, say e and f, respectively. Let v_k be the end of e distinct from v_1, let v_l denote the end of f distinct from v_{2n}. Then, by (5.9)and by definition of i and j, we have that $1 < i < k$ and $l < j < 2n$. Edges e and f cannot cross. To see this, assume the contrary. By (5.8), each of the three segments $S[1, l]$, $S[k, 2n]$ and $S[l, k]$ has even length. But the sum of the legnths of these three segments is odd, $(2n - 1)$. This is a contradiction. We conclude that $1 < i < k < l < j < 2n$ (Figure 9(c)).

Suppose that at least one of $S[1, i]$ or $S[j, 2n]$ has even length. Adjust the notation so that the length of $S[1, i]$ is even . In that case, the assertion holds, with $s = 3$, $Q_1 := S[l, 2n] \cdot (v_{2n}, f, v_l)$, $Q_2 := (v_j, e_j, v, e_i, v_i) \cdot S[i, l]$, and $Q_3 := (v_k, e, v_1) \cdot S[1, i]$.

Finally, if each of $S[1, i]$ and $S[j, 2n]$ has odd length then so too have paths $S[i, k]$ and

$S[l,j]$. Then, the assertion holds with $s=3$,

$$Q_1 := S[1,i]\cdot(v_i,e_i,v,e_j,v_j)\cdot S[j,2n]\cdot(v_{2n},f,v_l)\cdot S[l,k]\cdot(v_k,e,v_1),$$

$Q_2 := S[i,k]$ and $Q_3 := S[l,j]$.

As asserted, graph $G_2-\overline{x}$ has a critical spanning subgraph. Therefore, $G_2-\overline{x}$ is also critical. □

As asserted, both G_1-x and $G_2-\overline{x}$ are critical. The proof of Lemma 5.6 is complete. □

Proposition 5.11
Let $D := \nabla(Y)$ be any odd cut of G, $H := G\{Y;\overline{y}\}$. If graph $H-\overline{y}$ is critical then H is matching covered and bicritical.

Proof: For any vertex w of H distinct from $\overline{y}$, graph $H-\overline{y}-w$ has a perfect matching. Thus, each edge of H incident with vertex $\overline{y}$ is admissible in H. Therefore, graph H has perfect matchings. Moreover, no nontrivial barrier of H contains vertex $\overline{y}$. Every barrier of H that does not contain vertex $\overline{y}$ is a barrier of G, therefore it is trivial. We conclude that H is bicritical. □

Graph G_1 is a C-contraction of G in which the vertex of contraction is x and graph G_1-x is critical, by (5.6). Likewise, graph G_2 is a C-contraction of G in which the vertex of contraction is $\overline{x}$ and graph $G_2-\overline{x}$ is critical. Thus, both G_1 and G_2 are bicritical matching covered graphs, by(5.11). We conclude that cut C is a nontrivial separating cut of G. Cut C is not tight, because G is a brick. Therefore, G is not solid. The analysis of the last case of the Theorem on Odd Wheels is complete. □

6 Proof of Theorem 1.1

Theorem 1.1
The characteristic $\lambda_G(C)$ of any separating cut C of any near-brick G lies in $\{3,5,\infty\}$. Moreover, if $\lambda_G(C)=5$ then graph G has a tight cut minor P that is the Petersen graph, up to multiple edges, and cut C is strictly separating in P.

Proof: By induction on the size of G. Let λ denote the characteristic of C in G. We may assume C to be nontrivial and G to be free of multiple edges.

Case 1 *Graph G has a nontrivial tight cut D that does not cross cut C.*

Let H_1 and H_2 denote the two D-contractions of G. By hypothesis, graph G is a near-brick, therefore one of H_1 and H_2 is bipartite, the other is a near-brick. By hypothesis, cuts C and D do not cross, therefore C is a cut of one of H_1 and H_2. Adjust notation, so that C is a cut of graph H_1.

Cut D is tight in G, therefore collection $\{C, D\}$ is cohesive in G. By (4.5), cut C is separating in H_1. Moreover, the characteristic of C in H_1 is equal to λ. If graph H_1 is bipartite then cut C is tight in H_1, whence it is tight in G. In that case, the assertion holds.

Assume thus that H_1 is not bipartite. Then, H_1 is a near-brick. By induction hypothesis, with graph H_1 playing the role of G, λ lies in $\{3, 5, \infty\}$. If λ lies in $\{3, \infty\}$ then we are done in this case. Assume thus that $\lambda = 5$. By induction hypothesis, graph H_1 has a tight cut minor P that is the Petersen graph, up to multiple edges, and cut C is strictly separating in P. But graph H_1 is a tight cut minor of G. Therefore graph P is also a tight cut minor of G. The analysis of this case is complete.

Case 2 *Graph G is not a brick and every nontrivial tight cut of G crosses C.*

We assert that $\lambda = 3$ in this case. Graph G is a near-brick that is not a brick, therefore it has nontrivial tight cuts. Let D be a nontrivial tight cut of G. Every nontrivial tight cut of G crosses cut C, therefore cuts C and D cross.

Let X be a shore of C, Y be a shore of D. Adjust notation so that $|X \cap Y|$ is odd. Let

$$I := \nabla(X \cap Y), U := \nabla(\overline{X} \cap \overline{Y}), H_1 := G\{Y; \overline{y}\}, H_2 := G\{\overline{Y}; y\}.$$

Collection $\{C, D\}$ is cohesive. By (4.6), so too is collection $\{C, D, I, U\}$. Moreover, $\lambda = \min\{\lambda_I, \lambda_U\}$, where λ_I denotes the characteristic of cut I in H_1 and λ_U denote the characteristic of cut U in H_2. Graph G is a near-brick and cut D is nontrivial and tight in G. Therefore, one of H_1 and H_2 is bipartite, the other is a near-brick. Adjust notation so that H_2 is bipartite, whereupon H_1 is a near-brick.

Graph H_2 is bipartite and cut U is separating in H_2. Therefore, cut U is tight in H_2. That is, $\lambda_U = \infty$, whence $\lambda = \lambda_I$.

Cut I cannot be tight in H_1, otherwise C would be a nontrivial tight cut that does not cross itself. If $\lambda_I = 3$ then $\lambda = 3$ and the assertion holds. Assume, to the contrary, that $3 < \lambda_I < \infty$. By induction hypothesis, with I playing the role of C and H_1 that of G, graph H_1 has a tight cut minor P that is the Petersen graph up to multiple edges, and cut I is strictly separating in P.

We assert that the subgraph Q of G spanned by $X \cap Y$ is a pentagon. To see this, let T be any tight cut of H_1 that has a shore Z that is a subset of $X \cap Y$. Then, T is a tight cut of G. Moreover, cuts C and T do not cross because Z is a subset of X. Thus, cut T is trivial. That is, set Z is a singleton. We conclude that the vertices of Q are all vertices of P. As asserted, the vertices of Q span a pentagon in G.

The Petersen graph is cubic, therefore precisely one vertex of Q is adjacent in G to vertices of $\overline{Y}$. Let i denote that vertex. Let $Y' := \{i\} \cup (X \cap \overline{Y})$, $D' := \nabla(Y')$. The modularity relating cuts C, D, I and U implies that no edge of G joins vertices of $X \cap \overline{Y}$ with vertices of $\overline{X} \cap Y$. Observe that

$$Y \cap \overline{Y'} = Y - i = (Q - i) \cup (\overline{X} \cap Y) \text{ and } \overline{Y} \cap Y' = X \cap \overline{Y}.$$

Therefore, no edge of G joins vertices of $Y \cap \overline{Y'}$ with vertices of $\overline{Y} \cap Y'$. Note that $Y \cap Y' = \{i\}$ and $\overline{Y} \cap \overline{Y'} = \overline{X} \cap \overline{Y}$. Thus, modularity also relates cuts D, D', $\nabla(i)$ and U. But cuts

D, $\nabla(i)$ and U are each tight in G, therefore cut D' is also tight in G. We conclude that cut D' is a nontrivial tight cut of G that does not cross cut C, a contradiction. As asserted, $\lambda = \lambda_I = 3$. The analysis of the case is complete.

In view of Cases 1 and 2, we may assume graph G to be a brick. We now introduce a concept that will be quite important to reduce the proof further, to the case in which each C-contraction of G is a solid near-brick.

A cut D of G is a *witness* for C if D is robust in G, collection $\{C, D\}$ is cohesive and cuts C and D are not matching-equivalent,

CASE 3 *Graph G is a brick and it has a witness for C that does not cross cut C.*

We assert that $\lambda = 3$ in this case. Let X be a shore of C. By hypothesis of the case, at least one C-contraction of G contains a cut that is a witness. Adjust notation so that $G\{X\}$ has a cut that is a witness. Among the witnesses in $G\{X\}$, choose one, D, such that the shore Y of D that is a subset of X is maximal.

Let $H := G\{\overline{Y}; y\}$. Then, H is the D-contraction of G that contains C. By definition of witness, cut D is robust and collection $\{C, D\}$ is cohesive, whence graph H is a near-brick and cut C is separating in H. Let λ_H denote the characteristic of C in H. By induction hypothesis, with H playing the role of G, λ_H lies in $\{3, 5, \infty\}$.

We assert that cut C is not tight in H. Assume, to the contrary, that C is tight in H. Cut C is not tight in G because it is nontrivial in G and G is a brick. By (3.1), cuts C and D are matching-equivalent, in contradiction to the definition of witness. As asserted, C is not tight in H. We conclude that λ_H lies in $\{3, 5\}$.

By (4.5), $\lambda \leq \lambda_H$. If $\lambda_H = 3$ then $\lambda = 3$ and the assertion holds. Assume thus that $\lambda_H = 5$. By induction hypothesis, graph H has a tight cut minor P such that cut C is strictly separating in P and P is the Petersen graph, up to multiple edges.

We assert that $H = P$. Let T be any (possibly trivial) tight cut of H that does not cross cut C. By the hypothesis of the case, G is a brick; if T is tight in G then it is trivial in G, therefore trivial in H. Assume thus that T is not tight in G. By (3.1), cut T is robust in G, matching-equivalent to D and the T-contraction of H that contains vertex y is bipartite. Cuts C and D are not matching-equivalent, whence neither are cuts C and T. Let Z be the shore of T in H that contains vertex y. Then, $H\{Z\}$ is bipartite. Every separating cut of H that lies in $H\{Z\}$ is tight in $G\{Z\}$, whence it is also tight in H. Cut C is not tight in H, therefore C is not a cut of $G\{Z\}$. Thus, $Y' := Y \cup (Z - y)$ is the shore of T in G that is a subset of X. By the maximality of Y, it follows that $Z = \{y\}$. That is, D and T coincide. We conclude that every tight cut of H that does not cross C is trivial in H. Graph P is a tight cut minor of H that has C as a cut. It follows that $H = P$, as asserted.

Cut D is a trivial cut of H, but a robust cut of G. In particular, cut D is not tight in G. By (2.7), $\lambda = 3$. The analysis of this case is complete.

CASE 4 *Graph G is a brick, every witness for C crosses C and G has a witness D.*

We assert that $\lambda = 3$. By hypothesis of the case, cuts C and D cross. Let X be a shore of

C, Y a shore of D. Adjust the notation so that $|X \cap Y|$ be odd. Let

$$I := \nabla(X \cap Y), U := \nabla(\overline{X} \cap \overline{Y}), H_1 := G\{Y; \overline{y}\}, H_2 := G\{\overline{Y}; y\}.$$

By definition of witness, collection $\{C, D\}$ is cohesive. By (4.6), so too is collection $\{C, D, I, U\}$. Thus, cut I is separating in H_1 and cut U is separating in H_2. Let λ_I denote the characteristic of I in H_1, let λ_U denote the characteristic of U in H_2. By (4.6), $\lambda \leq \min\{\lambda_I, \lambda_U\}$. If $\min\{\lambda_I, \lambda_U\} = 3$ then $\lambda = 3$ and the assertion holds.

Assume, to the contrary, that $\min\{\lambda_I, \lambda_U\} > 3$. Cut D is robust, therefore graphs H_1 and H_2 are both near-bricks. By induction hypothesis, with H_1 playing the role of G and I that of C, it follows that λ_I lies in $\{5, \infty\}$. Likewise, λ_U also lies in $\{5, \infty\}$. Let

$$\begin{aligned} \mathcal{C}_I &:= \{Z : Z \subseteq X \cap Y, \nabla(Z) \text{ is a witness for } C \text{ in } G\}, \text{ and} \\ \mathcal{C}_U &:= \{Z : Z \subseteq \overline{X} \cap \overline{Y}, \nabla(Z) \text{ is a witness for } C \text{ in } G\}. \end{aligned}$$

Proposition 6.1

Either set $\mathcal{C}_I$ is nonnull or set $X \cap Y$ contains a vertex, i, such that no vertex of $\overline{Y}$ is adjacent to any vertex of $X \cap Y - i$.

Proof: Consider first the case in which there exists a nontrivial subset Z of $X \cap Y$ such that cut $W := \nabla(Z)$ is tight in H_1. Graph G is a brick and cut W is nontrivial, therefore W is not tight in G. By (3.1), cut W is robust in G and matching-equivalent to D, whence it is a witnessfor C. That is, Z lies in $\mathcal{C}_{\mathcal{I}}$. The assertion holds in this case.

We may assume that for every separating cut $\nabla(Z)$, $Z \subset X \cap Y$, cut $\nabla(Z)$ is not tight in H_1. In particular, this implies that either I is trivial or it is not tight in H_1. If I is trivial then the assertion holds, with i the only vertex of $X \cap Y$.

We may thus assume that I is not tight in H_1. But λ_I lies in $\{5, \infty\}$. Therefore, $\lambda_I = 5$. By induction hypothesis, graph H_1 has a tight cut minor P that is the Petersen graph, up to multiple edges, and cut I is strictly separating in P. We have assumed above that no nontrivial separating cut of H_1 whose shore is a subset of $X \cap Y$ is tight in H_1. We conclude that $X \cap Y$ spans a pentagon in G. Moreover, precisely one vertex of $X \cap Y$ is adjacent to vertices in $\overline{Y}$. Let i be that vertex. The assertion holds. □

Likewise, either set $\mathcal{C}_U$ is nonnull or set $\overline{X} \cap \overline{Y}$ contains a vertex, u, such that no vertex of Y is adjacent to any vertex of $\overline{X} \cap \overline{Y} - u$.

If $\mathcal{C}_I$ and $\mathcal{C}_U$ are both empty then graph G has a 2-separation $\{i, u\}$, by (6.1). This is a contradiction to the hypothesis that graph G is a brick. If at least one of $\mathcal{C}_I$ or $\mathcal{C}_U$ is nonnull then G has a witness that does not cross C, in contradiction to the hypothesis of the case. In both alternatives, we derive a contradiction. As asserted, $\lambda = \min\{\lambda_I, \lambda_U\} = 3$. The analysis of the case is complete.

In view of Cases 1–4, we may assume that graph G is a brick free of witnesses for cut C. The next assertion implies then that cut C is robust and each C-contraction of G is solid.

Lemma 6.2

Let G be a brick, C a nontrivial separating cut of G. Either G has a witness for C or cut C is robust and each C-contraction of G is solid.

Proof: We first consider whether or not cut C is robust in G. Cut C is a nontrivial separating cut of G, in turn a brick. Therefore, cut C is not tight in G. Let M_0 be a perfect matching of G that contains more than one edge in C. Let $\mathcal{C}$ be the collection of those separating cuts D of G such that $D \preceq C$ and $|M_0 \cap D| > 1$. Collection $\mathcal{C}$ is cohesive and contains cut C. Let C_0 be a cut in $\mathcal{C}$ that is minimal with respect to the relation of precedence. By (3.3), cut C_0 is robust in G. Moreover, $\{C, C_0\}$, a subcollection of $\mathcal{C}$, is cohesive. If cuts C and C_0 are not matching-equivalent then cut C_0 is a witness for C. If cuts C and C_0 are matching-equivalent then cut C is a cut of $\mathcal{C}$ that is also minimal with respect to the relation of precedence; in that case, C is robust in G. We conclude that either G has a witness for C or C is robust in G.

We may thus assume that C is robust in G. We now consider whether or not a C-contraction H of G is solid. Assume that it is not. Let D be a strictly separating cut of H. Then, collection $\{C, D\}$ is cohesive. Cut C is robust in G, therefore H is a near-brick. Cut D is strictly separating in H, therefore it is not tight. Let M_D be a perfect matching of H that contains more than one edge in D. Let M_1 be an extension of M_D to a perfect matching of G. Let $\mathcal{D}$ be the collection of those cuts W of G such that $W \preceq D$ and M_1 has more than one edge in W. Every perfect matching that contains just one edge in D contains also just one edge in each cut of $\mathcal{D}$. Moreover, collection $\{C, D\}$ is cohesive, therefore collection $\{C\} \cup \mathcal{D}$ is also cohesive. Let D_1 be a cut of $\mathcal{D}$ that is minimal with respect to the relation of precedence. Cut D_1 is robust in G and collection $\{C, D_1\}$, a subcollection of $\{C\} \cup \mathcal{D}$, is cohesive. Moreover, cut D_1 has more than one edge in M_1, whereas cut C has just one. Thus, D_1 is not matching-equivalent to C, whence it is a witness for C. We conclude that either near-brick H is solid or G has a witness for C. □

Case 5 *Graph G is a brick, cut C is robust and one of the C-contractions of G has a removable doubleton.*

We assert that $\lambda = 3$. For this, let X be a shore of C, $H := G\{X; \overline{x}\}$ be a C-contraction of G that has a removable doubleton. Let e and f be the edges of the doubleton. By (2.10), graph $H - e - f$ is bipartite. Let (A, B) be the bipartition of $H - e - f$. Adjust notation so that edge e has both ends in A, edge f has both ends in B and vertex $\overline{x}$ lies in A.

Note that set B is a barrier of $G - e$. Every perfect matching of G that does not contain any of the edges e and f contains precisely one edge in C. Every perfect matching of G that contains edge f contains also just one edge in C. Cut C is not tight in G, therefore there must exist in G a perfect matching that contains edge e but no edge f. Any such matching has precisely three edges in cut C. We conclude that $\lambda = 3$. The analysis of the case is complete.

Case 6 *Graph G is a brick and there exists a b-removable edge e of G such that edge e does not lie in C and cut C is separating in $G - e$.*

We assert that $\lambda = 3$ in this case. Cut C is not tight in G, therefore neither C-contraction of G is bipartite. Thus, no C-contraction of $G - e$ is bipartite. That is, cut C is strictly separating in $G - e$. We conclude that cut C is not tight in $G - e$. Let λ' denote the characteristic of C in $G-e$. By induction hypothesis, λ' lies in $\{3,5\}$. Every perfect matching of $G-e$ is a perfect matching of G. Thus, if $\lambda' = 3$ then $\lambda = 3$ and we are done.

We may thus assume that $\lambda' = 5$. By induction hypothesis, graph $G-e$ has a tight cut minor P that is the Petersen graph, up to multiple edges, and cut C is strictly separating in P.

Let X be the shore of C that contains both ends of e. Let $\mathcal{D}$ be the set of those nontrivial, disjoint subsets Y of any of X or $\overline{X}$, such that graph P is the result of the contraction of each set Y in $\mathcal{D}$ to a single vertex.

Proposition 6.3

Collection $\mathcal{D}$ has at most two members, each of which is a subset of X. Moreover, for each set Y in $\mathcal{D}$, graph $H := (G-e)[Y]$ has a bipartition (A,B) such that $|B| = |A|+1$, and edge e is the only edge of G that is incident with some vertex of A but not incident with any vertex of B.

Proof: Let Y be a member of $\mathcal{D}$, $D := \nabla(Y)$. By definition of $\mathcal{D}$, cut $D-e$ is tight in $G-e$, in turn a near-brick. Therefore one of the $(D-e)$-contractions of $G-e$ is bipartite. Cut C is not tight in $G-e$, but is separating in the $(D-e)$-contraction $(G-e)\{\overline{Y}\}$ in which it lies. Thus, $(G-e)\{\overline{Y}\}$ is not bipartite. Therefore, $(G-e)\{Y\}$ is bipartite. Thus, so too is graph H. Let (A,B) denote a bipartition of H. Adjust notation so that $|B| = |A|+1$. Then, no edge of $G-e$ joins a vertex of A to a vertex of $V(G)-B$. If edge e is not incident with a vertex of A, or if edge e is incident with a vertex of B, then cut D is tight in G, whence set Y is trivial, a contradiction. Thus, edge e must be incident with at least one vertex of A and to no vertex of B. Indeed, e is the only edge of G that satisfies this property. Finally, each member of $\mathcal{D}$ contains at least one end of edge e; both ends of e lie in X. We conclude that $\mathcal{D}$ has at most two members, each of which is a subset of shore X of C. □

Notation 6.4

Let $r := |\mathcal{D}|$. For $1 \le i \le r$, let Y_i denote an enumeration of the members of $\mathcal{D}$, $H_i := (G-e)[Y_i]$, (A_i, B_i) the bipartition of H_i such that $|B_i| = |A_i|+1$. Let v_i and w_i denote the ends of e in G, such that vertex v_i lies in A_i. Let y_i be the vertex of P obtained by the contraction of Y_i to a single vertex.

If collection $\mathcal{D}$ is empty, then graph G is $P+e$, up to multiple edges. By (2.8), $\lambda = 3$. We may thus assume that $\mathcal{D}$ is nonempty. Consider next the case in which collection $\mathcal{D}$ has just one member, Y_1, and either (i) edge e has both ends in A_1, or the end w_1 of e, not in Y_1, is adjacent to y_1 in P. If edge e has both ends in A_1, then graph P is $G\{\overline{Y_1}\}$; if w_1 is adjacent to y_1 in P then edge e is a multiple edge in $G\{\overline{Y_1}\}$. In both cases, graph P is $G\{\overline{Y_1}\}$, up to multiple edges. Moreover, every perfect matching of G that contains edge e has precisely three edges in $\nabla(Y_1)$. By (2.7), $\lambda = 3$ also in this case.

We are thus left with three cases to consider: either (i) $r = 1$ and vertex w_1 is not adjacent to vertex y_1 in P, or (ii) $r = 2$ and vertices y_1 and y_2 are not adjacent in P, or (iii) $r = 2$ and vertices y_1 and y_2 are adjacent in P. The three possibilities are depicted in Figure 10, up to automorphism that fix cut C.

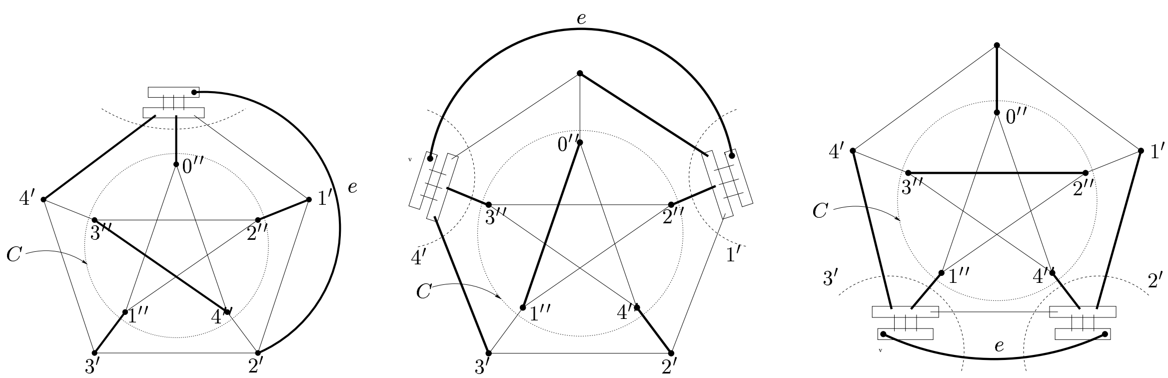


Figure 10: The three possibilities considered in Cases 6.1 and 6.2

Case 6.1 *Either $r = 1$ and vertex w_1 is not adjacent to vertex y_1 in P, or $r = 2$ and vertices y_1 and y_2 are not adjacent in P.*

Proposition 6.5
Under the hypothesis of Case 6.1, for each member Y_i of $\mathcal{D}$ and any two vertices x_i and y_i of the set Z_i of the three vertices of $V(G) \cap V(P)$ that are adjacent in G to vertices of B_i, the subgraph W_i of G spanned by $Y_i \cup \{x_i, y_i, w_i\}$ has a perfect matching, M_i, that contains precisely three edges in $\nabla(Y_i)$, incident, respectively, to x_i, y_i, and w_i.

Proof: Let z_i be the vertex of $Z_i - \{x_i, y_i\}$. Graph G a brick, is bicritical. Therefore, graph $G - \{z_i, v_i\}$ has a perfect matching, say, N_i. We have removed from G a vertex from A_i and a vertex not in Y_i, therefore N_i has precisely two edges in $\nabla(Y_i)$, each of which is incident with a vertex of B_i and a vertex of Z_i. The vertex outside Y_i removed from G is precisely one of the three vertices of Z_i. We conclude that those two edges necessarily are incident to x_i and y_i. Restrict N_i to $E(W_i)$ and add to that restriction edge e. It is easy to check that the resulting set, M_i, is a perfect matching of W_i that has the asserted properties. $\square$

We now apply the assertion just proved to the cases under consideration. In the case in which $r = 1$, we choose $\{x_i, y_i\}$ to be $\{0'', 4'\}$ (see Figure 10); it is easy to check that M_1 can be extended to a perfect matching of G that contains precisely three edges in C. In the case in which $r = 2$, the choices are $\{0', 2''\}$ and $\{3', 3''\}$ (see Figure 10), and again, it is easy to check that $M_1 \cup M_2$ may be extended to a perfect matching of G that contains precisely three edges in C.

Case 6.2 *Collection $\mathcal{D}$ has two members and vertices y_1 and y_2 are adjacent in P.*

For $i = 1, 2$, Let Z_i denote the set $\{x_i, y_i\}$ consisting of the two vertices of $V(G) \cap V(P)$ that are adjacent in P to y_i. Let $Z := Z_1 \cup Z_2$. (In Figure 10, $Z = \{1', 4', 1'', 4''\}$).

Proposition 6.6
The subgraph W of G spanned by $Y_1 \cup Y_2 \cup Z$ has a perfect matching that contains precisely four edges in $\nabla(Y_1 \cup Y_2)$, each of which is incident to one of the four vertices of Z.

Proof: Graph G, a brick, is 3-connected. Therefore, at least three vertices of $B_1 \cup B_2$ are adjacent to vertices of Z. Thus, either B_1 has at least two vertices that are adjacent to vertices of Z_1 or B_2 has at least two vertices that are adjacent to vertices of Z_2. Adjust notation so that B_1 has this property. Vertices x_1 and y_1 are both adjacent to vertices of B_1. Therefore, there exist two vertices in B_1, say x_1' and y_1', such that x_1' is adjacent to x_1 and y_1' is adjacent to y_1. Let e_1 and f_1 denote the corresponding edges that join those pair of vertices.

Graph G, a brick, is bicritical, therefore graph $G - \{x_1', y_1'\}$ has a perfect matching, N. Two vertices of B_1 have been removed from G, therefore edge e lies in N. Moreover, every vertex of A_1 other than the end v_1 of e is matched with a vertex of B_1. Therefore, edge e is the only edge of $N \cap Y_1$. In particular, no edge of $N - e$ joins any vertex of Y_1 to a vertex of Y_2. Moreover, edge e lies in N, and is incident to vertex v_2 of A_2. We conclude that $N \cap \nabla(Y_2)$ contains precisely three edges, one is edge e, the other two are edges incident with vertices x_2 and y_2. Restrict N to W and add to that restriction edges e_1 and f_1. The resulting matching has the asserted properties. □

It is now easy to check that the matching thus obtained may be (uniquely) extended to a perfect matching of G that contains precisely three edges in C. The analysis of this case is complete.

Case 7 *None of the previous cases apply.*

We show that either $\lambda = 3$ or G is the Petersen graph. Let X_1 be a shore of C, let X_2 be the other shore of C. For $i = 1, 2$, let

$$G_i := G\{X_i; v_i\}, n_i := |X_i|\,.$$

Proposition 6.7
For $i = 1, 2$, G_i is an odd wheel of hub v_i, up to multiple edges in $\nabla(v_i)$.

Proof: Cases 1 and 2 do not apply. Therefore, graph G is a brick. Cases 3 and 4 do not apply. Therefore, there are no witnesses for C in G. By (6.2), each C-contraction of G is a solid near-brick. Case 5 does not apply, therefore neither G_1 nor G_2 has a removable doubleton.

Cut C, a nontrivial separating cut of brick G, is not tight in G. Let M_0 be a perfect matching of G such that $|M_0 \cap C| > 1$. For $i = 1, 2$, let M_i denote $M_0 \cap E(G_i)$. Then, M_i is a v_i-matching of G_i, for $i = 1, 2$. By the Theorem on Odd Wheels, (5.1), one of the following alternatives hold, for each $i = 1, 2$:

(i) Either G_i is an odd wheel of hub v_i, up to multiple edges in $\nabla(v_i)$,

(ii) or G_i has a removable edge that does not lie in $M_i \cup C$.

Assume, to the contrary, that G_1 say, has a removable edge e that does not lie in $M_1 \cup C$. Graph G_1 is a solid near-brick. By (3.5), edge e is b-removable in G_1. That is, graph $G_1 - e$ is a near-brick. Edge e does not lie in C, therefore the two C-contractions of $G - e$ are near-bricks. Moreover, edge e does not lie in M_1, a subset of M_0. Therefore, cut C is not tight in $G - e$. In sum, C is a nontight cut of $G - e$ and both C-contractions of $G - e$ are near-bricks. By (3.2), graph $G - e$ is a near-brick. That is, edge e is b-removable in G. It follows that Case 6 applies, a contradiction. We conclude that for $i = 1, 2$, graph G_i is indeed an odd wheel of hub v_i, up to multiple edges in C. □

We have defined $n_i = |X_i|$ and, by previous lemma, each C-contraction is an odd wheel of hub v_i. Thus n_1 and n_2 are the order of wheels G_1 and G_2, respectively. If $|M_0 \cap C| = 3$ then $\lambda = 3$ and we are done. We may thus assume that M_0 contains at least 5 edges in C. In that case, both n_1 and n_2 are at least 5. Thus, each edge of C is removable in each of G_1 and G_2.

Let e be any edge of C. For $i = 1, 2$, let $p_i(e)$ denote the end of e in the shore X_i of C. Define graph $G'_i(e)$ as follows.

If the degree of $p_i(e)$ is greater than three then edge e is a multiple edge in G_i: in that case, graph $G_i - e$ is an odd wheel, let $G'_i(e)$ be $G_i - e$. If the degree of $p_i(e)$ is three then $p_i(e)$ and its two neighbors in $G_i - e$ constitute the shore of a tight cut D of $G_i - e$. The nonbipartite D-contraction of $G_i - e$ is thus an odd wheel of hub v_i, up to multiple edges incident in v_i, having two vertices less than G_i. Let $G'_i(e)$ be that odd wheel. In the first case, let $p'_i(e)$ be $p_i(e)$. In the second case, let $p'_i(e)$ be the vertex of the contraction. In both cases, $G'_i(e)$ is an odd wheel of hub v_i, up to multiple edges incident with v_i. Finally, let $G'(e)$ denote the graph whose $(C - e)$-contractions are $G'_1(e)$ and $G'_2(e)$.

Let H denote the (bipartite) subgraph of G spanned by the edges of C.

PROPOSITION 6.8

Let e be any edge of C. Then, graph $G'(e)$ is bicritical. Moreover, if the degree of one of $p_1(e)$ and $p_2(e)$ in H is at least two, or if edge e does not lie in M_0, then graph $G'(e)$ is a brick and graph $G - e$ is a near-brick.

Proof: The $(C - e)$-contractions of $G'(e)$ are both odd wheels. Odd wheels are bicritical graphs. Therefore, graph $G'(e)$ is bicritical, by (2.11).

Assume further that either edge e does not lie in M_0 or the degree in H of one of its ends is at least two.

Consider first the case in which edge e does not lie in M_0. Then, M_0 is a perfect matching of $G - e$ that contains more than one edge in $G - e$. Moreover, each $(C - e)$-contraction of $G - e$ is a near-brick. Thus, cut $C - e$ is robust in $G - e$. By (3.2), graph $G - e$ is a near-brick. Graph $G'(e)$ is obtained from $G - e$ by the contraction of the bipartite shores of two tight cuts. Therefore, graph $G'(e)$ is a near-brick. But $G'(e)$ is bicritical. Therefore, $G'(e)$ is a brick.

Consider next the case in which one of the ends of e has degree at least two in H. Let $H'(e)$ denote the bipartite subgraph of $G'(e)$ spanned by the edges of $C - e$. Cut $C - e$ has at least four edges in M_0. If the degree of $p_2(e)$ in H is also greater than one, then those four edges constitute a matching of $H'(e)$. If the degree of $p_2(e)$ in H is precisely one then two of the four edges might be incident with $p_2'(e)$ in $H'(e)$. In both cases, we conclude that graph $H'(e)$ has a matching of at least three edges. By (2.11), graph $G'(e)$ is a brick. □

Recall that n_1 and n_2 are the order of wheels G_1 and G_2, respectively. We now adjust notation, so that $n_1 \geq n_2$.

PROPOSITION 6.9
If a vertex of X_1 has degree at least four in G then $\lambda = 3$.

Proof: Let v denote vertex of X_1 that has degree at least four in G. Then, cut C contains at least two edges incident with vertex v. Let e be an edge of C that is incident with vertex v. If possible, choose an edge e such that $p_2(e)$ has degree three in G.

By (6.8), graph $G'(e)$ is a brick. Let $\lambda'(e)$ denote the characteristic of cut $C - e$ in $G'(e)$. Every perfect matching of $G'(e)$ may be extended to a perfect matching of G that has the same set of edges in C. Thus, if $\lambda'(e) = 3$, then $\lambda = 3$ and we are done.

Assume, to the contrary, that $\lambda'(e) > 3$. Graph $G'(e)$ is a brick and cut $C - e$ is separating but not tight in $G'(e)$. Therefore, brick $G'(e)$ is the Petersen graph, up to multiple edges, by induction hypothesis.

Graph $G_1'(e)$ is equal to $G_1 - e$. Therefore, $n_1 = 5$. We have assume that $n_1 \geq n_2$. Therefore, $n_2 \leq 5$and then graph $G_2'(e)$ has order 5. Therefore, $G_2'(e)$ is equal to $G_2 - e$. Thus, vertex $p_2(e)$ has degree four in G. We conclude that $G'(e) = G - e$, whence $G - e$ is the Petersen graph, without multiple edges. Let f be any edge of $C - e$ incident with v. The end $p_2(f)$ of f in X_2 has degree three in G. This is a contradiction to the definition of e. As asserted, $\lambda = 3$. □

PROPOSITION 6.10
If a vertex v of X_2 has degree at least four in G then $\lambda = 3$.

Proof: If a vertex of X_1 has degree four in G then $\lambda = 3$, by (6.9). We may thus assume that each vertex of X_1 has degree three in G, whereupon $|C| = n_1$.

For each edge e of C that is incident with vertex v, let w and x denote the two vertices of X_1 that are adjacent to $p_1(e)$. Let f and g denote the edges of C incident with w and x, respectively. If possible, choose edge e such that edges f and g are not adjacent in G.

By (6.8), graph $G'(e)$ is a brick. Let $\lambda'(e)$ denote the characteristic of cut $C - e$ in $G'(e)$. Every perfect matching of $G'(e)$ may be extended to a perfect matching of G that has the same set of edges in C. Thus, if $\lambda'(e) = 3$, then $\lambda = 3$ and we are done.

Assume, to the contrary, that $\lambda'(e) > 3$. Graph $G'(e)$ is a brick and cut $C - e$ is separating but not tight in $G'(e)$. Therefore, brick $G'(e)$ is the Petersen graph, up to multiple edges, by induction hypothesis.

Vertex $p_1(e)$ has degree three in G. Vertex $p_2(e)$ has degree greater than three in G, therefore $G_2'(e) = G_2 - e$. We conclude that $n_1 = 7$ and $n_2 = 5$. Moreover, since $|C| = n_1$, it follows that either X_2 has precisely two vertices of degree greater than three, each of which has degree 4, or X_2 has just one vertex of degree greater than three, and it has degree 5.

The ends of f and g in $G'(e)$ coincide in the shore of $C - e$ resulting from the contraction of $\{p_1(e), w, x\}$ and graph $G'(e)$ is the Petersen graph, up to multiples edges. Therefore, edges f and g are multiple edges in $G'(e)$. But $G_2'(e) = G_2 - e$, therefore, the ends of f and g in X_2 coincide. Let v' denote that common end of f and g.

If $v' = v$ then the degree of v in G is 5, every vertex of G distinct from v has degree three. In that case, both edges f and g contradict the choice of edge e. If $v' \neq v$, then v' and v each have degree 4 in G, every vertex of G distinct from v and v' has degree three. In that case, the edge of C that is distinct from e and is incident with vertex v also contradicts the choice of e. In both alternatives, we derive a contradiction. As asserted, $\lambda = 3$. □

Proposition 6.11
If cut C has a b-removable edge e then $\lambda = 3$.

Proof: By (6.9) and (6.10), we may assume G to be cubic. Thus, $n_1 = |C| = n_2$. If $G'(e)$ is not the Petersen graph, up to multiple edges, then $\lambda = 3$ and we are done. Assume thus that $G'(e)$ is the Petersen graph, up to multiple edges. Then, $n_1 = n_2 = 7$.

Let w_1 and x_1 denote the two vertices of X_1 that are adjacent to $p_1(e)$. Let f and g denote the edges of C that are incident with w_1 and x_1, respectively. Edges f and g share a common end of degree four in $G'(e)$. The underlying simple graph of $G'(e)$ is the Petersen graph. Thus, edges f and g are multiple in $G'(e)$. It follows that the ends $p_2(e)$, $p_2(f)$ and $p_2(g)$ of edges e, f and g are cyclically consecutive in that order, in the heptagon spanned by X_2. The same property holds for the ends of these three edges in G. Therefore, $\{e, f, g\}$ may be extended to a perfect matching of G, by adding two edges in each of the heptagons spanned by X_1 and X_2, respectively. □

We may thus assume that graph G is cubic and no edge of C is b-removable in G. We assert that G is the Petersen graph. For this, we observe first that every edge of C lies in M_0, for any edge of $C - M_0$ is b-removable in G, by (6.8). We conclude that $M_0 = C$

Let e be any edge of C. Graph $G'(e)$ is not a brick. Each $(C - e)$-contraction of $G'(e)$ is an odd wheel, a brick. By (2.11), no matching of $H'(e)$ has more than two edges, that is the bipartite graph $H'(e)$ has a vertex cover od edges cosisiting of at most two edges.

Perfect matching M_0 has at least 5 edges. Therefore, $C - e$ has at least 4 edges. It follows that C has just 5 edges, and $p_1'(e)$ and $p_2'(e)$, the vertices resulting from contractions in $G_1 - e$ and $G_2 - e$, constitute a 2-separation of $H'(e)$. This conclusion holds for each edge e of C.

Let us number the vertices of the pentagons spanned by X_1 and X_2,

$$(0', 1', 2', 3', 4') \text{ and } (0'', 1'', 2'', 3'', 4''),$$

respectively. Adjust notation, by changing the origin of those enumerations, if necessary, so that edge e is $(0', 0'')$. Then, the edge f of C incident with vertex $2''$ is incident with one of $1'$ and $4'$. Adjust notation, by adjusting the orientation of the enumeration of the vertices of X_1, so that $f = (1', 2'')$. The edge of C incident with $3''$ is thus incident with $4'$. The edge of C incident with vertex $1''$ cannot be incident with vertex $2'$, otherwise $(4', 3'')$ would be b-removable in G. We conclude that the edges of C are of the form (i', j''), where $j = 2i \mod 5$. Indeed, graph G is the Petersen graph. □

7 Proof of Theorem 1.2

THEOREM 1.2
The characteristic of any separating cut $C := \nabla(X)$ of any matching covered graph G lies in $\{3, 5, \infty\}$. Moreover, $\lambda_G(C) = 5$ if, and only if, graph G has a tight cut minor H, in which cut C is strictly separating, such that one of the following two alternatives holds:

(i) *Either graph H is the Petersen graph, up to multiple edges, or*

(ii) *graph H is not a near-brick and there exist two cuts in H, $D_1 := \nabla_H(Y_1)$ and $D_2 := \nabla_H(Y_2)$, each of which has characteristic 5 in G, set Y_1 is a subset of X and set Y_2 is a subset of $\overline{X}$, collection $\{D_1, D_2, C\}$ is cohesive, cuts D_1 and D_2 are not disjoint, and the graph obtained by the contraction of Y_1 and Y_2 is the Petersen graph, up to multiple edges.*

Proof: By induction on the size of G. Let λ denote the characteristic of C in G. We may assume C to be nontrivial and G to be free of multiple edges.

CASE 1 *Graph G has a nontrivial tight cut that does not cross C.*

Let D be a nontrivial tight cut of G that does not cross C. Let H' be the C-contraction of G that includes C.

Cut D is tight in G, therefore collection $\{C, D\}$ is cohesive. By (4.5), cut C is separating in H' and $\lambda_{H'}(C) = \lambda$. By induction hypothesis with H' playing the role of G, we conclude that λ lies in $\{3, 5, \infty\}$. If λ lies in $\{3, \infty\}$ then the assertion holds. We may assume that $\lambda = 5$.

By induction hypothesis, $\lambda = 5$ if, and only if, H' has a tight cut minor H, in which C is a strictly separating cut. Cut D is a tight cut and H is a tight cut minor of H', a D-contraction of G. Therefore, H is a tight cut minor of G. Moreover, by induction hypothesis, one of the following alternatives holds:

(i) Either graph H is the Petersen graph up to multiple edges, or

(ii) graph H is not a near-brick and there are two cuts in H, $D_1 := \nabla_H(Y_1)$ and $D_2 := \nabla_H(Y_2)$, each of which has characteristic 5 in G, set Y_1 is a subset of X and set Y_2 is a subset of $\overline{X}$, collection $\{D_1, D_2, C\}$ is cohesive, cuts D_1 and D_2 are not disjoint, and the graph obtained by the contraction of Y_1 and Y_2 is the Petersen graph, up to multiple edges.

Therefore, the analysis of this case is complete.

CASE 2 *Every nontrivial tight cut of G crosses C.*

Let $D := \nabla(Y)$ be a nontrivial tight cut of G. Adjust the notation so that $|X \cap Y|$ be odd. Among all nontrivial tight cuts of G choose one such that Y is minimal. Therefore, graph $G\{Y; \overline{y}\}$ is free of nontrivial tight cuts.

Let $I := \nabla(X \cap Y)$ and $U := \nabla(\overline{X} \cap \overline{Y})$. Collection $\{C, D\}$ is cohesive. Thus, by (4.6), $\{C, D, I, U\}$ is also cohesive, C, D, I and U are related by modularity, I and U are separating in $G\{Y; \overline{y}\}$ and $G\{\overline{Y}; y\}$, respectively, and $\lambda \leq \min\{\lambda_{G\{Y\}}(I), \lambda_{G\{\overline{Y}\}}(U)\}$. By (4.5),

$$\lambda_G(I) = \lambda_{G\{Y\}}(I) \quad \text{and} \quad \lambda_G(U) = \lambda_{G\{\overline{Y}\}}(U).$$

Let $\lambda_I := \lambda_G(I)$ and $\lambda_U := \lambda_G(U)$. Suppose that I is nontrivial, thus $G\{Y\}$ is nonbipartite, whence a brick. By induction hypothesis, with $G\{Y; \overline{y}\}$ playing the role of G and I playing the role of C, λ_I lies in $\{3, 5, \infty\}$. In fact, we conclude that λ_I lies in $\{3, 5\}$ because $G\{Y\}$ is free of nontrivial tight cuts. Thus, λ lies in $\{3, 5\}$ because $\lambda \leq \lambda_I$. If $\lambda_I = 3$ then $\lambda = 3$ and the assertion holds. We may assume that $\lambda_I = 5$

By induction hypothesis, there exists a tight cut minor H of $G\{Y; \overline{y}\}$, such that:

(i) Either graph H is the Petersen graph up to multiple edges, or

(ii) graph H is not a near-brick and there are two cuts in H, $D_1 := \nabla_H(Y_1)$ and $D_2 := \nabla_H(Y_2)$, each of which has characteristic 5 in G, set Y_1 is a subset of $X \cap Y$ and set Y_2 is a subset of $\overline{X \cap Y}$, collection $\{D_1, D_2, I\}$ is cohesive, cuts D_1 and D_2 are not disjoint, and the graph obtained by the contraction of Y_1 and Y_2 is the Petersen graph, up to multiple edges.

Graph $G\{Y; \overline{y}\}$ is a brick, thus, by induction hypothesis, it is the Petersen graph, up to multiple edges. Therefore I separates two pentagons in $G\{Y; \overline{y}\}$. Moreover, Petersen graph is a cubic graph, thus each vertex of each pentagon is adjacent to exactly one vertex of the other pentagon. Let v be the the vertex that is adjacent to $\overline{y}$ in the other pentagon. Figure 11 depicts graph G in this case.

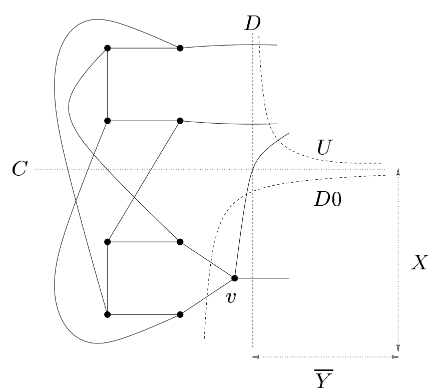


Figure 11: Graph G when I is nontrivial.

Let $D_0 := \nabla((X \cap \overline{Y}) \cup \{v\})$. The modularity relating cuts C, D, I and U implies that no edge of G join vertices of $X \cap \overline{Y}$ with vertices of $\overline{X} \cap Y$. Moreover, $\nabla(v) \cap \nabla(\overline{X} \cap Y) = \emptyset$. Therefore, modularity relates D_0, D, $\nabla((X \cap Y) \setminus \{v\})$ and U. That is, for each perfect matching M of G

$$|M \cap D_0| + |M \cap D| = |M \cap \nabla(\{v\})| + |M \cap U|$$

Both D and $\nabla(\{v\})$are tight cuts. Therefore, D_0 and U are matching equivalent. We conclude that D_0 is a separating cut of G and has the same characteristic of U.

Suppose first that U is a tight cut, whence, D_0 is also tight cut. By case hypothesis, both cuts are trivial. Thus, graph G is the Petersen graph, up to multiple edges and C is one of its strictly separating cuts.

Now, we may assume that $\lambda_U < \infty$. Cut U is a separating cut of $G\{\overline{Y}; y\}$. Thus, by induction hypothesis, λ_U lies in $\{3, 5, \infty\}$. Therefore, λ_U lies in $\{3, 5\}$. If $\lambda_U = 3$ then $\lambda = 3$ and we are done. So we may assume that $\lambda_U = 5$. Thus, $\lambda_{D_0} = 5$ because it is matching equivalent to U and $\lambda = 5$, by modularity. Moreover, after contraction of U and D_0 we have, up to multiple edges, the Petersen graph with C as a strictly separating cut of this graph. The assertion follows in this case.

Now, we may assume that I is a trivial cut. Let i be the vertex of $X \cap Y$. If U is a trivial cut then, by modularity, cut C is a tight cut and the assertion follows. Therefore, we may assume that U is nontrivial.

By induction hypothesis, with $G\{\overline{Y}; y\}$ playing the role of G and U playing the role of C, we conclude that λ_U lies in $\{3, 5\}$. If $\lambda_U = 3$ then the assertion holds. We may assume that $\lambda_U = 5$, whence, $\lambda = 5$.

By induction hypothesis, $\lambda_U = 5$ if, and only if, $G\{\overline{Y}; y\}$ has a tight cut minor H, in which U is a strictly separating cut. Moreover, by induction hypothesis, one of the following alternatives holds:

(i) Either graph H is the Petersen graph up to multiple edges, or

(ii) graph H is not a near-brick and there are two cuts in H, $D_1 := \nabla_H(Y_1)$ and $D_2 := \nabla_H(Y_2)$, each of which has characteristic 5 in $G\{\overline{Y}; y\}$, set Y_1 is a subset of $\overline{X} \cap \overline{Y}$ and set Y_2 is a subset of $X \cup Y$, collection $\{D_1, D_2, U\}$ is cohesive, cuts D_1 and D_2 are not disjoint, and the graph obtained by the contraction of Y_1 and Y_2 is the Petersen graph, up to multiple edges.

Graph H is a tight cut minor of G. Let $D' := \nabla(Z)$ be a tight cut of $G\{\overline{Y}; y\}$ used to obtain H that includes set Y in one of its shores. Adjust the notation so that $Y \subseteq \overline{Z}$.Cut D' does not cross D neither U, but it must cross C. Moreover D' separates D and U, that is $Y \subseteq \overline{Z}$ and $\overline{X} \cap \overline{Y} \subseteq Z$. Therefore, the relative position of these cuts are depicted in Figure 12.

Suppose first that H is the Petersen graph, up to multiple edges, with U a strictly separating cut of H. Then, U separates two pentagons in H. Moreover, Petersen graph is a cubic graph, whence each vertex of each pentagon is adjacent to exactly one vertex of the other pentagon. Let v be the vertex of $\overline{X} \cap \overline{Y}$ such that $\nabla(v) \cap D' \neq \emptyset$. Let H' be the splice of the $G\{\overline{Z}; z\}$ and H. Figure 13 shows this graph.

Set $\{i, v\}$ is a 2-separation of G. Cut D is a 2-separation cut. The other tight cut of this 2-separation is $\nabla((\overline{X} \cap Y) \cup \{v\})$ that does not cross C. Therefore, by case hypothesis $\overline{X} \cap Y$ must be empty, but in this case D is trivial. Contradiction.

Now, we may assume that H is not a near-brick and there exist two cuts in H, $D_1 := \nabla_H(Y_1)$ and $D_2 := \nabla_H(Y_2)$ satisfying ((ii)).

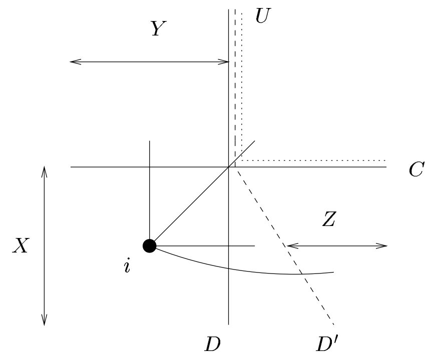


Figure 12: Sketch of the cuts when $\lambda_U = 5$ and I is trivial.

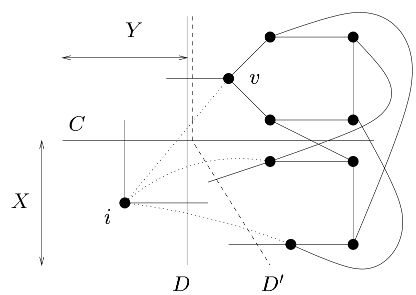


Figure 13: Graph H', splice of $G\{\overline{Z}; z\}$ and H.

Contracting D_1 and D_2 we have the Petersen graph. The contracted vertices are adjacent in the Petersen graph and they are in different shores of U. Petersen graph is cubic, therefore, each vertex in each pentagon of the shores of U is adjacent to only one vertex in the other. Therefore, determination of position of D_1 uniquely determines the position of D_2. We have two cases to analyze depending on whether the vertex y to be in Y_2 or in $\overline{Y_2}$ (see Figure 14). In both cases we derive a contradiction, whence the result follows.

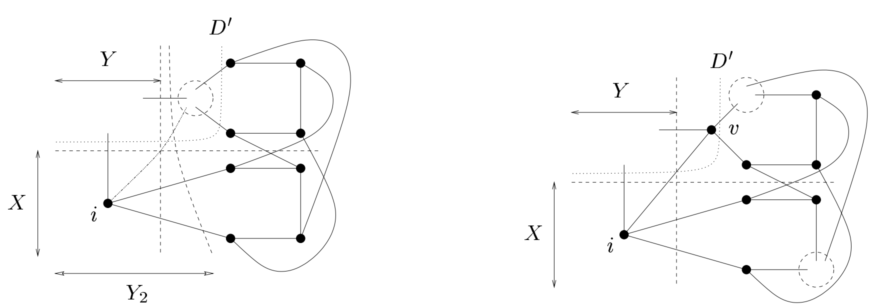


Figure 14: The two cases considered. In first $y \in Y_2$ and in second $y \notin Y_2$

In first case that $y \in Y_2$ (see Figure 14). Let $D' := \nabla((\overline{X} \cap Y) \cup Y_1)$. Note that there are only three vertices that are incident with edges of D' in shore $\overline{Y} \cup \{i\}$. Cut D' is nontrivial and does not cross C, therefore, by case hypothesis, D' can not be tight. Thus, $\lambda(D') = 3$. One of D'-contraction is the Petersen graph and D' is non-tight, then by (2.7) we conclude that $\lambda = 3$. Contradiction.

Suppose now that $y \notin D_1$. Let v be the vertex adjacent to y in the other pentagon. Thus, $\{i, v\}$ is a 2-separation of H'. One of the tight cuts associated with this 2-separation is D. The other is $\nabla((\overline{X} \cap Y) \cup \{v\})$. This cut is also a tight cut in G does not cross C. Therefore, by case hypothesis, this cut must be trivial, whence D is also trivial. Contradiction.

To complete the analysis of this case we must show that if G has a tight cut minor H, in which cut C is strictly separating cut and such that either (i) or (ii) holds, then $\lambda = 5$. Observe that in both alternatives, (i), or (ii), we have a Petersen graph as a minor and the characteristic of C in this minor is five, therefore λ lies in $\{3, 5\}$.

By hypothesis of the case every nontrivial tight cut of G crosses C. So, the only tight cut minor of G that includes C is G itself. If G is a near-brick then, by (i), G is the Petersen graph, up multiple edges, and C is a nontrivial separating cut in G, whence $\lambda = 5$.

We may assume that G is not a near-brick. By hypothesis, there are two separating cuts of characteristic 5 in G, say $D_1 := \nabla(Y_1)$ and $D_2 := \nabla(Y_2)$, such that D_1 and D_2 are not disjoint, C separates D_1 and D_2 and the graph obtained by the contraction of D_1 and D_2 is the Petersen graph, up to multiple edges. Adjust the notation so that $Y_1 \subset X$. Figure 15 shows this graph.

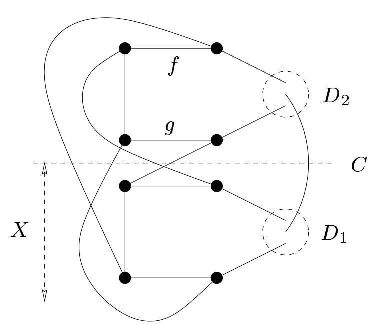


Figure 15: Graph G is not a near-brick.

Graph G is not a near-brick, therefore it has a nontrivial tight cut D. By hypothesis, every nontrivial tight cut crosses C. Let Y be a shore of D. Graph $G[Y]$ must be connected because $G\{Y\}$ is matching covered. Adjust the notation so that $|X \cap Y|$ be odd. Let $I := \nabla(X \cap Y)$ and $U := \nabla(\overline{X} \cap \overline{Y})$. Cut D is a tight cut of G and C is a separating cut of G, thus $\{C, D\}$ is a cohesive collection. By (4.6), modularity property holds with cuts C, D, I and U and $\lambda = \min\{\lambda(I), \lambda(U)\}$.

Suppose Y_1 is a (proper) subset of $X \cap \overline{Y}$. In this case there are edges from $\overline{X} \cap Y$ to $X \cap \overline{Y}$, contradicting modularity property. Therefore, $Y_1 \cap (X \cap Y) \neq \emptyset$.

Suppose that Y_2 is a proper subset of $\overline{X} \cap \overline{Y}$. Thus, because D crosses C edges f and g lie in D. There exists a perfect matching of the Petersen graph that uses edges f and g

(see Figure 15). This perfect matching can be extended to a perfect matching of G and has three edges in D. Contradiction.

Therefore, we conclude that $E(G[Y_1]) \cap D \neq \emptyset$ and $E(G[Y_2]) \cap D \neq \emptyset$. Let $Y_1 := Y_{11} \cup Y_{12}$ and $Y_2 := Y_{21} \cup Y_{22}$. Adjust the notation so that $Y_{11} \subset X \cap Y$ and $Y_{22} \subset \overline{X} \cap \overline{Y}$. Figure 16 shows graph G and cuts C and D. By counting, $|Y_{11}| \equiv 1 \mod 2$ and $|Y_{21}| \equiv 0 \mod 2$

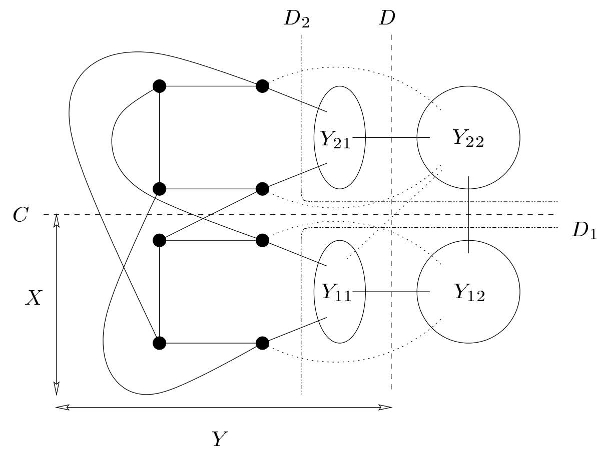


Figure 16: Graph G and cuts C and D.

Let e be an edge of $\nabla(Y_{22}) \cap C$. By hypothesis, collection $\{C, D_1, D_2\}$ is cohesive. Thus, there exists a perfect matching of G that uses e and has exactly one edge in each cut of $\{C, D_1, D_2\}$. Cut D is a tight cut, therefore $|M_e \cap \nabla(Y_{21})| = 0$ because $|M_e \cap \nabla(Y_{11}) \cap D| = 1$ and this edge is an edge of D, whence $M_e \cap E(G[Y_{21}])$ is a perfect matching of $G[Y_{21}]$.

Suppose, by absurd, that $\lambda = 3$. Let M be a perfect matching of G with three edges in C. In this matching,

$$|M \cap D_1 \cap C| = 3 \quad \text{and} \quad |M \cap \nabla(Y_{21}) \cap \nabla(Y_{22})| = 0.$$

Thus, $M \cap E(G[X \cup Y_{22}])$ is a perfect matching of $G[X \cup Y_{22}]$. Therefore,

$$(M_e \cap E(G[Y_{21}])) \cup (M \cap E(G[X \cup Y_{22}]) \cup \{f, g\})$$

is a perfect matching of G with three edges in D_1. Contradiction, the characteristic of D_1 is five. Thus $\lambda = 5$ as we have asserted and the analysis of this case is complete.

CASE 3 *Previous cases do not apply*

We may assume now that graph G is a brick. By (1.1), λ lies in $\{3, 5, \infty\}$ and, if $\lambda = 5$ then graph G has a tight cut minor H that is the Petersen graph, up to multiple edges.

To complete the proof we need analyze the case in which H, that is a tight cut minor of G, is isomorphic to Petersen graph, up to multiple edges, and C is a strictly separating cut of H. Each cut used to obtain H is a tight cut. Therefore by (4.5), $\lambda = 5$ and the proof of the theorem is complete. □

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Contents

List of Assertions

Theorem 1.1 {`main:near-bricks, MAIN:NEAR-BRICKS`} 3

The characteristic $\lambda_G(C)$ of any separating cut C of any near-brick G lies in $\{3, 5, \infty\}$. Moreover, if $\lambda_G(C) = 5$ then graph G has a tight cut minor P that is the Petersen graph, up to multiple edges, and cut C is strictly separating in P.

Theorem 1.2 {`main, MAIN, alt:Petersen, alt:cohesive`} 4

The characteristic of any separating cut $C := \nabla(X)$ of any matching covered graph G lies in $\{3, 5, \infty\}$. Moreover, $\lambda_G(C) = 5$ if, and only if, graph G has a tight cut minor H, in which cut C is strictly separating, such that one of the following two alternatives holds:

(i) Either graph H is the Petersen graph, up to multiple edges, or

(ii) graph H is not a near-brick and there exist two cuts in H, $D_1 := \nabla_H(Y_1)$ and $D_2 := \nabla_H(Y_2)$, each of which has characteristic 5 in G, set Y_1 is a subset of X and set Y_2 is a subset of $\overline{X}$, collection $\{D_1, D_2, C\}$ is cohesive, cuts D_1 and D_2 are not disjoint, and the graph obtained by the contraction of Y_1 and Y_2 is the Petersen graph, up to multiple edges.

Lemma 2.1 {`induced:contracted`} 5

Let $C := \nabla(X)$ be a separating cut of a matching covered graph G. Then, the C-contraction $G\{X\}$ is bipartite if, and only if, induced subgraph $G[X]$ of G is also bipartite. Moreover, if $G[X]$ is bipartite, the cardinality of two parts of the bipartition differ by one unit.

Corollary 2.2 {`shore:bip`} 5

Let C be a separating cut of a matching covered graph G. If one of the C-contractions of G is bipartite then cut C is tight in G.

In a bipartite graph G, a cut is tight if, and only if, it is separating in G.

A matching covered graph G is bipartite if, and only if, it has $b(G) = 0$.

A non-bipartite matching covered graph G is a near-brick if, and only if, graph G is free of strictly separating tight cuts.

Graph G is a near brick if, and only if, for each nontrivial tight cut of G one of the C-contractions is a bipartite graph and the other is a near-brick. □

Let G be a matching covered graph, D be a non-tight cut of G. If a D-contraction H of G is the Petersen graph, up to multiple edges, then every nontrivial separating cut of H is a separating cut of G with characteristic three in G.

Let G be the simple graph obtained from the Petersen graph P by adding an edge e. Let C be a nontrivial separating cut of G such that $C - e$ is separating in $G - e$. Then, the characteristic of C in G is equal to three.

For any maximal barrier B of a matching covered graph G, every bipartite (odd) component of $G - B$ is trivial.

For every 3-edge-connected near-brick G, every equivalence class Q with respect to the dependence relation contains at most two edges, with equality only if graph $G - Q$ is bipartite.

Let G be a matching covered graph, $C := \nabla(X)$ a separating cut of G. If each C-contraction of G is bicritical then graph G is bicritical. Moreover, if each C-contraction of G is a brick then, G is a brick if, and only if, subgraph $G[C]$ of G spanned by C has a matching with at least three edges.

Let G be a brace with at least four vertices. If G has at least six vertices then every edge is removable in G. If G has just four vertices and is free of vertices of degree two then, for every vertex v of G, at most one edge of $\nabla(v)$ is not removable in G.

Let G be a matching covered graph, $D := \nabla(Y)$ a separating cut of G that is either tight or robust in G, $H := G\{Y;\overline{y}\}$ a D-contraction of G, C a tight cut of H. Then, either (i) cut C is tight in G or (ii) cuts C and D are matching-equivalent in G, cut C is robust in G and the C-contraction of H that contains vertex $\overline{y}$ is bipartite.

(See [2], Theorem 4.3) If a matching covered graph has a robust cut then it is a near-brick. □

(See [3], Corollary 2.4) Let C be a separating cut of a brick G, let M_0 be a perfect matching of G that contains more than one edge in C. Let $\mathcal{C}$ be the collection of separating cuts D of G such that $|M_0 \cap D| > 1$ and $D \preceq C$. Then, every cut of $\mathcal{C}$ that is minimal with respect to the relation of precedence is robust in G. □

(See the proof of Theorem 2.23 in [2]) Let e be a removable edge of a matching covered graph G, let C be a cut of G such that $C - e$ is strictly separating in $G - e$. Let $\mathcal{C}$ be the collection of those cuts D of G such that $D - e$ is strictly separating in $G - e$ and $D \preceq C$. Then, every cut of $\mathcal{C}$ that is minimal with respect to the relation of precedence is strictly separating in G.

If a near-brick G is solid then every removable edge of G is b-removable in G.

A cut C of a matching covered graph G is separating if, and only if, every edge of G lies in a perfect matching of G that contains precisely one edge in C.

Every tight cut of a matching covered graph is separating.

A cut C of a matching covered graph G is separating, if, and only if, collection $\{C\}$ is cohesive.

For each cohesive collection $\mathcal{C}$ of a matching covered graph G and every tight cut C of G, collection $\{C\} \cup \mathcal{C}$ is also cohesive.

For any cohesive laminar collection $\{C, D\}$ of cuts of a matching covered graph G, let H denote the D-contraction of G that contains cut C. Then, cut C is separating in H. Moreover, $\lambda_G(C) \leq \lambda_H(C)$, with equality if cut D is tight in G.

Let $C := \nabla(X)$ and $D := \nabla(Y)$ be two crossing cuts of a matching covered graph G. Adjust notation so that $|X \cap Y|$ be odd. Let $I := \nabla(X \cap Y)$, let $U := \nabla(\overline{X} \cap \overline{Y})$. If collection $\{C, D\}$ is cohesive, then the following properties hold:

(i) For every set F of edges of G, the following relation of modularity holds:

$$|F \cap C| + |F \cap D| = |F \cap I| + |F \cap U|.$$

(ii) Collection $\{C, D, I, U\}$ is cohesive.

(iii) Let λ_I denote the characteristic of cut I in $G\{Y\}$ and λ_U denote the characteristic of cut U in $G\{\overline{Y}\}$. Then, $\lambda_C(G) \leq \min\{\lambda_I, \lambda_U\}$, with equality if cut D is tight in G.

Theorem 5.1 {`oddWheels`} 17

(Theorem on Odd Wheels) For any vertex v of a 3-edge-connected near-brick G and any v-matching M of G, one of the following properties holds:

(i) Either graph G is an odd wheel of hub v, up to multiple edges incident with vertex v,

(ii) or graph G is not solid,

(iii) or graph G has a removable singleton or doubleton that is disjoint with $M \cup \nabla(v)$.

Theorem 5.2 {`thm:lovaszCritical`} 18

Every critical graph G can be represented as

$$G = P_0 + P_1 + \cdots + P_r, \tag{4}$$

where P_0 is K_1 and each P_i $(1 \leq i \leq r)$ is either an odd path or an odd circuit having precisely its origin and terminus in common with $P_0 + P_1 + \cdots + P_{i-1}$.

Proposition 5.3 {`converse:lovaszCritical`} 18

If a graph G has an ear decomposition as in (4), then it is critical. □

Proposition 5.4 {`allInM`} 19

For each integer i, $(q < i \leq r)$, ear P_i has length one, its only edge, p_i, lies in M. Therefore, for any permutation $(P'_{q+1}, \cdots, P'_r)$ of $(P_{q+1}, \cdots, P_r)$, sequence $(P_0, P_1, \cdots, P_q, P'_{q+1}, \cdots, P'_r)$ is an ear decomposition of $G - v$ of index q.

Proposition 5.5 19

No edge f of S lies in $M \cup \nabla(v)$.

Lemma 5.6 {`both:critical`} 20

Each of $G_1 - x$ and $G_2 - \overline{x}$ is critical.

Proposition 5.7 {`bothEndsInPq`} 20

Let e be any upper edge of $\mathcal{P}$. Then, both ends of edge e are internal vertices of $V(P_q)$. Moreover, if v_j and v_k denote the two ends of e in $V(P_q)$, path $S[j,k]$ has even length.

Proposition 5.8 {`cross`} 21

Let v_i, v_j be the ends of upper edge e and v_k, v_l the ends of upper edge f such that $i < k < j < l$. Then, each of $S[i,k]$, $S[k,j]$ and $S[j,l]$ has even length.

Proposition 5.9 {`adjTo:v`} 21

Let e be an upper edge of $\mathcal{P}$, v_i and v_j its ends in $V(P_q)$. Then, at least one internal vertex of $S[i,j]$ is adjacent to vertex v.

Proposition 5.10 {`GtwoMinusxCritical`} 21

Graph $G_2 - \overline{x}$ is critical.

Proposition 5.11 {`contra:bicritical`} 23

Let $D := \nabla(Y)$ be any odd cut of G, $H := G\{Y;\overline{y}\}$. If graph $H - \overline{y}$ is critical then H is matching covered and bicritical.

Proposition 6.1 {`CInonnull`} 26

Either set $\mathcal{C}_I$ is nonnull or set $X \cap Y$ contains a vertex, i, such that no vertex of $\overline{Y}$ is adjacent to any vertex of $X \cap Y - i$.

Lemma 6.2 {`noWitnesses`} 27

Let G be a brick, C a nontrivial separating cut of G. Either G has a witness for C or cut C is robust and each C-contraction of G is solid.

Proposition 6.3 28

Collection $\mathcal{D}$ has at most two members, each of which is a subset of X. Moreover, for each set Y in $\mathcal{D}$, graph $H := (G-e)[Y]$ has a bipartition (A,B) such that $|B| = |A|+1$, and edge e is the only edge of G that is incident with some vertex of A but not incident with any vertex of B.

Notation 6.4 28

Let $r := |\mathcal{D}|$. For $1 \leq i \leq r$, let Y_i denote an enumeration of the members of $\mathcal{D}$, $H_i := (G-e)[Y_i]$, (A_i, B_i) the bipartition of H_i such that $|B_i| = |A_i| + 1$. Let v_i and w_i denote the ends of e in G, such that vertex v_i lies in A_i. Let y_i be the vertex of P obtained by the contraction of Y_i to a single vertex.

Proposition 6.5 29

Under the hypothesis of Case 6.1, for each member Y_i of $\mathcal{D}$ and any two vertices x_i and y_i of the set Z_i of the three vertices of $V(G) \cap V(P)$ that are adjacent in G to vertices of B_i, the subgraph W_i of G spanned by $Y_i \cup \{x_i, y_i, w_i\}$ has a perfect matching, M_i, that contains precisely three edges in $\nabla(Y_i)$, incident, respectively, to x_i, y_i, and w_i.

Proposition 6.6 30

The subgraph W of G spanned by $Y_1 \cup Y_2 \cup Z$ has a perfect matching that contains precisely four edges in $\nabla(Y_1 \cup Y_2)$, each of which is incident to one of the four vertices of Z.

Proposition 6.7 30

For $i = 1, 2$, G_i is an odd wheel of hub v_i, up to multiple edges in $\nabla(v_i)$.

Proposition 6.8 {`splicing:oddWheels`} 31

Let e be any edge of C. Then, graph $G'(e)$ is bicritical. Moreover, if the degree of one of $p_1(e)$ and $p_2(e)$ in H is at least two, or if edge e does not lie in M_0, then graph $G'(e)$ is a brick and graph $G - e$ is a near-brick.