Non-Homogeneous Spline Bases for Approximation on the Sphere

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Abstract. A *spherical polynomial* is the restriction to the sphere S^2 of a polynomial in the three coordinates x, y, z of $\mathbb{R}^3$. Let T be an arbitrary triangulation on the sphere, and let $\mathcal{P}_k^d[T]/S^2$ (resp $\mathcal{H}_k^d[T]/S^2$) be the space of all $\boldsymbol{C}_k$-continuous functions f from S^2 to $\mathbb{R}$ such that the restriction of f to each triangle of T is a spherical polynomial (resp. homogeneous). These are the *spherical polynomial* (resp *homogeneous*) *splines* of degree $\leq d$ (resp. exactly d) and continuity k.

In a previous paper, we have shown that $\mathcal{P}_k^d[T]/S^2 = \mathcal{H}_k^d[T]/S^2 \oplus \mathcal{H}_k^{d-1}[T]/S^2$. Alfeld, Neamtu and Schumaker have recently constructed explicit bases for the spaces $\mathcal{H}_k^d[T]/S^2$. Combining these two results, we obtain explicit constructions for bases of $\mathcal{P}_k^d[T]/S^2$.

We believe that the general spline spaces $\mathcal{P}_k^d[T]/S^2$ provide better approximations than the homogeneous spaces $\mathcal{H}_k^d[T]/S^2$ when used over the relatively large regions (radius 10^{-1} to 10^{-2}) that are likely to occur in pratice. In this paper we report numerical experiments in least squares approximation which offer some evidence for this claim.

§1. Introduction

The problem of modeling or approximating a real function defined on the sphere S^2 arises in many applications, such as geophysics, meteorology, computer graphics, etc. [8]. Such functions are usually represented as polynomials on the spherical coordinates ϕ, θ, (longitude and latitude). This approach, however, has several drawbacks: the resulting functions are often discontinuous at the poles, the geodesic lines correspond to curves in the (ϕ, θ) plane, the resolution of (ϕ, θ) grids is not uniform over the sphere, and so on. These problems are particularly annoying for applications that require irregular or adaptive meshes.

These difficulties have recently led some researchers to consider the modeling of spherical functions as functions of the spatial cartesian coordinates (x, y, z), restricted to the sphere. Alfeld, Neamtu e Schumaker [1,2,3] investigated the use of homogeneous spherical polynomial splines as an approximation space for functions defined on S^2. We showed in a previous work [7] that the general (non-homogeneous) spherical polynomial splines of degree $\leq d$ may be written as the direct sum of homogeneous splines of degree d and $d-1$.

In this work, we explore the use of such non-homogeneous splines for least square approximation of functions. We compare the accuracy of the results obtained with both kinds of splines, and show evidence that the non-homogenous splines seem to offer more uniformly accurate approximations than their homogeneous subspaces alone.

§2. Spherical Polynomials

A spherical polynomial is a polynomial in the three coordinates x, y, z of $\mathbb{R}^3$, restricted to the unit sphere S^2. Let $\mathcal{P}^d/S^2$ be the space of spherical polynomials with degree $\leq d$. It can be shown [7,6] that

$$\mathcal{P}^d/S^2 = \mathcal{H}^d/S^2 \oplus \mathcal{H}^{d-1}/S^2, \tag{1}$$

where $\mathcal{H}^d/S^2$ denotes the space of homogeneous polynomials of degree d in x, y, z. (Recall that function f from $\mathbb{R}^n$ to $\mathbb{R}$ is said to be homogeneous of degree m if $f(ax) = a^m f(x)$, for any $a \in \mathbb{R}$ and $x \in \mathbb{R}^n$.)

2.1. Spherical Harmonics

There is a strong relationship between the spherical polynomials and the well-known spherical harmonics [4]. We denote by $\mathcal{Y}^d$ the space of the real spherical functions from S^2 to $\mathbb{R}$ generated by the real and imaginary parts of the spherical harmonics Y_k^m of degree $k \leq d$. We also denote by $\mathcal{W}^d$ the subspace of $\mathcal{Y}^d$ generated by the spherical harmonics of degree d, $d-2$, ..., $d - 2\lfloor \frac{d}{2} \rfloor$. Fausshauer and Schumaker [6] prove the following results:

Theorem 1. *The space $\mathcal{W}^d$ coincides with the space $\mathcal{H}^d/S^2$.*

Theorem 2. *The space $\mathcal{Y}^d$ coincides with the space $\mathcal{P}^d/S^2$.*

§3. Spherical Splines

It turns out that the decomposition of $\mathcal{P}^d/S^2$ (1) can be extended to the splines defined on a geodesic triangulation T on the sphere. Let $\mathcal{P}_k^d[T]/S^2$ be the space of all functions f from S^2 into $\mathbb{R}$ such that (i) the restriction of f to each triangle of T coincides with a function of $\mathcal{P}^d/S^2$; and (ii) the function f has continuity of order k across the edges of T. Let also $\mathcal{H}_k^d[T]/S^2$ be the subspace of $\mathcal{P}_k^d[T]/S^2$ that consists of the functions that are homogeneous of degree d (i.e, $\mathcal{H}^d/S^2$) in each triangle of T. These are the spherical polynomial (resp homogeneous) splines of degree $\leq d$ (resp. exactly d) and continuity k.

We have shown [7] that

Theorem 3. $\mathcal{P}_k^d[T]/S^2 = \mathcal{H}_k^d[T]/S^2 \oplus \mathcal{H}_k^{d-1}[T]/S^2$.

3.1 Spline Bases

Alfeld, Neamtu, and Schumaker [2] obtained an explicit basis, with local support, for each space $\mathcal{H}_k^d[T]/S^2$, when $d \geq 3k+2$, in terms of the Bernstein-Bézier polynomials. (Their construction assumes that the triangulation T is non-degenerate, in the sense that no two edges incident to the same vertex are coplanar. Every triangulation T which is mentioned in this paper is assumed to have this property.)

The Alfeld-Neamtu-Schumaker (ANS) construction implies that the dimension of the space $\mathcal{H}_k^d[T]/S^2$, for $d \geq 3k+2$, is

$$\begin{aligned}\dim \mathcal{H}_k^d[T]/S^2 &= (d^2 - 3dk + 2k^2)v - 2d^2 + 6dk - 3k^2 + 3k + 2 \\ &= (d^2 - 3dk + 2k^2)t/2 + k^2 + 3k + 2,\end{aligned}$$

where v and t are respectively the number of vertices and triangles of the triangulation. Therefore, by Theorem 3, a basis for $\mathcal{P}_k^d[T]/S^2$ is obtained through the concatenation of a basis of $\mathcal{H}_k^d[T]/S^2$ and a basis of $\mathcal{H}_k^{d-1}[T]S^2$. Thus we get the dimension of the space $\mathcal{P}_k^d[T]/S^2$, for $d \geq 3k+3$:

$$\begin{aligned}\dim \mathcal{P}_k^d[T] &= (2d^2 - 6dk + 4k^2 - 2d + 3k + 1)v - 4d^2 + 12dk + 4d - 6k^2 + 2 \\ &= (2d^2 - 6dk + 4k^2 - 2d + 3k + 1)t/2 + 2k^2 + 6k + 4.\end{aligned}$$

The lowest-degree spaces $\mathcal{P}_k^d[T]/S^2$ that have ANS bases are $\mathcal{P}_0^3[T]/S^2$ for continuity class $\boldsymbol{C}_0$, and $\mathcal{P}_1^6[T]/S^2$ for continuity class $\boldsymbol{C}_1$. Table 1 gives the dimensions of those spaces and of their homogeneous components, as well as of the homogeneous spaces $\mathcal{H}_0^4[T]/S^2$ and $\mathcal{H}_1^7[T]/S^2$ which have approximately the same dimensions.

$\boldsymbol{C}_0$ *spaces*	*Dimensions*		
$\mathcal{H}_0^2[T]/S^2$	$4v-6$	$=$	$2t+2$
$\mathcal{H}_0^3[T]/S^2$	$9v-16$	$=$	$9t/2+2$
$\mathcal{H}_0^4[T]/S^2$	$16v-30$	$=$	$8t+2$
$\mathcal{P}_0^3[T]/S^2$	$13v-22$	$=$	$13t/2+4$

$\boldsymbol{C}_1$ *spaces*	*Dimensions*		
$\mathcal{H}_1^5[T]/S^2$	$12v-18$	$=$	$6t+6$
$\mathcal{H}_1^6[T]/S^2$	$20v-34$	$=$	$10t+6$
$\mathcal{H}_1^7[T]/S^2$	$30v-54$	$=$	$15t+6$
$\mathcal{P}_1^6[T]/S^2$	$32v-52$	$=$	$16t+12$

Tab. 1. Dimensions of some general and homogeneous spline spaces.

Each element of an ANS basis is associated with a face, an edge, or a vertex of the triangulation. Figure 1 shows some ANS basis elements for the space $\mathcal{H}_1^6[T]/S^2$, where T is the central projection of a regular icosahedron onto the sphere.

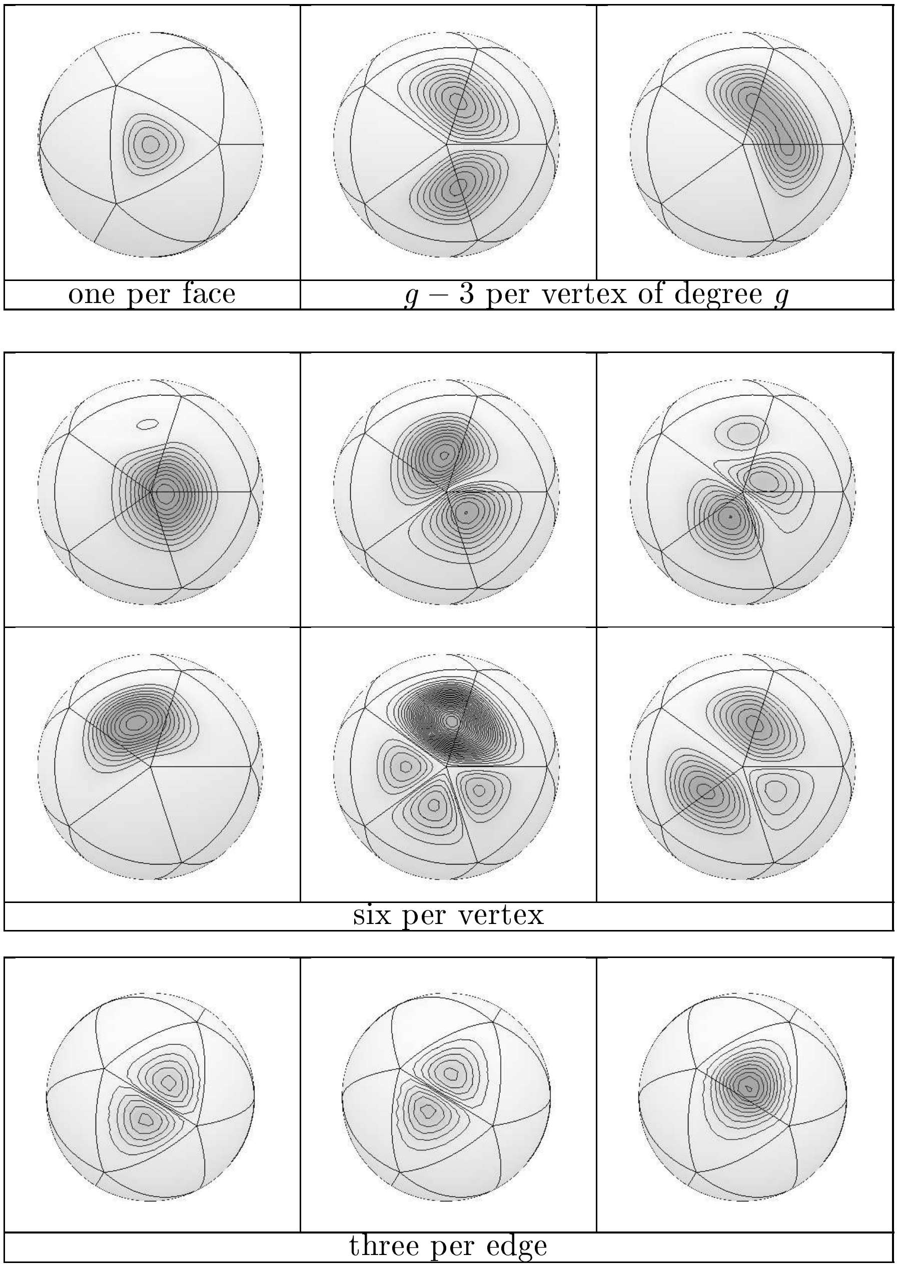

Fig. 1. Typical Alfeld-Neamtu-Schumaker basis elements.

§4. Approximation with Spherical Splines

In order to test the effective accuracy obtainable with these splines spaces, we have performed some numerical experiments in least squares approximation.

Let f be a real function on the sphere, and $\{\phi_i\}_1^n$ a basis for some space $\mathcal{F}$ of real functions on S^2. Let $\langle , \rangle$ be an inner product for spherical functions, and $\|f\|^2 = \langle f, f \rangle$ its associated squared norm. We want to find a function $u(p) = \sum_{i=1}^{n} x_i \phi_i(p)$ that best approximates f in the sense that $\|u(p) - f(p)\|^2$ is minimum.

It is well known that u can be obtained by solving the normal system of linear equations $Gx = b$. In this system, the unknowns $x = (x_1, x_2 \ldots x_n)$ are the coefficients of the approximation $u(p)$ in the basis $\{\phi_i\}_1^n$; the elements of the matrix G are the inner products $G_{ij} = \langle \phi_i, \phi_j \rangle$, $i, j = 1 \ldots n$; and the elements of the independent vector b are $b_i = \langle f, \phi_i \rangle$, $i = 1 \ldots n$.

In our tests, we used the spline spaces shown in table 1. Each of these spaces was used to approximate the functions $f(p) = x^2$, $f(p) = x^3$, $f(p) = \exp(x)$, $f(p) = \sin(x)$ and $f(p) = \cos(x)$, where $p = (x, y, z) \in S^2$. We also the tested the function $f^*(p) = 1 + x^8 + \exp(2y^3) + \exp(2z^2) + 10xyz$ with was used by Alfeld, Neamtu and Schumaker [3]. Tables 2 and 3 below summarize the approximation errors obtained in these tests.

	$\boldsymbol{C}_0$ *spaces (dimensions)*			
Functions	$\mathcal{H}_0^2[T]$ (34)	$\mathcal{H}_0^3[T]$ (74)	$\mathcal{H}_0^4[T]$ (130)	$\mathcal{P}_0^3[T]$ (108)
x^2	1.5×10^{-15}	3.9×10^{-2}	9.0×10^{-15}	4.8×10^{-13}
x^3	1.0×10^{-1}	2.0×10^{-15}	1.4×10^{-2}	5.8×10^{-13}
$\exp(x)$	1.2×10^{-1}	1.7×10^{-1}	3.3×10^{-2}	1.2×10^{-3}
$\sin(x)$	1.2×10^{-1}	3.7×10^{-4}	2.9×10^{-2}	2.8×10^{-4}
$\cos(x)$	6.6×10^{-3}	1.8×10^{-1}	4.5×10^{-5}	9.0×10^{-4}
$f^*(p)$	2.3×10^{-1}	1.2×10^{-1}	3.7×10^{-2}	3.2×10^{-2}

Tab. 2. Errors of $\boldsymbol{C}_0$ least squares aproximations.

	$\boldsymbol{C}_1$ *spaces (dimensions)*			
Functions	$\mathcal{H}_1^5[T]$ (102)	$\mathcal{H}_1^6[T]$ (166)	$\mathcal{H}_1^7[T]$ (246)	$\mathcal{P}_1^6[T]$ (268)
x^2	1.1×10^{-2}	4.4×10^{-12}	2.2×10^{-3}	1.7×10^{-8}
x^3	1.7×10^{-12}	3.7×10^{-3}	2.7×10^{-11}	1.0×10^{-8}
$\exp(x)$	6.1×10^{-2}	9.6×10^{-3}	1.7×10^{-2}	1.6×10^{-7}
$\sin(x)$	2.9×10^{-6}	9.8×10^{-3}	7.1×10^{-9}	1.0×10^{-7}
$\cos(x)$	5.7×10^{-2}	1.2×10^{-7}	1.6×10^{-2}	4.3×10^{-8}
$f^*(p)$	4.1×10^{-2}	9.4×10^{-3}	8.9×10^{-3}	1.4×10^{-3}

Tab. 3. Errors of $\boldsymbol{C}_1$ least squares aproximations.

The mesh T (with 24 edges, 10 vertices and 16 triangles) was a Delaunay triangulation of 10 irregularly distributed points. The approximation error was estimated by evaluating $\|u(p_k) - f(p_k)\|$ for a set of 7216 points p_k, with an approximately uniform distribution over the sphere S^2.

Observe that with the space $\mathcal{P}_0^3[T]/S^2$ we usually obtain better approximations to general functions than with the space $\mathcal{H}_0^4[T]/S^2$, even though the the latter has more degrees of freedom (130 against 108). Likewise, the space $\mathcal{P}_1^6[T]/S^2$ usually provides better approximations than $\mathcal{H}_1^7[T]/S^2$, which is only a little smaller (246 degrees of freedom against 268). The exceptions seem to be functions which happen to lie in the homogeneous space, either exactly (such as x^2 for $\mathcal{H}_0^4$, and x^3 for $\mathcal{H}_0^7$), or nearly so (such as $\cos(x)$ for $\mathcal{H}_0^4$ and $\sin(x)$ for $\mathcal{H}_0^7$).

§5. Conclusion

In light of Theorems 1—3, it is obvious that the space $\mathcal{P}_k^d[T]/S^2$ can approximate exactly any spherical harmonic function of degree d. The same is not true of the space $\mathcal{H}_k^d[T]/S^2$, which is the subspace of $\mathcal{P}_k^d[T]/S^2$ generated by the spherical harmonics whose degrees have the same parity as d.

Our numerical experiments seem to indicate that, for the relatively large triangles one is likely to use in practice (with radii around 10^{-1} or 10^{-2}), the approximation errors obtained with the splines $\mathcal{H}_k^{d'}[T]/S^2$ are usually larger than those otained with splines $\mathcal{P}_k^{d''}[T]/S^2$, even when the degrees d' and d'' are chosen so that the two spaces have approximately the same dimension. (Similar results were obtained in the integration of partial differential equations on the sphere by finite element methods; these results will be reported elsewhere.)

The experiments by Alfeld, Neamtu and Schumaker [3] with local interpolation methods for homogeneous splines confirm the expected approximation order $O(r^{d+1})$ of those spaces for meshes of vanishing triangle radius r. Unfortunately, we were unable to perform the analogous experiments for non-homogeneous splines, since we were unable to devise local interpolation methods that would generate the full space $\mathcal{P}_k^d[T]/S^2$—even for continuity $k = 0$. (One difficulty is the coincidence of two Bézier knots ($c_{d,0,0}$ and $c_{d-1,0,0}$) at each vertex, which call for two independent scalar data values at that point, depending only on those coefficients.) Thus the asymptotic approximation order of the non-homogeneous splines is still an open question.

§Appendix A

A.1. Numerical Integration on the Sphere

In order to apply the least square approximation method, it is necessary to compute the dot products $\langle \phi_i, \phi_j \rangle$ and $\langle f, \phi_i \rangle$ where the ϕ_i are splines basis functions and f is an arbitrary given function. We chose $\langle f, g \rangle = \int_{S^2} f(p)g(p)\,dp$ as the inner product of spherical functions.

To evaluate the integral, we first break it into a sum of integrals over individual triangles of T. For each spherical triangle $t \in T$, let $\bar{t}$ be the plane triangle with the same vertices as t. Then, we can write:

$$\int_t h(p)\,dp = \int_{\bar{t}} h(\frac{q}{|q|}) w(q)\,dq, \tag{2}$$

where $w(q) = dp/dq$ is the spherical correction, the ratio between the area of an element dq of $\bar{t}$ around the point q, and the area of its central projection dp onto the sphere—that is, $w(q) = n \circ q/\|q\|^2$ where n is the normal of the plane triangle $\bar{t}$. The integral on the right-hand-side of the equation (2) is then approximated by a 13-point, seventh-order Gauss cubature formula for a plane triangle, described by G. R. Cowper [5].

Notice that the functions h that we have to integrate may be the products of two polynomials of degree 6 (that is, polynomials of degree 12) times a non-polynomial spherical correction factor. For that reason, we found that the 7th-order cubature formula was not sufficient to calculate $\langle f, g \rangle$ with the necessary accuracy, on the triangulation T used in our tests. To get around this problem, we had to partition each triangle t into four sub-triangles, and apply the cubature formula to each part.

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