Non-Homogeneous Polynomial $\mathbf{C}_k$ Splines on the Sphere $\mathbf{S}^n$

Anamaria Gomide *Jorge Stolfi*

Relatório Técnico IC–00-13

Julho de 2000

Non-Homogeneous Polynomial $\mathbf{C}_k$ Splines on the Sphere $\mathbf{S}^n$

Anamaria Gomide
Jorge Stolfi
Institute of Computing, University of Campinas
Cx. Postal 6176 - 13081-970 Campinas, SP - Brazil
{anamaria,stolfi}@dcc.unicamp.br

Abstract

A *homogeneous spherical polynomial* (HSP) is the restriction to the sphere $\mathbf{S}^{n-1}$ of a homogeneous polynomial on the cartesian coordinates $x_1, x_2, \ldots, x_n$ of $\mathbf{R}^n$. A *homogeneous spherical spline* is a function that is an HSP within each element of a geodesic triangulation of $\mathbf{S}^{n-1}$. There has been considerable interest recently in the use of such splines for approximation of functions defined on the sphere. In this paper we introduce the *general* (non-homogeneous) *spherical splines* and argue that they are a more natural approximating spaces for spherical functions than the homogeneous ones. It turns out that the space of general spherical polynomials of degree d is the direct sum of the homogeneous spherical polynomials of degrees d and $d-1$. We then generalize this decomposition result to polynomial splines defined on a geodesic triangulation (spherical simplicial decomposition) T of the sphere $\mathbf{S}^{n-1}$, of arbitrary degree d and continuity order k.

For the particular case $n = 3$, the homogeneous spline spaces were extensively studied by Alfeld, Neamtu, and Schumaker, who showed how to construct explicit local bases when $d \geq 3k+2$. Combining their construction with our decomposition theorem, we obtain an explicit construction for a local basis of the general polynomial splines when $d \geq 3k+3$.

1 Introduction

The problem of modeling or approximating a real function defined on the sphere $\mathbf{S}^2$ arises in many applications, such as geophysics, meteorology, computer graphics, etc.. Such functions are usually represented as polynomials on the spherical coordinates ϕ, θ, (longitude and latitude). This approach, however, has several drawbacks: the resulting functions are often discontinuous at the poles, the geodesic lines correspond to curves in the (ϕ, θ) plane, the resolution of (ϕ, θ) grids is not uniform over the sphere, and so on. These problems are particularly annoying for applications that require irregular or adaptive meshes.

These difficulties have recently led some researchers to consider the modeling of spherical functions as piecewise polynomial on the spatial cartesian coordinates (x, y, z), restricted to the sphere. In particular, Alfeld, Neamtu and Schumaker [1, 2, 3] have proposed the use of the so-called *homogeneous spherical splines* as an approximation space for functions defined on $\mathbf{S}^2$. Here we define an alternative space for this same purpose the *general* (*non-homogeneous*) *spherical polynomial splines.* We show that the general splines of any degree d are the direct sum of the homogeneous splines of degrees d and $d-1$, for any continuity k. Alfeld, Neamtu, and Schumaker gave an explicit construction for a basis of the homogeneous splines, provides $d \geq 3k+2$. Combining their construction with our decomposition theorem, we obtain an explicit construction for a local basis of the general polynomial splines when $d \geq 3k+3$.

This concept can be extended to functions defined on the sphere $\mathbf{S}^{n-1}$ of arbitrary dimension (although explicit basis constructions for such spaces is still an open problem). This result allows us to obtain a characterization for the bases of the space $\mathcal{P}_k^{d,3}[T]/\mathbf{S}^2$.

2 Polynomial Function on $\mathbf{R}^n$

Let $\mathcal{P}^{d,n}$ be the space of polynomials on n variables of degree $\leq d$, viewed as functions from $\mathbf{R}^n$ to $\mathbf{R}$. A function p belongs to $\mathcal{P}^{d,n}$ if and only if it can be written in the form

$$p(x) = \sum_{\substack{0 \leq i_1+i_2+\dots+i_n \leq d \\ i_1,\dots,i_n \geq 0}} c_{i_1 i_2 \dots i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$, and the *coefficients* $c_{i_1 i_2 \dots i_n}$ are real coefficients. The set $\mathcal{P}^{d,n}$ is obviously a vector space, of dimension

$$\dim \mathcal{P}^{d,n} = \begin{pmatrix} d+n \\ n \end{pmatrix}$$

We say that a function f defined on $\mathbf{R}^n$ is *homogeneous of degree* d if $f(ax) = a^d f(x)$, for all $a \in \mathbf{R}$ and all $x \in \mathbf{R}^n$. Let $\mathcal{H}^{d,n}$ the space of the polynomials on $\mathbf{R}^n$ of degree $\leq d$ which are homogeneous of degree d. Obviously $\mathcal{H}^{d,n}$ is a subspace of $\mathcal{P}^{d,n}$. A function h belongs to $\mathcal{H}^{d,n}$ if and only if it can be written in the form

$$h(x) = \sum_{\substack{0 \leq i_1+i_2+\dots+i_n \leq d \\ i_1,\dots,i_n \geq 0}} c_{i_1 i_2 \dots i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$$

It follows that

$$\dim \mathcal{H}^{d,n} = \begin{pmatrix} d+n-1 \\ n-1 \end{pmatrix}$$

As a special case, it is convenient to define $\mathcal{H}^{d,n}$ for all $d < 0$, as the trivial 0-dimensional space $\{0\}$. It easy to see that, if $d \neq d'$, the spaces $\mathcal{H}^{d,n}$ and $\mathcal{H}^{d',n}$ are linearly independent; that is, $\mathcal{H}^{d,n} \cap \mathcal{H}^{d',n} = \{0\}$.

3 Spherical polynomials

If a function f is defined on $\mathbf{R}^n$, and $X \subseteq \mathbf{R}^n$, we denote by f/X the restriction of f to the set X. By extension, we define the restriction of a function space $\mathcal{F}$ to the set X as $\mathcal{F}/X = \{\, f/X : f \in \mathcal{F} \,\}$ If $f/X = g/X$, we will also write $f \equiv g \pmod{X}$; or just $f \equiv g$, when X is implicit in the context. It is obvious that '$\equiv$' is an equivalence relation.

We are interested in the space $\mathcal{P}^{d,n}/\mathbf{S}^{n-1}$, consisting of the polynomial functions on $\mathbf{R}^n$ of degree d, restricted to the sphere $\mathbf{S}^{n-1} = \{\, x \in \mathbf{R}^n : |x| = 1 \,\}$. Observe that polynomials which are distinct in $\mathbf{R}^n$ can be identical when restricted to the sphere $\mathbf{S}^{n-1}$. Therefore, the dimension of $\mathcal{P}^{d,n}/\mathbf{S}^{n-1}$ is generally smaller than that of $\mathcal{P}^{d,n}$. The following lemmas are fundamental for the caracterization of $\mathcal{P}^{d,n}/\mathbf{S}^{n-1}$:

Lemma 1 *For any d and any $n \geq 1$, $\mathcal{H}^{d,n}/\mathbf{S}^{n-1} \subseteq \mathcal{H}^{d+2,n}/\mathbf{S}^{n-1}$.*

Proof:

If $(x_1, x_2, \ldots, x_n)$ is a point on $\mathbf{S}^{n-1}$, then $x_1^2+x_2^2+\cdots+x_n^2 = 1$ by definition. Therefore, if h is any poynomial from $\mathcal{H}^{d,n}$, then the polynomial

$$h'(x_1, x_2, \ldots, x_n) = h(x_1, x_2, \ldots, x_n)(x_1^2 + x_2^2 + \cdots + x_n^2)$$

is a homogeneous polynomial of degree $d+2$ wich coincides wich h on $\mathbf{S}^{n-1}$.
□

Lemma 2 *For any d and any $n \geq 1$, if $q \in \mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n}$ and $q \equiv 0 \pmod{\mathbf{S}^{n-1}}$ then $q = 0$.*

Proof:

Let q be a polynomial in $\mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n}$ with $q/\mathbf{S}^{n-1} = 0$. Since $\mathbf{S}^{n-1}$ is an algebric variety [6], the minimal equation of $\mathbf{S}^{n-1}$ must be a factor of q that is,

$$q = r \cdot (x_1^2 + x_2^2 + \ldots + x_n^2 - 1)$$

where r is some polynomial on $\mathbf{R}^n$ of degree $d-2$. Since the polynomial $x_1^2 + x_2^2 + \ldots + x_n^2 - 1$ has terms whose degrees differ by 2, and, on the other hand, q has only terms of degree d and $d-1$, we can conclude that $r = 0$, i.e. $q = 0$.
□

Corollary 3 *For any d and any $n \geq 1$, $\mathcal{H}^{d-1,n}/\mathbf{S}^{n-1} \cap \mathcal{H}^{d,n}/\mathbf{S}^{n-1} = \{0\}$*

Corollary 4 *For any d and any $n \geq 1$, if $p, q \in \mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n}$ and $p \equiv q \pmod{\mathbf{S}^{n-1}}$, then $p = q$.*

We can now give a characterization of the general spherical polynomials:

Theorem 5 *For any d and any $n \geq 1$, $\mathcal{P}^{d,n}/\mathbf{S}^{n-1} = (\mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n})/\mathbf{S}^{n-1} = \mathcal{H}^{d-1,n}/\mathbf{S}^{n-1} \oplus \mathcal{H}^{d,n}/\mathbf{S}^{n-1}$.*

Proof:

Since

$$\mathcal{P}^{d,n} = \mathcal{H}^{0,n} \oplus \mathcal{H}^{1,n} \oplus \ldots \oplus \mathcal{H}^{d,n}$$

we have

$$\mathcal{P}^{d,n}/\mathbf{S}^{n-1} = \mathcal{H}^{0,n}/\mathbf{S}^{n-1} + \mathcal{H}^{1,n})/\mathbf{S}^{n-1} + \ldots + \mathcal{H}^{d,n}/\mathbf{S}^{n-1}$$

by lemma 1, $\mathcal{H}^{0,n}/\mathbf{S}^{n-1} \subseteq \mathcal{H}^{2,n})/\mathbf{S}^{n-1} \ldots$ and $\mathcal{H}^{1,n}/\mathbf{S}^{n-1} \subseteq \mathcal{H}^{3,n})/\mathbf{S}^{n-1} \ldots$ Therefore

$$\mathcal{P}^{d,n}/\mathbf{S}^{n-1} = \mathcal{H}^{d-1,n}/\mathbf{S}^{n-1} \oplus \mathcal{H}^{d,n})/\mathbf{S}^{n-1}$$

By corollary 3, the theorem follows. □

As a consequence of theorem 5,

$$\dim((\mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n})/\mathbf{S}^{n-1}) = \dim(\mathcal{H}^{d-1,n} \oplus \mathcal{H}^{d,n}) = (d+1)^2$$

4 Derivatives of spherical polynomials

4.1 Spherical gradient

If f is a function from $\mathbf{S}^{n-1}$ to $\mathbf{R}$, we denote by $\overline{\nabla} f$ its gradient with respect to directions tangent to $\mathbf{S}^{n-1}$. For this paper, we can define $\overline{\nabla} f$ as a vector of $\mathbf{R}^n$, tangent to $\mathbf{S}^{n-1}$, such that the derivative of f at a point $u \in \mathbf{S}^{n-1}$, in the direction of a unit vector v tangent at u, is $v \cdot (\overline{\nabla} f(u))$.

If $f = F/\mathbf{S}^{n-1}$ for some differentiable function F from $\mathbf{R}^n$ to $\mathbf{R}$, then $\overline{\nabla} f$ turns out to be merely the projection of ∇F (the ordinary gradient of F) onto the sphere. That is, for any point $u \in \mathbf{S}^{n-1}$,

$$(\overline{\nabla} f)(u) = ((\nabla F)(u) - u((\nabla F)(u) \cdot u))/\mathbf{S}^{n-1} \tag{1}$$

Let's denote by $[v]_\alpha$ the αth component of a vector $v \in \mathbf{R}^n$.

Theorem 6 *For any d, any $n \geq 1$, and any $\alpha \in \{1, \ldots, n\}$, if f belongs to $\mathcal{H}^{d,n}/\mathbf{S}^{n-1}$, then $[\overline{\nabla} f]_\alpha \in \mathcal{H}^{d+1,n}/\mathbf{S}^{n-1}$*

Proof:

If f belongs to $\mathcal{H}^{d,n}/\mathbf{S}^{n-1}$, for $d \geq 1$, then $f = F/\mathbf{S}^{n-1}$ for some $F \in \mathcal{H}^{d,n}$. It is easy to check that $[\nabla F]_\alpha = \partial F/\partial x_\alpha$ is in $\mathcal{H}^{d-1,n}$. Therefore, the right-hand side of formula (1) lies in $\mathcal{H}^{d-1,n}/\mathbf{S}^{n-1} + \mathcal{H}^{d+1,n}/\mathbf{S}^{n-1}$. By lemma 1, $\mathcal{H}^{d-1,n}/\mathbf{S}^{n-1}$ is actually a subspace of $\mathcal{H}^{d+1,n}/\mathbf{S}^{n-1}$. □

Corollary 7 *For any d, any $n \geq 1$, and any $\alpha \in \{1, \ldots, n\}$, if f belongs to $\mathcal{P}^{d,n}$, then $[\overline{\nabla} f]_\alpha \in \mathcal{P}^{d+1,n}/\mathbf{S}^{n-1}$,*

4.2 Spherical harmonics

For the special case $n = 3$, there is a strong relationship between the spherical polynomials $\mathcal{P}^{d,3}/\mathbf{S}^2$ and the well-known *spherical harmonics* [4].

We denote by $\mathcal{Y}^d$ the space of the real spherical functions from $\mathbf{S}^2$ to $\mathbf{R}$ generated by the real and imaginary parts of the spherical harmonics Y_k^m of degree $0 \leq k \leq d$. We also denote by $\mathcal{W}^d$ the subspace of $\mathcal{Y}^d$ generated by the spherical harmonics of degree d, $d-2$, $\ldots$, $d - 2\lfloor \frac{d}{2} \rfloor$. Fausshauer and Schumaker [5] prove the following results:

Theorem 8 *The space $\mathcal{W}^d$ coincides with the space $\mathcal{H}^{d,3}/\mathbf{S}^2$.*

Theorem 9 *The space $\mathcal{Y}^d$ coincides with the space $\mathcal{P}^{d,3}/\mathbf{S}^2$.*

In appendix A we provide another direct proof of theorems 8 and 9.

5 Spherical splines

5.1 Triangulations of the sphere $\mathbf{S}^{n-1}$

A *simplicial cone of* $\mathbf{R}^n$ is a convex subset of $\mathbf{R}^n$, with non-empty interior, delimited by n hyperplanes that go through the origin. The intersection of a simplicial cone of $\mathbf{R}^n$ with the unit sphere $\mathbf{S}^{n-1}$ will be called a *spherical simplex.*

Let T be a decomposition of $\mathbf{R}^n$ into simplicial cones $T_1, T_2, \ldots, T_m$, with pairwise disjoint interiors. The collection T induces a *spherical simplicial subdivision decomposition* of $\mathbf{S}^{n-1}$ into spherical simplexes $T_i \cap \mathbf{S}^{n-1}$, which we denote by $T/\mathbf{S}^{n-1}$. For brevity, we will also use the terms *trihedral decomposition* for T, and *spherical triangulation* for $T/\mathbf{S}^{n-1}$, for any dimension n.

5.2 Spherical splines spaces

Given a trihedral decomposition T of $\mathbf{R}^n$, we define the following spaces of functions from $\mathbf{R}^n$ to $\mathbf{R}$:

$$\begin{aligned} \mathcal{P}^{d,n}[T] &= \{\, p : (\forall i)\ p/T_i \in \mathcal{P}^{d,n}/T_i \,\} \\ \mathcal{H}^{d,n}[T] &= \{\, h : (\forall i)\ h/T_i \in \mathcal{H}^{d,n}/T_i \,\} \end{aligned}$$

The restriction of these functions to the sphere $\mathbf{S}^{n-1}$ gives the *spherical splines* $\mathcal{P}^{d,n}[T]/\mathbf{S}^{n-1}$ and $\mathcal{H}^{d,n}[T]/\mathbf{S}^{n-1}$, respectively *general* and *homogeneous.*

5.3 Characterization of spherical splines

Theorem 5 can be easily extended to spherical splines. First, we need a few lemmas:

Lemma 10 *For all $n \geq 1$, $\mathcal{P}^{d,n}[T]/\mathbf{S}^{n-1} = \mathcal{H}^{d-1,n}[T]/\mathbf{S}^{n-1} + \mathcal{H}^{d,n}[T]/\mathbf{S}^{n-1}$*

This result follows directly from the definition and from theorem 5. Moreover, we have

Lemma 11 *Let $P \in \mathcal{P}^{d,n}$ for $n \geq 2$, and W a non-empty subset of $\mathbf{S}^{n-1}$ with dimension $n-1$. If P vanishes in W, then P vanishes in all $\mathbf{S}^{n-1}$.*

Proof:

If $n \geq 2$ then $\mathbf{S}^{n-1}$ is a irreducible variety of $\mathbf{R}^n$, and the thesis follows from a classic result of algebric geometry [6]. □

Lemma 12 *If $p \in \mathcal{H}^{d-1,n}[T] + \mathcal{H}^{d,n}[T]$ with $n \geq 2$, and $p \equiv 0 \pmod{\mathbf{S}^{n-1}}$ then $p = 0$.*

Proof:

Let $p \in \mathcal{H}^{d-1,n}[T] + \mathcal{H}^{d,n}[T]$ such that $n \geq 2$ and $p \equiv 0 \pmod{\mathbf{S}^{n-1}}$. For all $i \in \{1 \ldots k\}$, let $p_i \in \mathcal{H}^{d-1,n} + \mathcal{H}^{d,n}$ such that $p/T_i = p_i/T_i$. Then, we have

$$p_i/(\mathbf{S}^{n-1} \cap T_i) = 0$$

Since $T_i \cap \mathbf{S}^{n-1}$ is a subset of $\mathbf{S}^{n-1}$ with dimension $n-1$, by lemma 11 we can conclude that p_i is zero over the whole sphere $\mathbf{S}^{n-1}$. By theorem 5 p_i is zero over the whole domain $\mathbf{R}^n$. Since this equality is true for all T_i, we conclude that the composite function p is identically zero, too. □

This lemma has the following consequences:

Corollary 13 *If $p, q \in \mathcal{H}^{d-1,n}[T] + \mathcal{H}^{d,n}[T]$ and $n \geq 2$, then $p \equiv q \pmod{\mathbf{S}^{n-1}}$ if and only if $p = q$.*

Corollary 14 *For any $n \geq 2$, $\mathcal{H}^{d-1,n}[T]/\mathbf{S}^{n-1} \cap \mathcal{H}^{d,n}[T]/\mathbf{S}^{n-1} = \{0\}/\mathbf{S}^{n-1}$*

Corollary 15 *For any $n \geq 2$, $\mathcal{P}^{d,n}[T]/\mathbf{S}^{n-1} = \mathcal{H}^{d-1,n}[T]/\mathbf{S}^{n-1} \oplus \mathcal{H}^{d,n}[T]/\mathbf{S}^{n-1}$*

It should be noted that lemma 12 does not hold for $n = 1$; the proof fails, in this case, because $\mathbf{S}^0 = \{-1, 1\}$ is not irreducible. Indeed, the function

$$p(x) = \begin{cases} 1 - x & \text{if } x \geq 0 \\ 1 + x & \text{if } x \leq 0 \end{cases}$$

which belongs to $\mathcal{H}^{0,1} + \mathcal{H}^{1,1}$, is zero on $\mathbf{S}^0$ but not on $\mathbf{R}^1$.

5.4 Continuity Constraints

Finally, we extend Corollary 15 to spherical splines which are subject to continuity constraints.

We say that a function from $\mathbf{S}^{n-1}$ to $\mathbf{R}$ is *continuous to order zero* if it is continuous in the ordinary sense; and is *continuous to order* k, for $k > 0$, if it is continuous, differentiable, and each component of its spherical gradient is continuous to order $k-1$. We denote by $\mathbf{C}_k(\mathbf{S}^{n-1})$ the set of all functions from $\mathbf{S}^{n-1}$ to $\mathbf{R}$ that are continuous to order k.

For a trihedral decomposition T of $\mathbf{R}^n$ we define the function spaces

$$\begin{aligned}\mathcal{P}_k^{d,n}[T]/\mathbf{S}^{n-1} &= \{\, p : p \in \mathcal{P}^{d,n}[T]/\mathbf{S}^{n-1} \ \wedge\ p/\mathbf{S}^{n-1} \in \mathbf{C}_k(\mathbf{S}^{n-1}) \,\} \\ \mathcal{H}_k^{d,n}[T]/\mathbf{S}^{n-1} &= \{\, h : h \in \mathcal{H}^{d,n}[T]/\mathbf{S}^{n-1} \ \wedge\ h/\mathbf{S}^{n-1} \in \mathbf{C}_k(\mathbf{S}^{n-1}) \,\}\end{aligned}$$

Our goal is to show that $\mathcal{P}_k^{d,n}[T]/\mathbf{S}^{n-1}$ is the direct sum of $\mathcal{H}_k^{d-1,n}[T]/\mathbf{S}^{n-1}$ and $\mathcal{H}_k^{d,n}[T]/\mathbf{S}^{n-1}$. In other words, imposing kth-order continuity on $\mathcal{P}^{d,n}[T]/\mathbf{S}^{n-1}$ is equivalent to independently imposing kth-order continuity on each of the two subspaces $\mathcal{H}^{d-1,n}[T]/\mathbf{S}^{n-1}$ and $\mathcal{H}^{d,n}[T]/\mathbf{S}^{n-1}$. For that we need the following results:
To prove the main result of this section, we start with case $k = 0$, namely a characterization of the space $\mathcal{P}_0^{d,n}[T]/\mathbf{S}^{n-1}$:

Theorem 16 *If* $n \geq 3$, *then* $\mathcal{P}_0^{d,n}[T]/\mathbf{S}^{n-1} = \mathcal{H}_0^{d-1,n}[T]/\mathbf{S}^{n-1} \oplus \mathcal{H}_0^{d,n}[T]/\mathbf{S}^{n-1}$.

Proof:

($\supseteq$): Trivial.

($\subseteq$): Let p be a function in $\mathcal{P}_0^{d,n}[T]/\mathbf{S}^{n-1}$. Let T_i and T_j be adjacent cones of T, and let p_i and p_j be functions of $\mathcal{P}^{d,n}/\mathbf{S}^{n-1}$ such that $p/T_i = p_i/T_i$ and $p/T_j = p_j/T_j$. By corollary 10,

$$p = h' + h''$$

with $h' \in \mathcal{H}^{d-1,n}[T]/\mathbf{S}^{n-1}$ and $h'' \in \mathcal{H}^{d,n}[T]/\mathbf{S}^{n-1}$. Moreover,

$$p_i = h_i' + h_i'' \qquad p_j = h_j' + h_j''$$

where h_i' and h_j' are in $\mathcal{H}^{d-1,n}/\mathbf{S}^{n-1}$, and h_i'' and h_j'' are $\mathcal{H}^{d,n}/\mathbf{S}^{n-1}$.

Let W be the common boundary of the spherical triangles $T_i \cap \mathbf{S}^{n-1}$ and $T_j \cap \mathbf{S}^{n-1}$. We can assume, without loss of generality, that W is contained in the hyperplane π with equation $x_n = 0$. Let C be the sphere $\mathbf{S}^{n-2}$ contained in π, defined by the equation $x_1^2 + x_2^2 + \ldots + x_{n-1}^2 = 1$. Since $p \in \mathbf{C}_0(\mathbf{S}^{n-1})$, we have $p/W = p_i/W = p_j/W$, and therefore $(p_i - p_j)/W = 0$. Given that W is a subset of $\mathbf{S}^{n-2}$ with dimension $n-2$, by lemma 11 we conclude that $(p_i - p_j)/C = 0$.

Note that $(p_i - p_j)/C$ belongs to $\mathcal{P}^{d,n-1}/\mathbf{S}^{n-2}$. On the other hand, $(p_i - p_j)/C = (h_i' - h_j')/C + (h_i'' - h_j'')/C$. Since $(h_i' - h_j')/C \in \mathcal{H}^{d-1,n-1}/\mathbf{S}^{n-2}$, and $(h_i'' - h_j'')/C \in \mathcal{H}^{d,n-1}/\mathbf{S}^{n-2}$, by theorem 5

$$(h_i' - h_j')/C = 0 \qquad (h_i'' - h_j'')/C = 0$$

Therefore,

$$h_i^{'}/C = h_j^{'}/C \qquad h_i^{''}/C = h_j^{''}/C$$

Since these identities hold for any two adjacent trihedra of T, we conclude that $h^{'} \in \mathcal{H}_0^{d-1,n}[T]/\mathbf{S}^{n-1}$ and $h^{''} \in \mathcal{H}_0^{d,n}[T]/\mathbf{S}^{n-1}$. □

Let's now prove the general case:

Theorem 17 *For $n \geq 3$ and any $k \geq 0$,*

$$\mathcal{P}_k^{d,n}[T]/\mathbf{S}^{n-1} = \mathcal{H}_k^{d-1,n}[T]/\mathbf{S}^{n-1} \oplus \mathcal{H}_k^{d,n}[T]/\mathbf{S}^{n-1}$$

Proof:

($\supseteq$): Trivial.

($\subseteq$): We prove this part by induction on k. The case $k = 0$ is theorem 16, so let's assume $k > 0$.

Let p be a function in $\mathcal{P}_k^{d,n}[T]/\mathbf{S}^{n-1}$. By definition, p is continuous, and ∇p is continuous of order $k-1$. By corollary 7, $[\nabla p]_\alpha$ belongs to $\mathcal{P}^{d+1,n}[T]/\mathbf{S}^{n-1}$, and therefore to $\mathcal{P}_{k-1}^{d+1,n}[T]/\mathbf{S}^{n-1}$. By induction,

$$[\nabla p]_\alpha \in \mathcal{H}_{k-1}^{d,n}[T]/\mathbf{S}^{n-1} \oplus \mathcal{H}_{k-1}^{d+1,n}[T]/\mathbf{S}^{n-1} \qquad (2)$$

On the other hand, by corollary 10, $p = h^{'} + h^{''}$, where $h^{'} \in \mathcal{H}^{d-1,n}[T]/\mathbf{S}^{n-1}$ and $h^{''} \in \mathcal{H}^{d,n}[T]/\mathbf{S}^{n-1}$. In the interior of each triangle of T, the spherical gradient of p is then $\nabla p = \nabla h^{'} + \nabla h^{''}$ By theorem 6,

$$\begin{aligned} [\nabla h^{'}]_\alpha &\in \mathcal{H}^{d,n}[T]/\mathbf{S}^{n-1} && (3) \\ [\nabla h^{''}]_\alpha &\in \mathcal{H}^{d+1,n}[T]/\mathbf{S}^{n-1} && (4) \end{aligned}$$

Comparing equation (2) with equations (3) and (4), we conclude that

$$\begin{aligned} [\nabla h^{'}]_\alpha &\in \mathcal{H}_{k-1}^{d,n}[T]/\mathbf{S}^{n-1} && (5) \\ [\nabla h^{''}]_\alpha &\in \mathcal{H}_{k-1}^{d+1,n}[T]/\mathbf{S}^{n-1} && (6) \end{aligned}$$

Since p is continuous, theorem 16 implies that $h^{'}$ and $h^{''}$ are continuous, too. Equations (5-6) imply that $h^{'} \in \mathcal{H}_k^{d-1,n}[T]/\mathbf{S}^{n-1}$ and $h^{''} \in \mathcal{H}_k^{d,n}[T]/\mathbf{S}^{n-1}$. We conclude that

$$\mathcal{P}_k^{d,n}[T]/\mathbf{S}^{n-1} \subseteq \mathcal{H}_k^{d-1,n}[T]/\mathbf{S}^{n-1} + \mathcal{H}_k^{d-1,n}[T]/\mathbf{S}^{n-1} \qquad (7)$$

On the other hand,

$$\begin{aligned} (\mathcal{H}_k^{d-1,n}[T]/\mathbf{S}^{n-1}) \cap (\mathcal{H}_k^{d-1,n}[T]/\mathbf{S}^{n-1}) &\subseteq && (8) \\ (\mathcal{H}^{d-1,n}[T]/\mathbf{S}^{n-1}) \cap (\mathcal{H}^{d-1,n}[T]/\mathbf{S}^{n-1}) &= \{0\}/\mathbf{S}^{n-1} && (9) \end{aligned}$$

by corollary 3. □

Note that since lemma 12 fails for $n = 1$, theorem 17 cannot be extended to $n = 2$. For instance, the function $p(x, y) = p'(x, y) + p''(x, y)$, where

$$P'(x,y) = \begin{cases} y & if \quad x > 0, y > 0 \\ 1 & if \quad x < 0, y > 0 \\ -x & if \quad x < 0, y < 0 \\ 0 & if \quad x > 0, y < 0 \end{cases} \qquad P''(x,y) = \begin{cases} 0 & if \quad x > 0, y > 0 \\ 1 & if \quad x < 0, y > 0 \\ 0 & if \quad x < 0, y < 0 \\ 0 & if \quad x > 0, y < 0 \end{cases}$$

Observe that p is $\mathbf{C}_0$ on $\mathbf{S}^1$, but its homogeneous components p', p'' aren't.

6 Bases for Spherical Splines

Let's now turn our attention to the sphere $\mathbf{S}^2$. The results of the previous section show that $\mathcal{P}_k^{d,3}[T]/\mathbf{S}^2$, the space of piecewise polynomial functions restricted to the sphere with order-k continuity, is the direct sum of the spaces $\mathcal{H}_k^{d,3}[T]$ and $\mathcal{H}_k^{d-1,3}[T]$, restricted to $\mathbf{S}^2$.

Alfeld, Neamtu and Schumaker [2] obtained an explicit basis, with local support, for the space $\mathcal{H}_k^{d,3}[T]/\mathbf{S}^2$, in terms of Bernstein-Bézier polynomials for $d \geq 3k + 2$. In view of theorem 17, their construction also gives a basis for $\mathcal{P}_k^{d,3}[T]/\mathbf{S}^2$ when $d \geq 3k + 3$, through the concatenation of a basis of $\mathcal{H}_k^{d,3}[T]/\mathbf{S}^2$ and a basis of $\mathcal{H}_k^{d-1,3}[T]/\mathbf{S}^2$.

For $k = 0$ or $k = 1$ (wich are the cases most likely to be used in pratice) the dimension of the spaces are

$$\begin{aligned} \dim \mathcal{H}_k^{d,3}[T]/\mathbf{S}^2 &= (d^2 - 3dk + 2k^2)v - 2d^2 + 6dk - 3k^2 + 3k + 2 \\ &= (d^2 - 3dk + 2k^2)t/2 + k^2 + 3k + 2, \end{aligned}$$

and

$$\begin{aligned} \dim \mathcal{P}_k^{d,3}[T] &= (2d^2 - 6dk + 4k^2 - 2d + 3k + 1)v \\ &\quad -4d^2 + 12dk + 4d - 6k^2 + 2 \\ &= (2d^2 - 6dk + 4k^2 - 2d + 3k + 1)t/2 + 2k^2 + 6k + 4. \end{aligned}$$

where v and t are respectively the number of vertices and triangles of the triangulation.

The lowest-degree spaces $\mathcal{P}_k^{d,3}[T]/\mathbf{S}^2$ that have ANS bases are $\mathcal{P}_0^{3,3}[T]/\mathbf{S}^2$ for continuity class $\mathbf{C}_0$, and $\mathcal{P}_1^{6,3}[T]/\mathbf{S}^2$ for continuity class $\mathbf{C}_1$. Table 1 gives the dimensions of those spaces and of their homogeneous components.

Spaces	*Dimensions*	
$\mathcal{H}_0^{2,3}[T]/\mathbf{S}^2$	$4v - 6$	$2t + 2$
$\mathcal{H}_0^{3,3}[T]/\mathbf{S}^2$	$9v - 16$	$9t/2 + 2$
$\mathcal{P}_0^{3,3}[T]/\mathbf{S}^2$	$13v - 22$	$13t/2 + 4$

Spaces	*Dimensions*	
$\mathcal{H}_1^{5,3}[T]/\mathbf{S}^2$	$12v - 18$	$6t + 6$
$\mathcal{H}_1^{6,3}[T]/\mathbf{S}^2$	$20v - 34$	$10t + 6$
$\mathcal{P}_1^{6,3}[T]/\mathbf{S}^2$	$32v - 52$	$16t + 12$

Table 1: The dimensions of some general and homogeneous spline spaces with $\mathbf{C}_0$ and $\mathbf{C}_1$ continuity.

Each element of an ANS basis is associated with a face, an edge, or a vertex of the triangulation. Figure 1 shows some ANS basis elements for the space $\mathcal{H}_1^{6,3}[T]/\mathbf{S}^2$, where T is the central projection of a regular icosahedron onto the sphere.

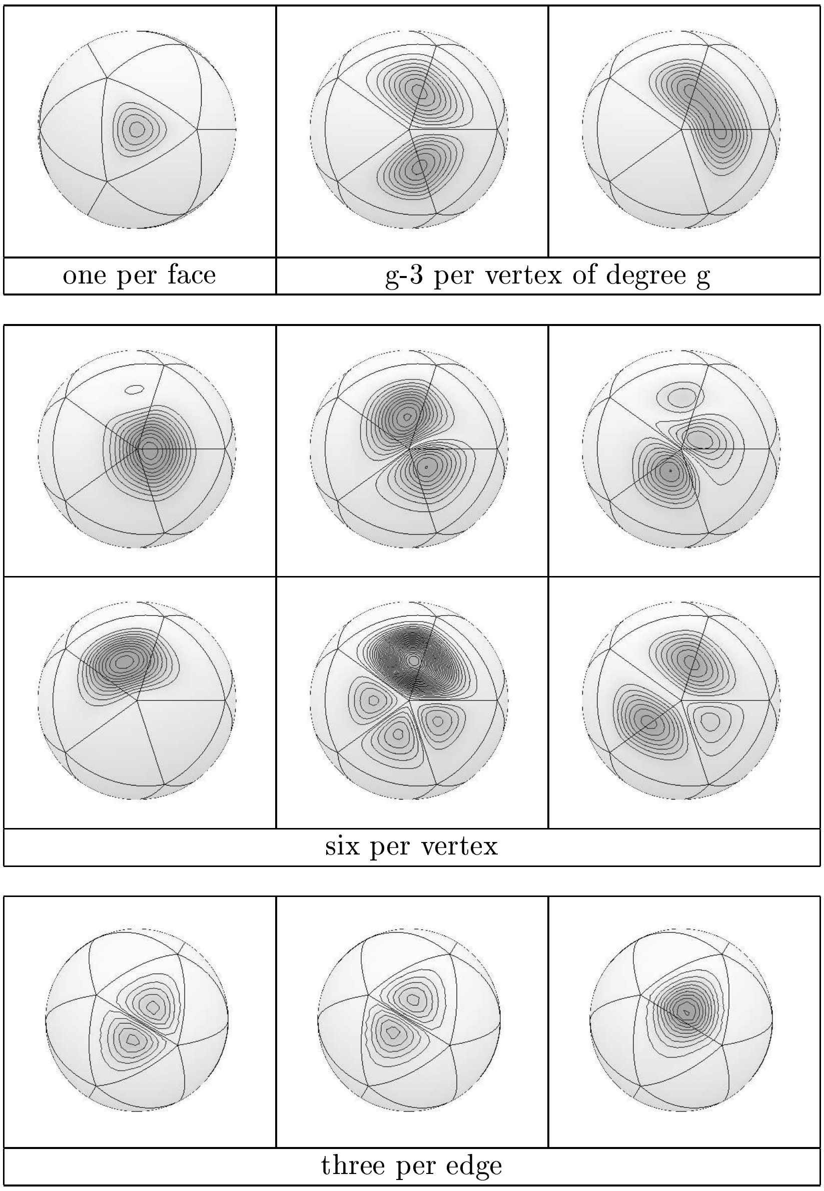

Figure 1: Typical Alfeld-Neamtu-Schumaker basis elements.

7 Conclusion

The general spherical spline space $\mathcal{P}_k^{d,n}[T]/\mathbf{S}^{n-1}$ which we defined here include the homogeneous splines $\mathcal{H}_k^{d,n}[T]/\mathbf{S}^{n-1}$ as a proper subspace; indeed, as we have shown, $\mathcal{P}_k^{d,n}[T]/\mathbf{S}^{n-1}$ is the direct sum of the corresponding of degree d and $d-1$, for all continuity order k.Therefore, for the particular case $n = 3$, and $d \geq 3k + 3$ we obtain an explicit basis for $\mathcal{P}_k^{d,n}[T]/\mathbf{S}^{n-1}$ by concatenating the bases for $\mathcal{H}_k^{d-1,n}[T]/\mathbf{S}^{n-1}$ and $\mathcal{H}_k^{d,n}[T]/\mathbf{S}^{n-1}$, as constructed by Alfeld Neamtu Schumaker [2].

In light of theorems 8 and 9, it is obvious that the space $\mathcal{P}_k^{d,3}[T]/\mathbf{S}^2$ can approximate exactly any spherical harmonic function of degree d. The same is not true of the space $\mathcal{H}_k^{d,3}[T]/\mathbf{S}^2$, which is the subspace of $\mathcal{P}_k^{d,3}[T]/\mathbf{S}^2$ generated by the spherical harmonics whose degrees have the same parity as d. For this reason, we believe that the space $\mathcal{P}_k^{d,3}[T]/\mathbf{S}^2$ is a better choice than $\mathcal{H}_k^{d,3}[T]/\mathbf{S}^2$ for function approximation on the sphere. For one thing, $\mathcal{P}_k^{r,3}[T]/\mathbf{S}^2 \subseteq \mathcal{P}_k^{d,3}[T]/\mathbf{S}^2$ when $r \leq d$, while $\mathcal{H}_k^{r,3}[T]/\mathbf{S}^2 \subseteq \mathcal{H}_k^{d,3}[T]/\mathbf{S}^2$ only when $d - r$ is even. In particular, $\mathcal{P}_k^{d,3}[T]/\mathbf{S}^2$ includes the functions which are constant on $\mathbf{S}^2$, for all d; whereas $\mathcal{H}_k^{d,3}[T]/\mathbf{S}^2$ only contains such functions when d is even.

These remains a host of practical problems to solve, such as determining the asymptotic approximation power of these spaces, and efficient approximation, interpolation methods based on them.

Appendix A

Theorem 18 *The space $\mathcal{W}^d$ coincides with the space $\mathcal{H}^{d,3}/\mathbf{S}^2$.*

Proof:

We will first show that the real and imaginary parts of any spherical harmonic Y_d^m of degree d is an element of $\mathcal{P}^{d,3}/\mathbf{S}^2$. We know that Y_d^m can be written as

$$Y_d^m = e^{\mathbf{i}m\phi} P_d^m(\theta), \qquad m = -d, \ldots, d \tag{10}$$

where $P_d^m(\theta)$ is the Legendre functions of the first kind of order m and degree d [4]. A classical analytical expression for P_d^m is Rodrigues's [7] formula.

$$P_d^m(\theta) = \frac{(1-z^2)^{|m|/2}}{2^d d!} \frac{\partial^{d+|m|}(1-z^2)^d}{\partial z^{d+|m|}} \tag{11}$$

where $z = \sin\theta$. Therefore

$$Y_d^m = C(d,m) e^{\mathbf{i}m\phi}(1-z^2)^{|m|/2} \frac{\partial^{d+|m|}(1-z^2)^d}{\partial z^{d+|m|}}$$

where $C(d,m)$ is a constant that depends on m and d. Let's first assume $m \geq 0$. Recalling that

$$e^{\mathbf{i}\phi} = \cos\phi + \mathbf{i}\sin\phi = \frac{x + \mathbf{i}y}{\sqrt{x^2+y^2}} = \frac{x+\mathbf{i}y}{(1-z^2)^{1/2}}$$

we have

$$(e^{\mathbf{i}\phi})^m = \frac{(x+\mathbf{i}y)^m}{(1-z^2)^{m/2}}$$

therefore

$$Y_d^m = C(d,m)(x + \mathbf{i}y)^m \frac{\partial^{d+m}(1-z^2)^d}{\partial z^{d+m}}$$

We now observe that $(1-z^2)^d = R(z)$ is a polynomial in the variable z, all of whose terms have degree between 0 and $2d$. It is easy to see that

$$\frac{\partial^{d+m}(1-z^2)^d}{\partial z^{d+m}}$$

is a polynomial $Q(z)$, in the variable z, all of whose terms have degree between 0 and $d-m$ inclusive, with the same parity as $d-m$. Therefore $Q(z) \in \mathcal{H}^{d-m,3}/\mathbf{S}^2 + \mathcal{H}^{d-m-2,3}/\mathbf{S}^2 + \ldots$

Since $\mathcal{H}^{d-2,3} \subseteq \mathcal{H}^{d,3}$, the restriction of $Q(z)$ to the sphere $\mathbf{S}^2$ is an element of $\mathcal{H}^{d-m,3}/\mathbf{S}^2$. Therefore, since

$$Y_d^m = C(d,m)(x + \mathbf{i}y)^m Q(z)/\mathbf{S}^2$$

and $Re[(x+\mathbf{i}y)^m]$ and $Im[(x+\mathbf{i}y)^m]$ are homogeneous polynomials in x e y of degree m, we conclude that $Re(Y_d^m)$ and $Im(Y_d^m)$ are also elements of $\mathcal{H}^{d,3}/\mathbf{S}^2$.

Since $Y_d^{-m} = (Y_d^m)^*$ [7],the result is also true for $m < 0$. It then follows that $\mathcal{W}^d \subseteq \mathcal{H}^{d,3}/\mathbf{S}^2$.

We will now demonstrate that both spaces have the same dimension. Notice that for each k, the functions $Re[Y_k^m]$, $m = 0 \ldots k$ and $Im[Y_k^m]$, $m = 1 \ldots k$ are linearly independent [4]. Observe also that for $m < 0$, $Re[Y_k^{-m}] = Re[Y_k^m]$ and $Im[Y_k^{-m}] = -Im[Y_k^m]$. Therefore, the dimension of the space $\mathcal{W}^d$ is at least $\binom{d+2}{2}$. As shown in this section 2 $\dim \mathcal{H}^{d,3}/\mathbf{S}^2$ is also $\binom{d+2}{2}$. □

Theorem 19 $\mathcal{Y}^d = \mathcal{P}^{d,3}/\mathbf{S}^2$

Proof:

$\mathcal{Y}^d = \mathcal{W}^0 + \mathcal{W}^1 + \mathcal{W}^2 + \ldots + \mathcal{W}^d = \mathcal{H}^{0,3}/\mathbf{S}^2 + \mathcal{H}^{1,3}/\mathbf{S}^2 + \ldots + \mathcal{H}^{d,3}/\mathbf{S}^2 = \mathcal{H}^{d-1,3}/\mathbf{S}^2 \oplus \mathcal{H}^{d,3}/\mathbf{S}^2 = \mathcal{P}^{d,3}/\mathbf{S}^2$ □

8 Acknowledgements

This research was partly funded by CNPq grant 301016/92-5, by FAPESP grants 93/0655-1, and by PRONEX/CNPq project n.76.97.1022.00

References

[1] Alfeld, P., Neamtu, M., and Schumaker, L.. Bernstein-Bézier polynomials on circle, spheres, and sphere-like surfaces. *Computer Aided Geometric Design Journal*, 13:333–349, 1996.

[2] Alfeld, P., Neamtu, M., and Schumaker, L.. Dimension and local bases of homogeneous spline spaces. *SIAM Journal of Mathematical Analysis*, 27(5):1482–1501, September 1996.

[3] Alfeld, P., Neamtu, M., and Schumaker, L.. Fitting scattered data on sphere-like surfaces using spherical splines. *Journal of Computational and Applied Mathematics*, 73:5–43, 1996.

[4] Courant, R, and Hilbert, D., *Methods of Mathematical Physics*, vol.I. Interscience, New York, 1953.

[5] Fausshauer, G. E. and M. and Schumaker, L. L. *Scattered data fitting on the sphere.* Mathematical Methods for Curves and Sufaces, pp. 117–166 Vanderbilt University Press, Nashville ,TN 1998.

[6] Fulton, W. *Algebraic Curves: An Introduction to Algebraic Geometry.* W. A. Benjamin, 1969.

[7] Jarraud, M. and Baede, A. P. M. *The use of spectral techniques in numerical weather prediction.* Lectures in Applied Mathematics, Vol 22, pp. 1-41, 1985.

[8] Kaderstuncer H. and Norrier D. H. *Finite Element Handbook.* McGraw-Hill, 1987.

[9] Spivak, M. *Calculus on Manifolds - A Modern Approach to Classical Theorems of Advanced Calculus.* W. A. Benjamin, 1965.