On a Conjecture of Lovász Concerning Bricks

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Abstract

In 1987, Lovász conjectured that every brick G different from K_4, $\overline{C}_6$, and the Petersen graph has an edge e such that $G-e$ is a matching covered graph with exactly one brick. Lovász and Vempala announced a proof of this conjecture in 1994. Their paper is under preparation. We present here an independent proof of their theorem. We shall in fact prove that if G is any brick different from K_4 and $\overline{C}_6$ and does not have the Petersen graph as its underlying simple graph, then it has an edge e such that $G-e$ is a matching covered graph with exactly one brick, with the additional property that the underlying simple graph of that one brick is different from the Petersen graph. Our proof involves establishing an interesting new property of the Petersen graph.

1 Introduction

The study of matching covered graphs originated in the works of Kotzig and Lovász. It developed into a beautiful theory, mainly through the efforts of Lovász and his co-workers. The matching lattice of a matching covered graph is the lattice generated by the set of incidence vectors of perfect matchings of the graph. The crowning achievement of this theory is the characterization of the matching lattice by Lovász in 1987 [6].

Motivated by his work on the matching lattice, Lovász proposed the following conjecture in 1987:

Conjecture 1: *Every brick different from K_4, $\overline{C}_6$, and the Petersen graph has an edge whose deletion yields a matching covered graph with exactly one brick.*

He noted that the proof of his theorem which characterizes the lattice could be simplified significantly if the above conjecture were true. Carvalho and Lucchesi, in their attempts

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to find ear decompositions of matching covered graphs with the least possible number of double ears, were led to the following conjecture in 1993:

Conjecture 2: *Every brick different from K_4 and $\overline{C}_6$ and not having the Petersen graph as its underlying simple graph has an edge whose deletion yields a matching covered graph with exactly one brick, with the additional property that the underlying simple graph of that one brick is different from the Petersen graph.*

Clearly, Conjecture 2 implies Conjecture 1. But the two conjectures in fact turn out to be equivalent. In 1994, Lovász and Vempala announced ([7]) a proof of Conjecture 1. Their paper is under preparation. Marcelo Carvalho's Ph.D. thesis [1], written under the supervision of Cláudio Lucchesi, and submitted to the University of Campinas, Brasil, in December 1996, contains a proof of Conjecture 2. In this paper, we present a revised version of that proof.

This new proof relies on establishing an interesting new property of the Petersen graph, which in turn involves the notion of the characteristic of a brick. An odd cut C of a brick G is a *separating* cut of the brick if the two C-contractions of G are matching covered. If C is nontrivial, then, since bricks do not have any nontrivial tight cuts, there must exist at least one perfect matching of G which meets C in more than one edge, and we define the *characteristic* of C to be the smallest odd integer $\lambda(C)$ such that $\lambda(C) = \min|M \cap C|$, where the minimum is taken over all perfect matchings M of G which meet C in more than one edge. Not all bricks have nontrivial separating cuts; we call such bricks *solid* bricks. For example, odd wheels are solid bricks. If G is a nonsolid brick, then the *characteristic* $\lambda(G)$ of G is the smallest odd integer for which there is a nontrivial separating cut C of G with $\lambda(C) = \lambda(G)$. For example, the characteristic of the triangular prism is three and the characteristic of the Petersen graph is five. The property of the Petersen graph that we establish, and which forms an integral part of the proof of the Main Theorem, is that **the only nonsolid simple brick of characteristic greater than three is the Petersen graph**. We give a precise statement of the Main Theorem in section 3, after reviewing the basic notions of the subject in section 2.

Carvalho has used the Main Theorem in his thesis to provide answers to several long standing problems. For example, for each matching covered graph, he has determined the minimum number of double ears needed in an ear decomposition of the graph and has established the existence of a basis for its matching lattice consisting of incidence vectors of perfect matchings. These results will be presented in a separate paper [4]. Various other results from Carvalho's thesis pertaining to matching covered graphs appear in a paper [3] which has been submitted for publication.

2 Basic Notions

The graphs we consider in this paper may have multiple edges, but no loops. An edge $e = uv$ in a graph G is a *multiple edge* if G has more than one edge joining u and v. In this section, we review briefly some of the important notions used in this paper. For a history of the theory of matching covered graphs, and for notation and terminology not defined here, we refer the reader to [5], [6], and [8]. All the notions in this section, except for the notion of a separating cut and its characteristic, appear in the seminal work of Lovász [5].

2.1 Theorems of Hall and Tutte

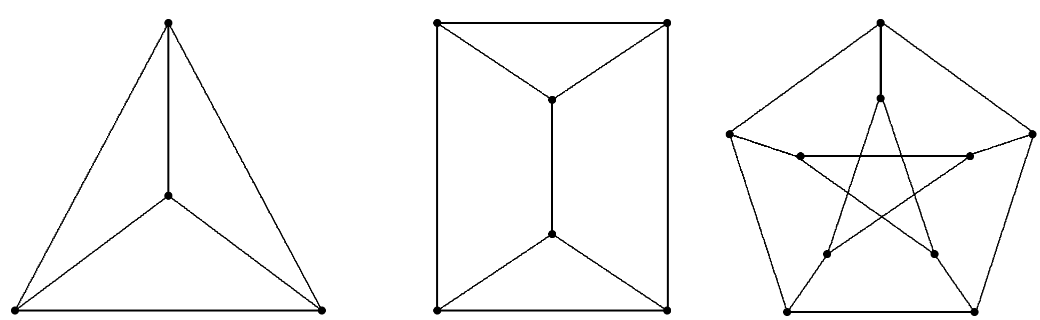
The basic problem in matching theory is the determination of necessary and sufficient conditions for a graph to have a perfect matching. This problem was solved for bipartite graphs by Hall in 1935 and, in general, by Tutte in 1947. We state below the well-known theorems of Hall and Tutte in the notation that is used in this paper.

Theorem 2.1 (Hall, See [5]) *A graph G with bipartition (A, B) has a perfect matching if, and only if, $|A| = |B|$ and, for every subset X of B, $|I(G - X)| \leq |X|$, where $I(G - X)$ denotes the set of isolated vertices of $G - X$.*

Theorem 2.2 (Tutte, see [5]) *A graph G has a perfect matching if, and only if, for every subset X of V, $C_{odd}(G-X) \leq |X|$, where $C_{odd}(G-X)$ denotes the number of odd components of $G - X$.*

2.2 Matching Covered Graphs

An edge e in a graph G is *admissible* in G, if there is some perfect matching M of G such that $e \in M$. A nontrivial connected graph in which every edge is admissible is called a *matching covered graph*. Using Tutte's theorem, it is possible to show that every 2-connected cubic graph is a matching covered graph. Three cubic graphs, K_4, $\overline{C}_6$, and the Petersen graph, see Figure 2.2, play special roles in this theory.



K_4 $\overline{C_6}$ The Petersen Graph

Figure 1: Three important bricks

From Hall's theorem it is easy to deduce the following characterization of bipartite matching covered graphs:

Theorem 2.3 *A bipartite graph G with bipartition (A, B) that has a perfect matching is matching covered if, and only if, for every nontrivial partition (A', A'') of A and every partition (B', B'') of B such that $|A'| = |B'|$, at least one edge of G joins some vertex of A' to some vertex of B'' and at least one edge of G joins some vertex of A'' to some vertex of B'.*

2.3 Barriers

Let G be a graph with a perfect matching. Then, a subset B of V is a *barrier* of G if $C_{odd}(G - B) = |B|$. Clearly, the empty subset of V is a barrier of G. But, henceforth, by a barrier we shall mean a nonnull barrier. All singleton subsets of V are barriers of G. We shall refer to such barriers as *trivial* barriers.

The following theorem characterizes the set of admissible edges in a graph with a perfect matching.

Theorem 2.4 (See [8]) *Let G be a graph which has a perfect matching. Then an edge e of G is admissible if, and only if, there there is no barrier which contains both the ends of e.*

The following theorem characterizes the maximal barriers of a matching covered graph.

Theorem 2.5 (See [8]) *Let G be a matching covered graph, and let $\sim$ denote the binary relation on V where $u \sim v$ if $G - \{u, v\}$ has no perfect matching. Then, the relation $\sim$ is an equivalence relation on V and the equivalence classes are precisely the maximal barriers of G.*

The partition of V into maximal barriers is called the *canonical partition* of G.

A graph G is *critical* if, for any vertex v of G, the subgraph $G - v$ has a perfect matching. The following theorem can be proved using Tutte's theorem.

Theorem 2.6 *Let G be a matching covered graph, and let B be a maximal barrier of G. Then all components of $G - B$ are critical.*

A matching covered graph G is *bicritical* if, for any $u, v \in V$, $u \neq v$, the graph $G - \{u, v\}$ has a perfect matching. Equivalently, a matching covered graph G is bicritical if all its maximal barriers are singletons.

2.4 Cuts and Cut-Contractions

Let G be a graph. Then, for any subset S of V, $C = \nabla_G(S)$ (or simply $C = \nabla(S)$) denotes the (edge-) cut of G with S and $\overline{S} = V \setminus S$ as its *shores*; in other words, $\nabla(S)$ is the set of all edges of G which have precisely one end in S. Then, the graph obtained from G by contracting $\overline{S}$ to a single vertex $\overline{s}$ is denoted by $G\{S; \overline{s}\}$ and the graph obtained from G by contracting S to a single vertex s is denoted by $G\{\overline{S}; s\}$. We shall refer to these two graphs $G\{S; \overline{s}\}$ and $G\{\overline{S}; s\}$ as the *C-contractions* of G. If the names of the new vertices in the

C-contractions are irrelevant, we shall simply denote them by $G\{S\}$ and $G\{\overline{S}\}$. Observe that this notation is similar to the notation $G[S]$ used for the subgraph of G induced by S; $G\{S;\overline{s}\}$ is the subgraph induced by S, together with a new vertex $\overline{s}$ such that each edge in $\nabla_G(S)$ joins its end in S to the vertex $\overline{s}$.

A cut C is *trivial* if either of its shores is a singleton. A cut C is *odd* (*even*) if both its shores have odd (even) cardinality. If V is even, then every cut C is either odd or even, and if C is odd (even), then $|C \cap M|$ is odd (even) for every perfect matching M of G.

2.5 Tight Cuts

Let G be a matching covered graph. Then, a cut $C = \nabla(S)$ is a *tight cut* of G if $|C \cap M| = 1$ for every perfect matching M of G. The two C-contractions of a matching covered graph with respect to a tight cut C are also matching covered. Thus, given any matching covered graph G, and a nontrivial tight cut C in G, we can obtain two smaller matching covered graphs G_1 and G_2 which are the two C-contractions of G with respect to C. Moreover, inferences can be made about properties of G based on the properties of G_1 and G_2. For example, a vector w is in the matching lattice of G if, and only if, the restrictions of w to $E(G_1)$ and $E(G_2)$ are in the matching lattices of G_1 and G_2, respectively.

In any matching covered graph G, $\nabla(v)$ is a tight cut for any vertex v. All such cuts are *trivial*. A matching covered graph G may not have any nontrivial tight cuts. There are two types of tight cuts, known as barrier cuts, and 2-separation cuts which play important roles in the subject. They are defined below:

Barrier cuts: Let B be a nontrivial barrier of G and let H be a nontrivial component of $G - B$. Then, $\nabla(V(H))$ is a nontrivial tight cut of G. Such a cut is called a *barrier cut*. (See Figure 2(a).)

Note that a bicritical graph cannot have a barrier cut.

2-Separation cuts: By a *2-separation* of G we mean a 2-vertex cut of G that is not a barrier. Let $\{u, v\}$ be a 2-separation of G. Then, all components of $G - \{u, v\}$ are even. Write G as the union of G_1 and G_2 in the usual manner. Then $\nabla(V(G_1) - u)$ and $\nabla(V(G_1) - v)$ are both tight cuts in G. Such cuts are referred to as *2-separation cuts*. (See Figure 2(b).)

The two C-contractions of a bicritical graph G with respect to a 2-separation cut C of G are both bicritical.

A matching covered graph may have tight cuts which are not of the above two types. However, the following deep theorem shows the importance of the above types of tight cuts.

Theorem 2.7 (See [5]) *If a matching covered graph has a nontrivial tight cut, then it either has a barrier cut or a 2-separation cut.*

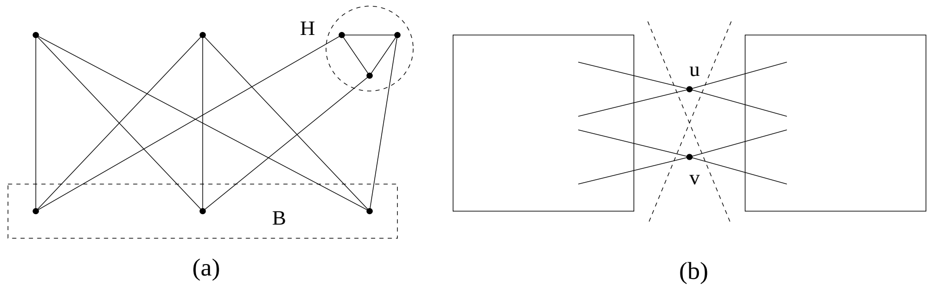


Figure 2: Barrier cuts and 2-separations cuts:
(a) A barrier cut $\nabla(V(H))$ associated with a barrier B.
(b) Two 2-separation cuts associated with a 2-separation $\{u, v\}$.

2.6 Bricks and Braces

A *brick* is a 3-connected bicritical graph. Using Tutte's theorem it can be shown that every cyclically 4-connected cubic graph is a brick. Bicritical graphs do not have any barrier cuts, 3-connected graphs do not have 2-separation cuts. Thus, in view of the above theorem, bricks do not have nontrivial tight cuts.

Let G be a bipartite matching covered graph with bipartition (A, B), and let (A_1, A_2) be a partition of A into nonempty sets A_1, and A_2, and (B_1, B_2) be a partition of B into nonempty sets B_1, and B_2, such that $|A_1| = |B_1| + 1$, $|B_2| = |A_2| + 1$, and there are no edges of G linking B_1 with A_2. Then, the cut $\nabla(A_1 \cup B_1)$ is a nontrivial tight cut of G. In fact, every tight cut in a bipartite matching covered graph must be of this form.

A bipartite matching covered graph G with bipartition (A, B) is called a *brace* if u_1, u_2 are any two distinct vertices in A, and v_1, v_2 are any two distinct vertices in B, then the graph $G - \{u_1, u_2, v_1, v_2\}$ has a perfect matching. It can be shown that braces do not have nontrivial tight cuts.

Theorem 2.8 (See [6]) *A matching covered graph has no nontrivial tight cuts if, and only if, it is either a brick or a brace.*

2.7 Uncrossing

The following theorem is an important tool in proving theorems concerning matching covered graphs

Theorem 2.9 (See [5]) *Let G be a matching covered graph and let $\nabla(X)$ and $\nabla(Y)$ be two tight cuts such that $|X \cap Y|$ is odd. Then $\nabla(X \cap Y)$ and $\nabla(\overline{X} \cap \overline{Y})$ are also tight. Furthermore, no edge connects $X \setminus Y$ to $Y \setminus X$.*

Two cuts $\nabla(X)$ and $\nabla(Y)$ are said to *cross* if the four sets $X \cap Y$, $X \setminus Y$, $Y \setminus X$ and $V \setminus (X \cup Y)$ are all non-empty. The above theorem says that, given any pair of crossing tight cuts, one can 'uncross' them and obtain two other tight cuts which do not cross. This uncrossing procedure is essential to many proofs in this theory.

2.8 Tight Cut Decomposition

Let G be a matching covered graph, and let $C = \nabla(X)$ be a nontrivial tight cut of G. Then, as already noted, the two C-contractions G_1 and G_2 of G are also matching covered. If either G_1 or G_2 has a nontrivial tight cut, we can take its cut-contractions, in the same manner as above, and obtain smaller matching covered graphs. Thus, given any matching covered graph G, by repeatedly applying cut-contractions, we can obtain a list of graphs which do not have nontrivial tight cuts (bricks and braces).

Theorem 2.10 (See [6]) *The results of any two applications of the tight cut decomposition procedure on a matching covered graph G are the same list of bricks and braces except possibly for the multiplicities of edges.*

In particular, the numbers of bricks and braces resulting from a tight cut decomposition of a matching covered graph G is independent of the tight cut decomposition; we shall call these the numbers of bricks and braces of G respectively. We shall let $b(G)$ denote the number of bricks of G. The number of bricks of G whose underlying simple graphs are Petersen graphs is also an invariant of G; we shall denote it by $p(G)$. The numbers $b(G)$, and $(b+p)(G) = b(G) + p(G)$ play important roles in this paper.

Note that $b(G) = 0$ if, and only if, G is bipartite, and $b(G) = 1$ if, and only if, for every tight cut C of G one of the C-contractions of G is bipartite and the other C-contraction has exactly one brick. We shall refer to a matching covered graph G with $b(G) = 1$ as a *near-brick*. Many useful properties that bricks satisfy are quite often satisfied more generally by near-bricks. Furthermore, for proving theorems concerning bricks, it is often convenient to consider the wider class of near-bricks.

2.8.1 The monotonicity of b and $b + p$.

An edge e of a matching covered graph G is *removable* in G if $G - e$ is also matching covered. The following simple theorem shows that the functions b and $b + p$ are monotonic under deletions of removable edges.

Theorem 2.11 *Let G be a matching covered graph, let e be a removable edge of G. Then $b(G - e) \geq b(G)$ and $(b + p)(G - e) \geq (b + p)(G)$.*

<u>Proof:</u> By induction. It is easy to see that if G is bipartite, $b(G - e) = b(G) = 0$, and $(b + p)(G - e) = (b + p)(G) = 0$. Let us consider the case in which G is a brick. In this case, it is easy to see that $G - e$ cannot be bipartite, and thus $b(G - e) \geq 1 = b(G)$. This establishes the first inequality for bricks. To prove the second inequality for bricks, let us note that if either the underlying simple graph of G is not the Petersen graph, or

if the underlying simple graph of G is the Petersen graph and e is a multiple edge of G, $p(G-e) \geq p(G)$, and therefore $(b+p)(G-e) \geq (b+p)(G)$. On the other hand, if the underlying simple graph of G is the Petersen graph, and e is not a multiple edge of G, it is easy to see that $b(G-e) = 2$, and $p(G-e) = 0$, and therefore, $(b+p)(G-e) = (b+p)(G) = 2$.

Thus, we may assume that G is neither a brace nor a brick. In this case, G has a nontrivial tight cut, say C, and let G_1 and G_2 denote the two C-contractions of G. Clearly, $G_1 - e$ and $G_2 - e$ are the two $(C-e)$-contractions of $G-e$. Moreover, $C-e$ is tight in $G-e$. By induction hypothesis,

$$\begin{aligned} b(G-e) = &\ b(G_1 - e) + b(G_2 - e) \geq b(G_1) + b(G_2) = b(G), \\ (b+p)(G-e) = &\ (b+p)(G_1 - e) + (b+p)(G_2 - e) \geq \\ &\ (b+p)(G_1) + (b+p)(G_2) = (b+p)(G), \end{aligned}$$

and the asserted inequalities hold. ■

The following corollary can be deduced easily from the monotonicity of the function b.

Corollary 2.12 *If G is a near-brick, and e is an edge such that $G+e$ is matching covered, then $G+e$ is also a near-brick.*

2.9 Separating Cuts and their Characteristic

Let G be a matching covered graph. Cut $C = \nabla(S)$ is a *separating cut* of G if both the C-contractions of G are matching covered. A separating cut C is *strictly separating* if each C-contraction of G is nonbipartite. The following lemma provides a useful characterization of separating cuts in a matching covered graph.

Lemma 2.13 *Let G be a matching covered graph. A cut C of G is separating if, and only if, for each edge e of G, there exists a perfect matching that contains e and just one edge in C.*

Thus, tight cuts are separating cuts. But, in general, a separating cut need not be a tight cut. For example, bricks do not have non-trivial tight cuts, but they may have non-trivial separating cuts. In case of bipartite graphs, however, every separating cut is a tight cut. This is a consequence of the following simple lemma.

Lemma 2.14 *Let S be a shore of a cut C in a graph G such that the subgraph $G[S]$ of G spanned by S has a bipartition, say, (A, B). Let C_A and C_B denote the set of edges of C that have one end in A, or in B, respectively. For each perfect matching M of G, $|M \cap C_A| - |M \cap C_B| = |A| - |B|$.*

It follows that if $C := \nabla(S)$ is separating and $G[S]$ is bipartite, then $G\{S\}$ is also bipartite and C is tight. In particular, for bipartite matching covered graphs G with bipartition (A, B), the tight cuts C are precisely those whose shores S satisfy the equality $|S \cap A| - |S \cap B| = \pm 1$ and all edges of C are incident to whichever of $S \cap A$ or $S \cap B$ is the largest.

Corollary 2.15 *In a bipartite matching covered graph, a cut is tight if, and only if, it is separating.*

2.9.1 The characteristic of a graph

Let G be a matching covered graph. A cut C is *good* if it is separating but not tight. If it is necessary to indicate a perfect matching M of G that contains more than one edge in C then we say that C is M*-good*. By Lemma 2.14, we have the following consequences.

Lemma 2.16 *If a cut is good then it is strictly separating.*

Lemma 2.17 *If G is a near-brick and C is a separating cut, then C is good if, and only if, it is strictly separating.*

The following result, although very elementary, plays an important role in the paper.

Lemma 2.18 *Let G be a brick, C a nontrivial separating cut of G, and e a removable edge of G. If cut $C - e$ is tight in $G - e$ then $C - e$ is strictly separating in $G - e$.*

Proof: Let H denote a $C-$contraction of G. Then, H is matching covered because C is a separating cut of G. Also H is nonbipartite because, if it were bipartite, C would be a nontrivial tight cut of G which is not possible because G is a brick.

By hypothesis, cut $C - e$ is tight in $G - e$. Thus, $H - e$ is matching covered. Since H is matching covered and nonbipartite, $H - e$ is also nonbipartite. Since this conclusion holds for each of the C-contractions H of G, cut $C - e$ is indeed a strictly separating in $G - e$. ■

We denote by $\mathcal{M}_G$ the set of perfect matchings of G, or simply by $\mathcal{M}$, if G is understood. For each odd cut C and each positive odd integer i, we define $\mathcal{M}_i(C)$, and $\mathcal{M}_{\leq i}(C)$ as follows:

$$\begin{aligned} \mathcal{M}_i(C) &:= \{M : M \in \mathcal{M}, |M \cap C| = i\}, \\ \mathcal{M}_{\leq i}(C) &:= \{M : M \in \mathcal{M}, |M \cap C| \leq i\}. \end{aligned}$$

For each separating cut C of G, the *characteristic* $\lambda(C)$ *of* C is defined as follows:

$$\lambda(C) := \begin{cases} \min\{i > 1 : \mathcal{M}_i(C) \neq \emptyset\}, & \text{if } C \text{ is not tight} \\ \infty, & \text{otherwise.} \end{cases}$$

The *characteristic* $\lambda(G)$ *of* a matching covered graph G is defined as follows:

$$\lambda(G) := min\{\lambda(C) : \ C \text{ is separating}\}.$$

A matching covered graph G is *solid* if $\lambda(G) = \infty$. In other words, G is solid if, and only if, each of its separating cuts is tight. Bipartite braphs are solid. An important class of nonbipartite solid bricks is defined below.

A *wheel* is a simple graph obtained from a circuit by adding to it a new vertex and joining that vertex to each vertex of the circuit; the circuit is called the *rim*, the new vertex the *hub* and each edge joining the hub to the rim a *spoke*. The *order* of the wheel is the number of vertices of its rim; a wheel of order n is denoted W_n. A wheel is *even* or *odd*, according to the parity of n. Note that the hub of a wheel is uniquely identified, except for

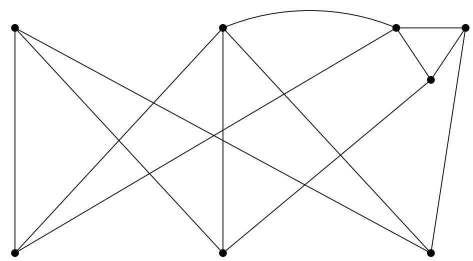


Figure 3: A solid graph that is neither bipartite nor an odd wheel

W_3, which is K_4, the complete graph on four vertices: in that case, we may say that any of its vertices is a hub. It is easy to see that every odd wheel is a solid brick.

There do exist solid bricks other than bipartite graphs and odd wheels. An example of such a brick is shown in Figure 3.

The quantity $\lambda(G)$ plays a significant role in this paper. It turns out that the only simple nonsolid brick of characteristic greater than three is the Petersen graph. This is a part of the statement of our main theorem and is an essential ingredient in its inductive proof.

The characteristic of a matching covered graph may be expressed as a function of the characteristics of its bricks and braces. For establishing this relationship, we need the following theorem which may be regarded as a generalization of Theorem 2.9.

Theorem 2.19 *Let G be a matching covered graph, let $C := \nabla(X)$ be a separating cut and $D := \nabla(Y)$ a tight cut such that $X \cap Y$ and $\overline{X} \cap \overline{Y}$ are both odd. Let $I := \nabla(X \cap Y)$, and $U := \nabla(\overline{X} \cap \overline{Y})$. Then*

- *no edge joins a vertex in $X \cap \overline{Y}$ to a vertex in $\overline{X} \cap Y$,*
- *both I and U are separating, and*
- *at least one of them has characteristic at most that of C.*

Proof: Let e be any edge of G, let M be a perfect matching of G that contains e and has just one edge in C. Since D is tight, M has just one edge in D also. It follows that e does not join a vertex in $X \cap \overline{Y}$ to a vertex in $\overline{Y} \cap X$. Moreover, each of the cuts I and U has just one edge in M. These conclusions hold for each edge e of G. It follows that each of I and U is separating, and, for each perfect matching M of G,

$$|M \cap I| + |M \cap U| = |M \cap C| + |M \cap D|. \tag{1}$$

If C is tight then so too are I and U, and the assertions holds. If C is not tight then, for any perfect matching M such that $|M \cap C| = \lambda(C)$, we have that

$$|M \cap I| + |M \cap U)| = \lambda(C) + 1,$$

whence at least one of I and U has characteristic at most $\lambda(C)$. ■

The following corollary is an immediate consequence of the above theorem.

Corollary 2.20 *The characteristic of a matching covered graph G is the minimum of the characteristics of its bricks and braces.*

Equation (1) appears in various guises in many proofs in the later sections. Whenever there are cuts C, D, I, and U in a graph G, such that the Equation (1) is valid for all perfect matchings M of G, we say that *modularity property* holds for these cuts.

2.9.2 The monotonicity of $\lambda(C)$.

We conclude this section by establishing the monotonicity of the function $\lambda(C)$.

Theorem 2.21 *Let G be a matching covered graph, and let e be a removable edge of G. Then $\lambda(G-e) \geq \lambda(G)$.*

Proof: If $G-e$ is solid then the assertion holds trivially. Assume thus that $G-e$ is not solid. By definition of characteristic, this means that there exist in $G-e$ separating cuts of characteristic $\lambda(G-e)$. In other words, the collection $\mathcal{C}$ of cuts of G defined to be

$$\mathcal{C} := \{C : C-e \text{ is separating in } G-e, \lambda_{G-e}(C-e) = \lambda(G-e)\}$$

is nonempty. For each $C \in \mathcal{C}$, define the *index* of C to be the smallest odd integer $i(C)$ such that, for some perfect matching M of G, edge e lies in M and $|M \cap C| = i(C)$. We assert that $i(C) = 1$ for some cut C in $\mathcal{C}$.

Let C be a cut in $\mathcal{C}$ with the smallest possible index, say i. Let M_0 be any perfect matching of $G-e$ having exactly $\lambda(G-e)$ edges in C. Let M_e denote a perfect matching of G that contains edge e and such that the number of edges of M_e in C is precisely its index i.

To prove that $i = 1$, assume the contrary. Clearly, edge e is thus inadmissible in some C-contraction of G, say $G_1 := G\{X; \overline{x}\}$. But cut $C-e$ is separating in $G-e$. Thus, graph $G_1 - e$ is matching covered. By Theorem 2.4, there exists a barrier B of G_1 such that edge e is the only edge having both ends in B. Since graph G is matching covered, set B is not a barrier of G. Therefore, vertex $\overline{x}$ lies in B.

Let K denote any component of $G_1 - B$, and let $C_K := \nabla_G(V(K))$. (See Figure 4.) For each edge f of $G-e$, there exists a perfect matching M_f of $G-e$ such that $f \in M_f$ and $|M_f \cap C| = 1$. A simple counting argument then shows that M_f has just one edge in C_K. We conclude that C_K is separating in $G-e$, for every component K of $G_1 - B$.

Again, a simple counting argument shows that for at least one component K of $G_1 - B$, the corresponding cut C_K satisfies the inequality

$$1 < |M_0 \cap C_K| \leq \lambda(G-e).$$

But cut C_K is separating in $G-e$, therefore $|M_0 \cap C_K| = \lambda(G-e)$. In other words, cut C_K lies in $\mathcal{C}$.

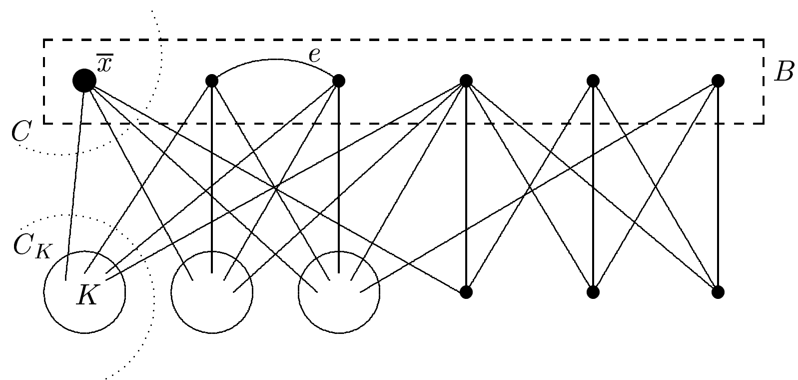


Figure 4: Illustration for the proof of Theorem 2.21

Clearly, the number of edges of M_e in C_K is at most $i-2$. We conclude that cut C_K not only lies in $\mathcal{C}$, but it also has index strictly smaller than that of C, a contradiction. As asserted, the index of cut C is 1.

Recall that cut $C-e$ is separating in $G-e$. Since its index is 1, we conclude that cut C is separating in G as well. Moreover, cut C has precisely $\lambda(G-e)$ edges in M_0. Therefore, $\lambda(G-e) \geq \lambda_G(C) \geq \lambda(G)$. As asserted, $\lambda(G-e) \geq \lambda(G)$. ■

3 The Main Theorem

To give a precise statement of our Main Theorem, and to give an outline of some of the main themes contained in our proof, we now introduce some useful terminology. Recall that an edge e of a matching covered graph G is removable in G if the graph $G-e$ is also matching covered. A removable edge e of G is *b-invariant* if $b(G-e) = b(G)$, and *$(b+p)$-invariant* if $(b+p)(G-e) = (b+p)(G)$. A *b-removable* edge is an edge which is removable and b-invariant, and a *$(b+p)$-removable* edge[1] is an edge which is removable and $(b+p)$-invariant If G is bipartite, then any removable edge of G is both b-removable and $(b+p)$-removable in G. If G is a brick then a removable edge of G is b-removable in G if $G-e$ is a near-brick. In the Petersen graph, every edge is removable, but no edge is b-removable; the deletion of any edge results in a matching covered graph with two bricks. However, every edge of the Petersen graph is $(b+p)$-removable in it. If G is a brick other than the Petersen graph, then a removable edge e of G is $(b+p)$-removable in G if $b(G-e) = 1$ and, up to multiple edges, the brick of $G-e$ is not the Petersen graph. Clearly, if G is a simple brick different from the Petersen graph, then any $(b+p)$-removable edge is also b-removable. If G is an

[1] In [4] we prove the conjecture which says that the minimum possible number of double ears an ear decomposition of a matching covered graph G may have is $b+p$. We were led to the notion of a $(b+p)$-removable edge in our attempts to prove that conjecture.

odd wheel of order greater than three, then any spoke of G is $(b+p)$-removable, but no edge in its rim is even removable.

We may now restate the main theorems as follows:

Theorem 3.1 (Lovász-Vempala) *Every brick different from K_4, $\overline{C}_6$, and the Petersen graph has a b-removable edge.*

Theorem 3.2 *Every brick different from K_4 and $\overline{C}_6$ has a $(b+p)$-removable edge.*

The first theorem above provides an answer to Conjecture 1, and the second theorem provides an answer to Conjecture 2. It turns out that, in fact, the two theorems are equivalent. In our efforts to find an inductive proof of Conjecture 2, we have found it convenient to prove the following apparently more general theorem which incorporates the property of the Petersen graph mentioned in the introduction.

Theorem 3.3 (The Main Theorem) *Let G be any brick different from K_4 and $\overline{C_6}$. Then G has a $(b+p)$-removable edge. Furthermore, if G is nonsolid, and its characteristic is greater than three, then the underlying simple graph of G is the Petersen graph.*

For clarity and future reference, we note that for nonsolid simple bricks, the above statement reduces to the following:

Theorem 3.4 *If G is any nonsolid simple brick different from $\overline{C_6}$, then*

- **either** $\lambda(G) = 3$ *and G has a $(b+p)$-removable edge*
- **or** *G is the Petersen graph (and each of its edges is $(b+p)$-removable).*

Our approach is to prove the above theorem by induction on $|V(G)| + |E(G)|$, and has many features in common with Lovász's proof of his main lemma in [6]. In order to prove the theorem by induction, we must find ways of obtaining from a given brick a smaller brick to which the induction hypothesis could be applied. Thus, it is natural to consider removable edges in Bricks.

The two small bricks K_4 and $\overline{C}_6$ do not have removable edges. Using the theory of ear decompositions of matching covered graphs, Lovász proved the following theorem.

Theorem 3.5 (See [6] and [3]) *If G is any brick different from K_4 and $\overline{C}_6$, then G has a removable edge.*

He used the above fact as a crucial tool for induction in his proof of the main lemma in [6]. However, that lemma was (as our main theorem is) a statement about bricks. So, the above theorem was inadequate for proving his lemma because a removable edge in a brick need not be a b-removable edge. In fact, the Petersen graph has no b-removable edges.

To circumvent the above difficulty, Lovász needed to find other means of breaking up a brick into smaller matching covered graphs. This led him to the notion of separating cuts introduced in the last section. (Lovász did not use this term 'separating cut'.) Large part of this work consists of establishing the existence of suitable separating cuts in bricks. A brief general outline of our proof is given below.

In section 4, we derive necessary and sufficient conditions for a removable edge e in a brick G to be b-removable. A barrier B in a matching covered graph G is *special* if $G - B$ has precisely one nontrivial odd component. We show that a necessary condition for a removable edge e in a brick G to be b-removable is that every barrier in $G - e$ is special. A good cut C of a matching covered graph G is a *robust* cut if the two C-contractions of G are near-bricks. We show that the existence of a robust cut in G is a sufficient condition for G to be a near-brick. Thus, if G is a brick, and e is a removable edge of G, then a sufficient condition for e to be b-removable in G is that $G - e$ has a robust cut.

Using the above conditions, we show, in section 5, that if e is a b-removable edge in a brick G, then either e is itself $(b + p)$-removable in G, or there is another edge f which is $(b+p)$-removable in G and, furthermore, that G has a separating cut of characteristic three (Theorem 5.4). A consequence of this theorem is that Conjecture 1 implies Conjecture 2. Another consequence is that in a solid brick, every b-removable edge is also $(b+p)$-removable.

Every nonsolid brick G, by definition, has good cuts; in fact has good cuts whose chracteristic is equal to that of G. Let G be a brick of G and let e be a removable edge of G. In section 6, we show that if e is not b-removable, then G has good cuts of characteristic three or five (Theorem 6.4). It follows from this that in a solid brick, every removable edge is also b-removable (hence, from the discussion above, is also $(b + p)$-removable). Thus, we need only consider bricks which are not solid.

Although nonsolid bricks have good cuts, it is not at all obvious that such bricks have robust cuts. Section 7 is devoted to proving the existence and properties of robust cuts in nonsolid bricks. Suppose that G is a nonsolid brick different from K_4 and $\overline{C_6}$, and suppose that it has a removable edge e which is not b-removable. Then, as noted above, G has good cuts of characteristic three or five. We deduce from this that G has a robust cut C with $\lambda(C) = \lambda(G)$ (Corollary 7.5).

In section 8, we present a proof of the Main Theorem by induction on $|V(G)| + |E(G)|$. It is easy to apply induction if G either has multiple edges or is solid. We deal with nonsolid simple bricks by means of the following lemma.

Lemma 3.6 (The Main Lemma) *Let G be any nonsolid simple brick different from $\overline{C}_6$. Suppose that G satisfies the following conditions:*

(I) If G' is any brick with $|V(G')|+|E(G')| < |V(G)|+|E(G)|$, then G' satisfies the statement of the Theorem 3.3, and

(II) Either $\lambda(G) > 3$ or G has no b-removable edges[2]*.*

Then G is the Petersen graph.

[2] If a brick has no $(b+p)$-removable edges, then it cannot have any b-removable edges either. This follows from Theorem 5.4. So, the hypothesis (II) is implied by the statement that "either $\lambda(G) > 3$ or G has no $(b + p)$-removable edges."

The starting point for our proof of the Main Lemma is the observation that, since G is not solid, G has robust cuts of characteristic three or five, by Corollary 7.5. For such a robust cut $C = \nabla(X)$ of G, let $G_1 = G\{X;\overline{x}\}$ and $G_2 = G\{\overline{X};x\}$ denote the two C-contractions of G. We first show that G has a robust cut C of characteristic $\lambda(G)$ such that G_1 is an odd wheel with hub $\overline{x}$ (up to multiple edges incident with the hub) and that G_2 is simple brick different from K_4, $\overline{C}_6$, and the Petersen graph. By the induction hypothesis, G_2 has b-removable edges. If there is a b-removable edge e of G_2 which is also b-removable in G, then we deduce that $\lambda(G) = 3$ and that G has a b-removable edge; which is not possible by the second hypothesis. From all this information we show that both G_1 and G_2 must be 5-wheels and that G must be a graph obtained by splicing these two 5-wheels at their hubs, and finally that G must in fact the Petersen graph.

4 Necessary and Sufficient Conditions for b-Removability

In this section, we establish necessary and sufficient conditions for a removable edge e in a brick G to be b-removable.

4.1 Special Barriers

A barrier B of a matching covered graph H is *special* if $H - B$ has exactly one nontrivial odd component. Using this notion, we shall now establish a necessary condition for a removable edge e in a brick G to be b-removable in G.

Theorem 4.1 *Let G be a brick, and let e be a b-removable edge in G. Then, every barrier B of $G - e$ is special.*

Proof: Let B be any barrier of $G - e$. Since e is b-removable in G, $b(G - e) = 1$. Therefore, $G - e - B$ has exactly one nonbipartite odd component, say K. Let L be any odd component of $G - e - B$ different from K. We claim that the cut $\nabla(V(L))$ is a separating cut of G. Clearly the cut $\nabla(V(L)) - e$ is a tight cut of $G - e$. Thus, if f is any edge of $G - e$, there is a perfect matching of $G - e$ containing f and having exactly one edge in $\nabla(V(L))$. To show that there is a perfect matching of G containing e and having exactly one edge in $\nabla(V(L))$, consider the cut $\nabla(V(K))$. Since G is a brick, $\nabla(V(K))$ is not tight in G. Let M be any perfect matching such that $|\nabla(V(K)) \cap M| \geq 3$. Then, an easy counting argument shows that we must in fact have $e \in M$, $|\nabla(V(K)) \cap M| = 3$, and $|\nabla(V(K')) \cap M| = 1$, for any odd component K' of $G - e - B$ different from $\nabla(V(K))$. In particular, $e \in M$ and $|\nabla(V(L)) \cap M| = 1$. Thus $\nabla(V(L))$ is a separating cut in the brick G. One of its shores is $V(G) \setminus V(L)$ which is nontrivial. The other shore is $V(L)$. If $|V(L)| > 1$, and L is bipartite, then $\nabla(V(L))$ would be a nontrivial tight cut of G. This is not possible because G is a brick. So, L is either nonbipartite or trivial. But, as already noted, K is the only nonbipartite component of $G - e - B$. Thus every odd component L of $G - e - B$ different from K is trivial. It follows that B is a special barrier of $G - e$. ■

Remark: The above necessary condition for b-removability of e is not sufficient. For example, if G is the Petersen graph, and e is any edge of G, then e is removable and every barrier of $G-e$ is special. However, e is not b-removable in G. In fact, the Petersen graph has no b-removable edges.

4.2 The Subadditivity of Function $b(G)$

The following inequality relates the number of bricks of a matching covered graph G with numbers of bricks in the cut-contractions of G with respect to a separating cut of G.

Theorem 4.2 (Subadditivity of function $b(G)$**)** *Let G be a matching covered graph. Let $C := \nabla(X)$ be a separating cut of G, and let $G_1 = G\{X\}$, and $G_2 = G\{\overline{X}\}$ be the two C-contractions of G. Then*

$$b(G) \leq b(G_1) + b(G_2),$$

with equality if, and only if, C is a tight cut.

Proof: If C is a tight cut of G, then, clearly, equality holds. Suppose that C is not a tight cut of G. We shall prove the desired inequality by induction on $|V|$.

Firstly observe that if either G_1 or G_2 is bipartite, then C is a tight cut. We may therefore assume that $b(G_1) \geq 1$ and $b(G_2) \geq 1$. Now, if G is free of nontrivial tight cuts then the desired inequality is satisfied. Therefore we may assume that G has a nontrivial tight cut, say $D = \nabla(Y)$. Let $H_1 = G\{Y\}$ and $H_2 = G\{\overline{Y}\}$ denote the two D-contractions of G.

Adjust notation, by interchanging Y with $\overline{Y}$ if necessary, so that $|X \cap Y|$ is odd. Let $I := \nabla(X \cap Y)$ and $U := \nabla(\overline{X} \cap \overline{Y})$. By Theorem 2.19, each of I and U is separating in G; moreover, no edge joins a vertex of $X \cap \overline{Y}$ to a vertex of $\overline{X} \cap Y$. Therefore, for any perfect matching M of G,

$$|M \cap I| + |M \cap U| = |M \cap C| + |M \cap D|.$$

But, as D is a tight cut, we in fact have

$$|M \cap I| + |M \cap U| = |M \cap C| + 1. \tag{2}$$

Let $G_{11} := G\{X \cap Y\}$. Observe that G_{11} is one of the I-contractions of G_1. Let G_{12} denote the other I-contraction of G_1. Similarly, let $G_{22} := G\{\overline{X} \cap \overline{Y}\}$. Observe that G_{22} is one of the U-contractions of G_2. Let G_{21} denote the other U-contraction of G_2. Thus, G_{11} and G_{12} are the I-contractions of G_1; similarly, G_{22} and G_{21} are the U-contractions of G_2. Note also that G_{11} and G_{21} are the I-contractions of H_1; similarly, G_{12} and G_{22} are the U-contractions of H_2.

We then have

$$\begin{aligned}
b(G) &= b(H_1) + b(H_2), && (3)\\
b(H_1) &\leq b(G_{11}) + b(G_{21}), && (4)\\
b(H_2) &\leq b(G_{12}) + b(G_{22}), && (5)\\
b(G_{11}) + b(G_{12}) &= b(G_1), && (6)\\
b(G_{21}) + b(G_{22}) &= b(G_2), && (7)
\end{aligned}$$

The validity of (3) follows from the fact that D is tight in G. Since cut I is separating in G and D is tight in G, it follows that I is separating in H_1. Likewise, cut U is separating in H_2. The validity of (4) and of (5) follows from the inductive hypothesis; moreover, since cut C is not tight in G, equation (2) implies that either I is not tight in H_1 or U is not tight in H_2. Therefore at least one of the inequalities (4) and (5) is strict. Finally, equation (2) implies that I is tight in G_1 and U is tight in G_2, which establishes the validity of (6) and of (7).

Adding up inequalities (3)-(7), and simplifying, yields the asserted (strict) inequality. ■

The subadditivity of b suggests that if one could find a separating but not tight cut C of a near-brick such that both C-contractions are near-bricks, then C would be a certificate to the effect that G is a near-brick. This is indeed how we often certify the property of being a near-brick.

Let G be a matching covered graph. For a perfect matching M of G, we say that an odd cut C of G is *M-robust* in G if C is M-good and each of the C-contractions of G is a near-brick. If it is not necessary to identify the perfect matching M we simply say that C is *robust* in G. The following important theorem is an immediate consequence of the subadditivity property proved above.

Theorem 4.3 *If a matching covered graph has a robust cut then it is a near-brick.*

Using the above theorem, we can now derive a sufficient condition for b-removability of a removable edge in a brick.

Corollary 4.4 *Let G be a brick, and let e be a removable edge of G. Then, the existence of a robust cut in $G - e$ is a sufficient condition for e to be b-removable in G.*

Suppose that e is a removable edge of a brick G, and C is a separating cut of $G - e$. If there is a perfect matching M of $G - e$ such that $|M \cap (C - e)| > 1$, then $C - e$ would be an M-robust cut of $G - e$ implying that e is b-removable in G. This is often the approach we use for establishing$b$-removability of edges in bricks.

Remark: The sufficient condition derived in Corollary 4.4 is not necessary. For example if G is a brick such that $G - e$ is a solid brick, then although e is b-removable in G, $G - e$ has no robust cuts.

5 Conjecture 1 implies Conjecture 2

In this section, we use the conditions derived in the previous section to show that Conjecture 1 implies Conjecture 2. For this purpose, the Three Case Lemma proved below is crucial. If e is a b-removable edge in a brick G, and H is the unique brick of $G - e$, it says that H can be obtained from $G - e$ by at most two (zero, one, or two) cut-contractions with respect to barrier cuts associated with special barriers of G.

5.1 The Three Case Lemma

For any barrier B of a matching covered graph G, we shall denote by $I_G(B)$, or by simply $I(B)$, if G is understood, the set of isolated vertices of $G - B$. Thus, if B is special, that is, if $G - B$ has just one nontrivial component, then $|I(B)| = |B| - 1$. Before stating and proving the Three Case Lemma, we prove an auxiliary lemma applicable to all matching covered graphs in which every barrier is special.

Lemma 5.1 *Let G be a matching covered graph. If every barrier of G is special then there exists a (possibly empty) collection $\mathcal{B}$ of nontrivial (special) barriers of G such that for any two distinct barriers B' and B'' in $\mathcal{B}$, the sets $B' \cup I(B')$ and $B'' \cup I(B'')$ are disjoint. Moreover, the graph obtained from G by contracting, for each barrier $B \in \mathcal{B}$, the set $B \cup I(B)$ to a single vertex, is bicritical. (See Figue 5.)*

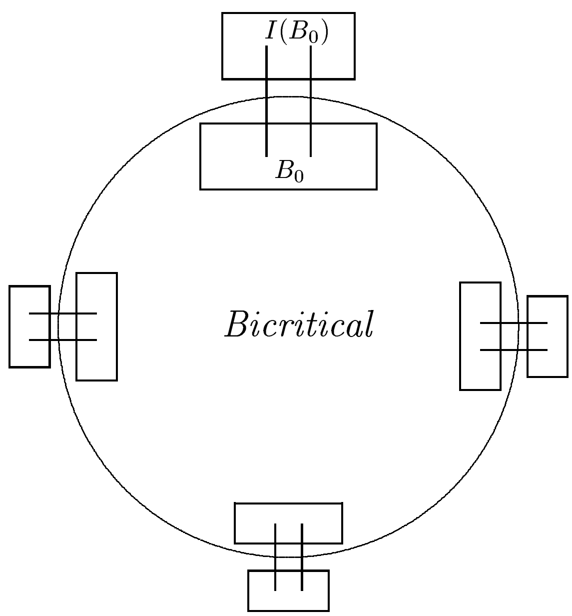


Figure 5: Illustration for Lemma 5.1

Proof: By induction on $|V(G)|$. If G is bicritical then the assertion holds trivially, with $\mathcal{B}$ the empty set. Note that G cannot be bipartite, because if G were bipartite, then one of the parts of the biparttion of G would be a barrier which is not special.

So, let us assume that G is not bicritical. In this case, it has nontrivial barriers. Let B_0 denote any maximal nontrivial barrier of G. By hypothesis, B_0 is special. Let K denote the only nontrivial component of $G - B_0$, and let $C := \nabla(V(K))$. Let $G' = G\{V(K); v'\}$ and $G'' = G\{\overline{V(K)}\}$ denote the two C-contractions of G. Clearly, C is a tight cut of G and G'' is bipartite. Thus, G' must be nonbipartite; otherwise, G would itself be bipartite.

We shall now show that G' satisfies the properties which allow us to apply induction. Towards this end, let B denote any barrier of G'. We claim that B satisfies the following properties:

- If vertex v' lies in B, then $B = \{v'\}$.
- If v' does not lie in B, then B is a special barrier of both G' and G, and $I_{G'}(B) = I_G(B)$.

Consider first the case in which vertex v' lies in B. Then clearly, set $B_0 \cup (B - v')$ is a barrier of G. By the maximality of B_0, it follows that $B = \{v'\}$. Now suppose that v' does not lie in B. In this case, set B is clearly a barrier of G. Let L' denote the component of $G' - B$ that contains vertex v', and let $L := G[\, B_0 \cup I(B_0) \cup (V(L') - v')\,]$. Clearly, $|V(L)|$ is odd, and L is nontrivial. Also, L must be an odd component of $G - B$; otherwise, either $C_{odd}(G-B) > |B|$ or $C_{odd}(G-B) = |B|$ and $C_{even}(G-B) > 1$, both of which are impossible because G is matching covered. Since, by hypothesis, all barriers of G are special, it follows that all components of $G - B$ other than L are trivial. But these components of $G - B$ are the same as the componets of $G' - B$ distinct from L'. So, all components of $G' - B$ other than L' are trivial. Now, if L' were also trivial, G would be bipartite. This, as noted is not possible. It follows that L' is the only nontrivial component of $G' - B$. Therefore, set B is a special barrier of G' and $I_{G'}(B) = I_G(B)$.

By the induction hypothesis, applied to G', there exists a collection $\mathcal{B}'$ of nontrivial barriers of G' satisfying the desired properties relative to G'. Let $\mathcal{B} := \mathcal{B}' \cup \{B_0\}$. We assert that $\mathcal{B}$ has the desired properties.

Clearly, $\mathcal{B}$ is a collection of nontrivial barriers of G. Let B and B' denote two distinct barriers in $\mathcal{B}$. We wish to prove that $B \cup I(B)$ and $B' \cup I(B')$ are disjoint. First suppose that one of B and B' is the barrier B_0. Adjust notation so that $B' = B_0$. Then, as noted above, the vertex v' of G' lies in the unique nontrivial component of $G' - B$, and so, the two sets indicated above are disjoint. Now suppose that both B and B' are in $\mathcal{B}'$. By induction hypothesis, sets $B \cup I_{G'}(B)$ and $B' \cup I_{G'}(B')$ are disjoint. But these two sets are equal, respectively, to $B \cup I(B)$ and $B' \cup I(B')$. Therefore $B \cup I(B)$ and $B' \cup I(B')$ are disjoint for any two distinct barriers B and B' in $\mathcal{B}$.

Finally, let H be the graph obtained from G' by contracting, for each barrier B in $\mathcal{B}'$, the set $B \cup I(B)$ to a single vertex. By induction hypothesis, graph H is bicritical. But v', the vertex resulting from the contraction of $B_0 \cup I(B_0)$ is a vertex of H. So, H is the same as the graph obtained from G by contracting, for each B in $\mathcal{B}$ $B \cup I(B)$ to a single vertex. We conclude that $\mathcal{B}$ has all the asserted properties. ■

In the remaining part of this section, we shall write, for brevity, I_1 for $I(B_1)$, and I_2 for $I(B_2)$.

Lemma 5.2 (The Three Case Lemma) *Let G be a bicritical matching covered graph, let e be a removable edge of G. If every barrier of $G - e$ is special then:*

(i) *either graph $G - e$ is bicritical,*

(ii) *or graph $G - e$ has a nontrivial (special) barrier, B_1, such that the graph obtained from $G - e$ by contracting $B_1 \cup I_1$ to a single vertex is bicritical, and edge e has at least one end in I_1,*

(iii) *or graph $G - e$ has two nontrivial (special) barriers, B_1 and B_2, such that sets $B_1 \cup I_1$ and $B_2 \cup I_2$ are disjoint, and the graph obtained from $G - e$ by contracting each of the sets $B_1 \cup I_1$ and $B_2 \cup I_2$ to single vertices is bicritical, and edge e has one end in I_1, the other in I_2.*

Proof: Since edge e is removable in G, graph $G - e$ is matching covered. By Theorem 5.1, there exists a collection $\mathcal{B}$ of nontrivial (special) barriers of $G - e$ such that for any two barriers B_1 and B_2 in $\mathcal{B}$, sets $B_1 \cup I_1$ and $B_2 \cup I_2$ are disjoint. Moreover,the graph obtained from $G - e$ by contracting, for each barrier B in $\mathcal{B}$, the set $B \cup I(B)$ to a single vertex, is bicritical.

It now suffices to show that $\mathcal{B}$ contains at most two barriers and also that edge e has at least one end in $I(B)$, for each $B \in \mathcal{B}$. By hypothesis, graph G is bicritical and matching covered, therefore free of nontrivial barriers. Each barrier B of $\mathcal{B}$ is nontrivial, therefore edge e has its ends in distinct components of $G - e - B$. By hypothesis, barrier B is special. We conclude that edge e has at least one end in $I(B)$. Since that conclusion holds for every barrier B in $\mathcal{B}$, we deduce that $\mathcal{B}$ contains at most two barriers. ■

If G is a brick, and e is a b-removable edge of G, then by Theorem 4.1, every barrier of $G - e$ is special. The following corollary is the approriate form in which the Three Case Lemma finds its use in proving the important theorem of this section.

Corollary 5.3 *Let e be a b-removable edge of a brick G, and let H be the brick of $G - e$. Then,*

- *either $G - e = H$,*
- *or there is a nontrivial special barrier B_1 of $G - e$ such that the graph obtained from $G - e$ by contracting $B_1 \cup I_1$ to a single vertex v_1 is H,*
- *or there are two nontrivial special barriers B_1 and B_2 of $G - e$ such that sets $B_1 \cup I_1$ and $B_2 \cup I_2$ are disjoint, and the graph obtained from $G - e$ by contracting these two sets to single vertices v_1 and v_2, respectively, is H.*

Proof: Since e is a b-removable edge of G, $b(G - e) = 1$. The Corollary follows from the Three Case Lemma, and the observation that any bicritical graph with one brick must in fact be a brick itself. ■

5.2 $(b + p)$-Removable Edges

A graph G is P_3-*transitive* if, for every pair of paths (u_1, u_2, u_3, u_4) and (u'_1, u'_2, u'_3, u'_4) of length three in G, there exists an automorphism σ of G such that $\sigma(u_i) = u'_i$, for $1 \le i \le 4$. It is well known that the Petersen graph has diameter two and is P_3-transitive (see [9]).

Theorem 5.4 *Let G be a brick, let e be a b-removable edge of G. If edge e is not $(b+p)$-removable then*

- *G has $(b+p)$-removable edges (in fact has at least two nonadjacent $(b+p)$-removable edges), and*
- *G has characteristic three.*

Proof: By hypothesis, edge e is b-removable in G. Therefore, $G-e$ is a near-brick. We note that the underlying simple graph of the brick of $G-e$ must be the Petersen graph. Otherwise, e is $(b+p)$-removable. This is not possible by the hypothesis. We also note that the underlying simple graph of G itself cannot be the Petersen graph. Because, if this were the case, e must be a multiple edge in order for it to be b-removable. But then, e is both b-removable and $(b+p)$-removable, which is precluded by the hypothesis.

In the remaining part of the proof, for clarity and brevity of expression, we shall refer to a brick whose underlying simple graph is the Petersen graph, as simply the Petersen graph. Similarly, we shall refer to a brick whose underlying simple graph is an odd wheel, as simply an odd wheel.

Since e is a b-removable edge of G, then by Theorem 4.1, every barrier of $G-e$ is special. And so, by Corollary 5.3 to the Three Case Lemma, our task reduces to examining the following three main cases, and the indicated subcases:

Case 1: $G-e$ is the Petersen graph. (See Figure 6)

Case 2: $G-e$ has a nontrivial barrier B_1 such that the graph obtained from $G-e$ by contracting $B_1 \cup I_1$ to a single vertex v_1 is the Petersen graph.

In this case, the edge e has at least one end in I_1. Let w_1 denote one end of e in I_1, and let w_2 denote the other end of e. We need to consider the following three subcases depending on where w_2 is:
(a): w_2 is in I_1,
(b): w_2 is in the Petersen graph, and is adjacent to v_1, and
(c): w_2 is in the Petersen graph, and is at distance two from v_1.

Taking into account the fact the Petersen graph is of diameter two, and is P_3-transitive, Case 2 reduces to examining the three subcases depicted in Figure 7.

Case 3: $G-e$ has two nontrivial special barriers B_1 and B_2 such that sets $B_1 \cup I_1$ and $B_2 \cup I_2$ are disjoint, and the graph obtained from $G-e$ by contracting these two sets to single vertices, v_1 and v_2, respectively, is the Petersen graph.

In this case, the edge e has one end in I_1 and one end in I_2. However, there are two subcases depending on whether or not v_1 and v_2 are adjacent in the Petersen graph:
(a): v_1 and v_2 are not adjacent in the Petersen graph, and
(b): v_1 and v_2 are adjacent in the Petersen graph.

Taking into account the fact the Petersen graph is of diameter two, and is P_3-transitive, Case 3 reduces to examining the two subcases depicted in Figure 8.

We now proceed to examine the cases indicated above. Before dealing with the individual cases, we describe the general idea that is common to the proof of the Theorem in all cases. To follow this general description, the reader might find it helpful to refer to the figures and the two simple special Cases 1 and 2a.

The starting point in all cases is the observation that, without loss of generality, we may assume that the vertices are labelled as shown in the figures. In each case we find a perfect matching of M_e of G, a Perfect matching N_e of G, and a cut $C = \nabla(X)$, such that

(i) $e \in M_e$, and $|M_e \cap C| = 1$, and

(ii) $e \in N_e$ and $|N_e \cap C| = 3$.

In fact, notation may be adjusted so that, in all cases, $X = \{0'', 1'', 2'', 3'', 4''\}$, and there are perfect matchings M_e and N_e satisfying the above properties.

Let $G_1 = G\{X, \overline{x}\}$, and $G_2 = \{\overline{X}, x\}$ denote the two C-contractions of G. Clearly G_1 is an odd wheel of order five, and $G_2 - e$ is a matching covered graph with an odd wheel of order five as its only brick. (We shall denote this brick of $G_2 - e$ by W.) Thus, C is a separating cut of $G - e$. The existence of matching M_e satisfying property (i) shows that C is also a separating cut of G, and the existence of matching N_e satisfying property (ii) shows that $\lambda(C) = 3$.

In the underlying simple graph of G, $|C \setminus N_e| = 2$. The two edges, say f_1 and f_2, of $C \setminus N_e$ are nonadjacent, and in Cases 2 and 3, they are not incident with the barriers B_1 and B_2. We shall show that f_1 and f_2 are $(b+p)$-removable in G. Consider first the edge f_1. This edge f_1 is a spoke of the 5-wheel G_1, and of the 5-wheel W, the unique brick of $G_2 - e$. Thus, f_1 is b-removable in both $G_1 - e = G_1$ and $G_2 - e$, and so $G - e - f_1$ is matching covered, and $C - f_1$ is a robust cut of $G - e - f_1$. Now, the existence of the perfect matching M_e shows that both $G_1 - f_1$ and $G_2 - f_1$ are matching covered. By Corollary 2.12 it follows that both $G_1 - f_1$ and $G_2 - f_1$ are near-bricks. And so, $C - f_1$ is a separating cut of $G - f_1$. Since f_1 does not lie in N_e, $C - f_1$ is not tight in $G - f_1$. Thus, by subadditivity, f_1 is b-removable in G. Finally, since $\lambda(C - f_1) = 3$, the brick of $G - f_1$ cannot be the Petersen graph. Thus f_1 is a $(b+p)$-removable edge of G. Similarly, f_2 is also a $(b+p)$-removable edge of G.

Thus, in each case, our task is to find the matchings M_e and N_e satisfying the properties described above. We give the details in a few typical cases, and sketch the procedure for finding these matchings in the other cases. In our figures, edges of M_e that can be explicitly shown are indicated by solid lines, and thoses of N_e are indicated by dashed lines.

Case 1:

$$M_e = \{e, (0', 0''), (1'', 2''), (3'', 4''), (2', 3')\},$$

$$N_e = \{e, (0', 0''), (3', 1''), (2', 4''), (2'', 3'')\},$$

$$X = \{0'', 1'', 2'', 3'', 4''\}, \quad C = \nabla(X),$$

and

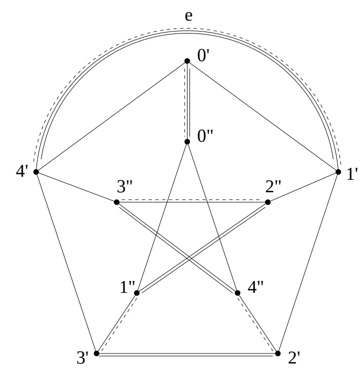


Figure 6: Case 1.

$$f_1 = (1', 2''), \text{ and } f_2 = (4', 3'').$$

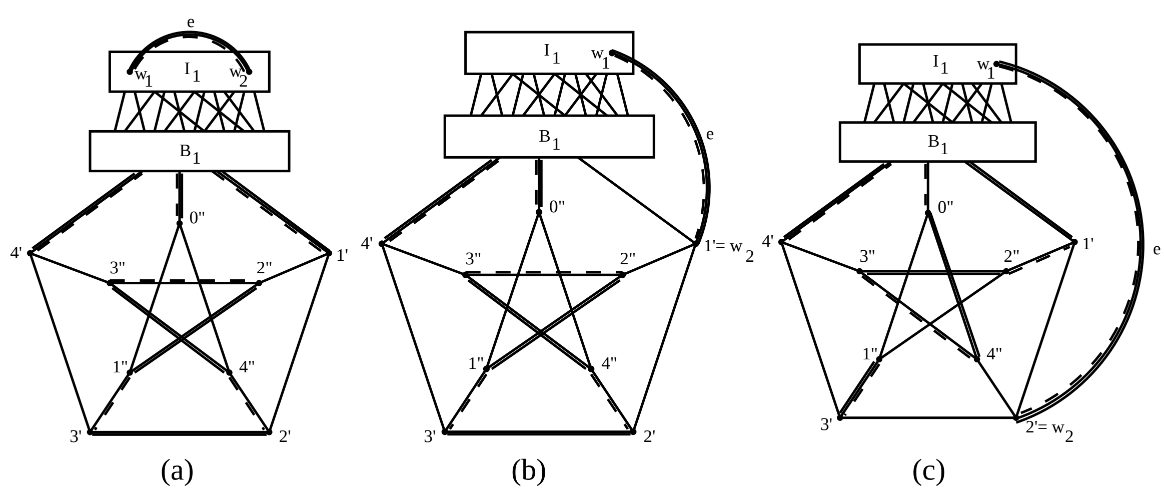


Figure 7: Case 2.

Case 2a: Let M denote any perfect matching of G that contains edge e. A simple counting argument shows that the vertices of $I_1 \setminus \{w_1, w_2\}$ are matched with $|B_1| - 3$ vertices of B_1 and the remaining three vertices of B_1 are matched with vertices $1', 0'', 4'$, respectively. We conclude that M includes a perfect matching, say M_1, of graph $G[\, B_1 \cup I_1 \cup \{1', 0'', 4'\}]$, that contains edge e. Now, we define M_e and N_e as follows:

$$M_e = M_1 \cup \{(2',3'),(1'',2''),(3'',4'')\}\ ,$$

$$N_e = M_1 \cup \{(2',4''),(3',1''),(2'',3'')\}\ .$$

Case 2b: Let M denote any perfect matching of G that contains edge e. A simple counting argument shows that the vertices of $I_1 \setminus \{w_1\}$ are matched with $|B_1|-2$ vertices of B_1 and the remaining two vertices are matched with vertices $0''$ and $4'$, respectively. We conclude that M includes a perfect matching, say M_1, of graph $G[\,B_1 \cup I_1 \cup \{1',0'',4'\}\,]$ that contains edge e. Now, we define M_e, N_e as follows:

$$M_e = M_1 \cup \{(2',3'),(1'',2''),(3'',4'')\}$$

and

$$N_e = M_1 \cup \{(2',4''),(3',1''),(2'',3'')\}\ .$$

Case 2c: Graph $G - \{w_1, 0''\}$ has a perfect matching. A simple counting argument shows that the vertices of $I_1 \setminus \{w_1\}$ are matched with $|B_1|-2$ vertices of B_1 and the remaining two vertices of B_1 are matched with vertices $1'$ and $4'$. We conclude that M includes a perfect matching, say M_1, of graph $G[\,B_1 \cup I_1 \cup \{1',2',4'\}]$ that contains edge e. We define M_e as follows:

$$M_e = M_1 \cup \{(3',1''),(2'',3''),(4'',0'')\}$$

Graph $G - \{w_1, 1'\}$ has a perfect matching. A simple counting argument shows that the vertices of $I_1 \setminus \{w_1\}$ are matched with $|B_1|-2$ vertices of B_1 and the remaining two vertices of B_1 are matched with vertices $0''$ and $4'$. We conclude that M includes a perfect matching, say M_2, of graph $G[\,B_1 \cup I_1 \cup \{0'',2',4'\}]$ that contains edge e. We define N_e as follows:

$$N_e = M_2 \cup \{(1',2''),(3',1''),(3'',4'')\}\ .$$

(Note that, in this case f_1 and f_2 are $(2',4'')$ and $(4',3'')$.)

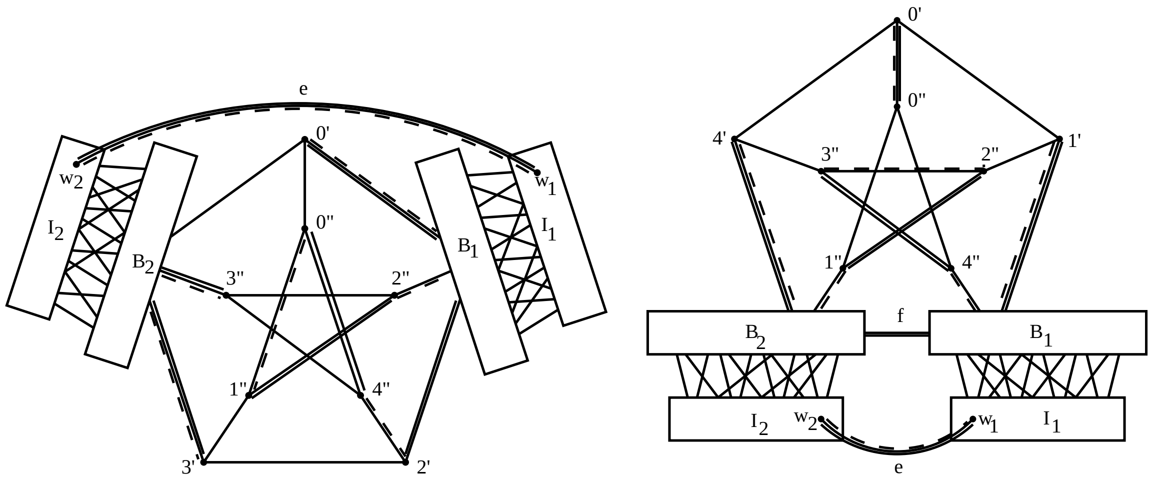
Case 3a: Every perfect matching of graph $G - \{w_1, 2''\}$ pairs two vertices of B_1 with vertices $0'$ and $2'$. Therefore, graph $G[B_1 \cup I_1 \cup \{0',2',w_2\}\,]$ has a perfect matching, say, M_1. Similarly, graph $G[B_1 \cup I_1 \cup \{0',2'',w_2\}\,]$ has a perfect matching, say, M_2. Finally, graph $G[B_2 \cup I_2 \cup \{3',3'',w_1\}\,]$ has a perfect matching, say M_3. Each of M_1, M_2 and M_3 contains edge e. We define M_e and N_e as follows:

$$M_e = M_1 \cup M_3 \cup \{(1'',2''),(4'',0'')\}\ ,$$

and

$$N_e = M_2 \cup M_3 \cup \{(2',4''),(0'',1'')\}\ .$$

Case 3b: Of all the cases, this is perhaps the most diffcult. So, we give more details. The following lemma is useful for establishing the existence of the required type of perfect matchings.



(a) (b)

Figure 8: Case 3.

Lemma 5.5 *Graph* $H := G[\, B_1 \cup I_1 \cup B_2 \cup I_2 \cup \{1', 4'', 1'', 4'\}]$ *has a perfect matching.*

Proof: The graph obtained from G by removing any two vertices of B_2 has a perfect matching. That perfect matching necessarily contains edge e and matches the remaining $|B_2| - 2$ vertices of B_2 to the vertices of $I_2 \setminus \{w_2\}$. Therefore, the vertices of $I_1 \setminus \{w_1\}$ are matched with $|B_1| - 2$ vertices of B_1. The remaining two vertices of B_1 are matched with $1'$ and $4''$. Restriction of this matching to the edges of $G[\, B_1 \cup I_1 \cup \{w_2, 1', 4''\}]$ yields a perfect matching, say M_1', for that graph. Moreover, M_1' contains edge e. By symmetry, graph $G[\, B_2 \cup I_2 \cup \{w_1, 1'', 4'\}]$ also has a perfect matching, say M_1'', that contains edge e. Then $M_1 = M_1' \cup M_2''$ is a perfect matching of H.

Let x_1' and x_1'' denote distinct vertices of B_1 that are adjacent to vertices $1'$ and $4''$, respectively. Similarly, let x_2' and x_2'' denote distinct vertices of B_2 that are adjacent to vertices $4'$ and $1''$, respectively.

By the hypothesis of the case, vertices v_1 and v_2 are adjacent. Let f denote an edge of G that has one end in B_1 and the other in B_2. Let y_1 and y_2 denote the ends of f in B_1 and in B_2, respectively.

By the P_3-transitivity of the Petersen graph, we may assume that y_1 and x_1' are distinct and y_2 and x_2' are also distinct: if necessary, exchange $1'$ with $4''$ or $4'$ with $1''$ (or both), preserving v_1 and v_2 as fixed points. Let $X := \{0'', 1'', 2'', 3'', 4''\}$, $C := \nabla(X)$. Clearly, cut C is separating in $G - e$.

Now the technique is similar to that used in the previous cases. Graph $G - \{x_1', y_1\}$ has a perfect matching. That perfect matching necessarily contains edge e

and matches the vertices of $B_1 \setminus \{x_1', y_1\}$ to the vertices of $I_1 \setminus \{w_1\}$. It follows that graph $G[\,B_1 \cup I_1 \cup \{w_2, y_2, 1'\}]$ has a perfect matching, say M_2', that contains edges e and f. Likewise, graph $G[\,B_2 \cup I_2 \cup \{w_1, y_1, 4'\}]$ has a perfect matching, say M_2'', that contains edges e and f. Then $M_2 = M_2' \cup M_2''$ is a perfect matching of the graph $G[\,B_1 \cup I_1 \cup B_2 \cup I_2 \cup \{1', 4'\}]$ that contains edges e and f. Now, define perfect matchings M_e and N_e of G as follows:

$$M_e = M_2 \cup \{(0', 0''), (1'', 2''), (3'', 4'')\},$$

and

$$N_e = M_1 \cup \{(0', 0''), (2'', 3'')\}.$$

The perfect matching M_e (solid lines in Figure 8) has the property that $|M_e \cap C| = 1$, and shows that C is a separating cut of G. And N_e (dashed lines in Figure 8) has the property that $|N_e \cap C| = 3$, and shows that $\lambda(G) = 3$.

Consider the edges $f_1 = (1', 2'')$, and $f_2 = (4', 3'')$. These edges are in C, and are not incident with the barriers B_1 and B_2 of $G - e$. Both these edges f_1 and f_2 are easily seen to be b-removable in $G_1 - e = G_1$ and in $G_2 - e$. Also neither M_e nor N_e contains either of these edges. The perfect matching M_e shows that f_1 and f_2 are removable in G_1 and in G_2, and N_e is a robust cut of chracteristic three in both $G - f_1$ and $G - f_2$. It follows that f_1 and f_2 are both $(b+p)$-removable in G. ■

The following important corollaries are obvious consequences of the above Theorem.

Corollary 5.6 *Conjecture 1 implies Conjecture 2.*

Corollary 5.7 *In a solid brick, every b-removable edge is also $(b+p)$-removable.*

Note that the second Corollary may also be deduced directly from the monotonicity of the function $\lambda(G)$.

6 Good Cuts

In this section, we study the structure of bricks which have removable edges that are not b-removable. In particular, we establish that every such brick has good cuts. This is an important first step towards establishing the existence of robust cuts crucial for our inductive proof of the Main Theorem.

6.1 Non-b-removable Edges and Good Cuts

Let G be a matching covered graph. Two odd cuts C and D of G are said to be *matching-equivalent* if, for every perfect matching M of G, $|M \cap C| = |M \cap D|$. The following lemma, which can be proved using a simple counting argument, demonstrates the manner in which matching-equivalent cuts often manifest themselves.

Lemma 6.1 *Let G be a matching covered graph, and let $C = \nabla(S)$ and $D = \nabla(T)$ denote two odd cuts of G. Suppose that by contracting S to a single vertex s and contracting T to a single vertex t, we obtain a bipartite graph H. If s and t belong to different parts of the bipartition of H, then C and D are matching-equivalent in G.*

The following lemma demonstrates a situation which gives rise to good cuts in bricks which are not matching-equivalent.

Lemma 6.2 *Let C and D be two distinct nontrivial odd cuts of a brick G, let e denote a removable edge of G. Assume that, for every perfect matching M of G,*

$$|M \cap C| + |M \cap D| \leq 2 + 2|\{e\} \cap M|. \tag{8}$$

Then cuts C and D are both good, not matching-equivalent and of characteristic 3. Moreover, each of $C - e$ and $D - e$ is tight in $G - e$.

Proof: Let M denote any perfect matching of $G - e$. Thus M is a perfect matching of G that does not contain edge e. Cuts C and D are both odd. Thus, by (8), perfect matching M has just one edge in each of C and D. We conclude that cuts $C - e$ and $D - e$ are both tight in $G - e$.

By hypothesis, cuts C and D are both nontrivial, and G is a brick. Thus, neither C nor D is tight in G. Let M_C and M_D be two perfect matchings of G such that $|M_C \cap C| > 1$ and $|M_D \cap D| > 1$. By (8), edge e lies in both M_C and M_D. Moreover,

$$|M_C \cap C| = 3, |M_C \cap D| = 1, |M_D \cap D| = 3 \text{ and } |M_D \cap C| = 1.$$

We conclude that each of C and D is a good cut of characteristic 3 of G. Moreover, the two cuts are not matching-equivalent. ■

Lemma 6.3 *Let G be a brick, e a removable edge of G, B a nonspecial barrier of $G-e$. Let K_C and K_D denote two nontrivial components of $G - e - B$. Then cuts $C := \nabla_G(V(K_C))$ and $D := \nabla_G(V(K_D))$ satisfy the hypothesis of Lemma 6.2.*

Proof: Graph G is a brick and set B is a nontrivial barrier of $G - e$. Therefore, edge e has its ends in distinct components of $G - e - B$. A simple counting argument shows that for every perfect matching M of G,

$$|M \cap C| + |M \cap D| \leq 2 + 2|\{e\} \cap M|.$$

■

Theorem 6.4 *Let G be a brick, let e be a removable edge of G. If edge e is not b-removable then G has two good cuts, C and D, each of which has characteristic at most 5. Moreover, cuts C and D are not matching-equivalent and each of $C - e$ and $D - e$ is tight in $G - e$.*

Proof: We first deal with a few special cases.

Lemma 6.5 *Assume that graph $G - e$ has a 2-separation, say $\{u, v\}$. Let K denote a component of $G - \{u, v\}$. Then cuts $C := \nabla(V(K) \cup \{u\})$ and $D := \nabla(V(K) \cup \{v\})$ satisfy the hypothesis of Lemma 6.2.*

Proof: Since G has no 2-separation, edge e has one end in $V(K)$, the other in $E(G) \setminus (V(K) \cup \{u, v\})$. A simple counting argument then shows that for every perfect matching M of G,

$$|M \cap C| + |M \cap D| = |M \cap \nabla(u)| + |M \cap \nabla(v)| + 2|\{e\} \cap M|.$$

Inequality (8) then holds for the two nontrivial cuts C and D. ■

Lemma 6.6 *Let B denote a nontrivial special barrier of $G - e$. If the graph H, obtained from $G - e$ by contracting set $B \cup I(B)$ to a single vertex v, has a 2-separation, then G has two cuts, C and D, that satisfy the hypothesis of Lemma 6.2.*

Proof: Let S denote any 2-separation of H. If vertex v does not lie in S then S is a 2-separation of $G - e$ and the assertion holds by Lemma 6.5. We may thus assume that vertex v lies in the 2-separation. Let u denote the other vertex of the 2-separation.

Let K denote the only nontrivial component of $G - e - B$ and let $C_K := \nabla_G(V(K))$. For every perfect matching M of G,

$$|M \cap C_K| = 1 + 2|\{e\} \cap M|.$$

Let L denote a component of $H - \{u, v\}$. Let $C := \nabla(V(L) \cup \{u\})$ and $D := \nabla(V(L) \cup \{v\})$. For every perfect matching M of G,

$$|M \cap C| + |M \cap D| = |M \cap C_K| + |M \cap \nabla(u)|.$$

We conclude that C and D satisfy the requirements of Lemma 6.2. ■

In order to complete the proof of Theorem 6.4, we may assume that the hypotheses of Lemmas 6.3, 6.5 and 6.6 are not applicable.

By Lemma 6.3, we may assume that every barrier of $G - e$ is special. By Lemma 6.5, we may assume that the graph $G - e$ has no 2-separation. By the hypothesis of the Theorem, $b(G - e) > 1$. Thus, $G - e$ is not bicritical.

For every special barrier B of $G - e$, the set $B \cup I(B)$ is a shore of a tight cut of $G - e$; moreover, the contraction $(G - e)\{B \cup I(B)\}$ is bipartite. Therefore the other contraction, say H', of $G - e$ along the tight cut satisfies $b(H') > 1$. The hypothesis of Lemma 6.6 is not applicable. Therefore graph H' has no 2-separation. We conclude that H' is not bicritical. This conclusion holds for every special barrier B of $G - e$. By the Three Case Lemma 5.2, graph $G - e$ has two nontrivial barriers, B' and B'', such that the sets $B' \cup I(B')$ and $B'' \cup I(B'')$ are disjoint. The graph H, obtained from $G - e$ by contracting each of the sets $B' \cup I(B')$ and $B'' \cup I(B'')$ to single vertices, v' and v'', respectively, is bicritical.

The contractions are done along tight cuts of $G - e$, contracting the shores that span bipartite graphs. Thus, $b(H) > 1$, and so, since H is bicritical, it has a 2-separation, say, $S = \{v', v''\}$. If none of v' and v'' lies in S then S is a 2-separation of $G - e$ itself, and

Lemma 6.5 would be applicable. If only one of v' and v'' lies in S, then Lemma 6.6 would be applicable. We conclude that $S = \{v', v''\}$.

Edge e has its ends in $I(B')$ and $I(B'')$. Thus, for every perfect matching M of G,

$$\begin{array}{rcl} |M \cap \nabla(v')| & = & 1 + 2|\{e\} \cap M| \text{ and} \\ |M \cap \nabla(v'')| & = & 1 + 2|\{e\} \cap M|. \end{array}$$

Let K denote a component of $H - \{v', v''\}$. Let $C := \nabla(V(K) \cup \{v'\})$, $D := \nabla(V(K) \cup \{v''\})$. Then, for every perfect matching M of G,

$$|M \cap C| + |M \cap D| = |M \cap \nabla(v')| + |M \cap \nabla(v'')|.$$

Therefore,

$$|M \cap C| + |M \cap D| = 2 + 4|\{e\} \cap M|. \tag{9}$$

Cuts C and D are both nontrivial and distinct. If M is any perfect matching of $G - e$, then the above equation implies that $|M \cap C| = |M \cap D| = 1$. Therefore $C - e$ and $D - e$ are tight in $G - e$. If C and D are separating cuts of G, then they are good cuts of G. It is easy to see that C and D are not matching-equivalent and are of characteristic at most 5, and hence are a pair of cuts of the required type.

We may thus assume that one of C and D, say C, is not a separating cut of G. Since $C - e$ is a tight cut of $G - e$, this implies that edge e is not admissible[3] in at least one C-contraction of G, say in $G_1 := G\{X; \overline{x}\}$, where $X = V(K) \cup \{v'\}$. But cut $C - e$ is tight in $G - e$. Therefore, graph $G_1 - e$ is matching covered. Thus, there exists a barrier B of G_1 such that e is the only edge of G_1 that has both ends in B. Since G is a brick, set B is not a barrier of G. Thus, vertex $\overline{x}$ lies in B.

Since H is bicritical, the cut $C - e$ is strictly separating in $G - e$. We conclude that at least one component of $G_1 - B$, say J, is nontrivial. Let $C' := \nabla_G(V(J))$. Clearly, for every perfect matching M of G,

$$|M \cap C'| \leq |M \cap C| - 2|\{e\} \cap M|. \tag{10}$$

From Equations (9) and (10), we now have:

$$|M \cap C'| + |M \cap D| \leq 2 + 2|\{e\} \cap M|.$$

Finally, cuts C' and D are distinct, for cuts C and D cross whereas cuts C and C' do not cross. The assertion follows, by Lemma 6.2. ■

The following corollary is an immediate consequence of the above Theorem.

Corollary 6.7 *In a solid brick, every removable edge is also b-removable.*

[3] This situation is similar to the one we encountered in the proof of Theorem 2.21.

7 Robust Cuts

Suppose that G is a brick and e is a removable edge of G. In section 4 we have seen that the existence of a robust cut in $G-e$ is a sufficient condition for e to be b-removable in G. If a brick is solid, then any removable edge in the brick is also b-removable in it. So, we only need to be concerned with nonsolid bricks. Although every nonsolid brick, by definition, has good cuts, it is not at all clear that nonsolid bricks have robust cuts. In this section, we show that every nonsolid brick G has robust cuts whose chracteristic is equal to that of G. We also establish several useful properties of robust cuts which are crucial in the proof of the Main Theorem.

7.1 Existence of Robust Cuts in Nonsolid Bricks

A good cut in a brick need not be a robust cut. However, if a brick has a good cut, then it also has a robust cut. This section is dedicated to proving this result. Indeed, it has a robust cut of characteristic equal to that of the brick.

Given a good cut in a brick, we define a family of good cuts related to the given good cut, and show that each member of this family is a robust cut. Underlying this proof of existence is a natural constructive procedure which can be used to find a robust cut starting from a good cut. The idea is as follows: Suppose that G is a brick and that C is a good cut in G. If C is not a robust cut, then one of the C-contractions of G, say G_1, has more than one brick. Then, G_1 has either a 2-separation cut associated with a 2-separation $\{u, v\}$ of G_1, or a barrier cut associated with a barrier B of G_1. Each of these tight cuts of G_1 is good in G. Moreover, one of them can be shown to be 'richer' in terms of the intersections of perfect matchings of G. If the new cut we find is not robust, then this procedure may be repeated, and eventually a robust cut may be obtained.

Recall that for any cut C of G, we let $\mathcal{M}_i(C)$ denote the set of all perfect matchings of G which have exactly i edges in common with C and we let $\mathcal{M}_{\leq i}(C)$ denote the set of all perfect matchings of G which have at most i edges in common with C.

Let G be a brick, M_0 a perfect matching of G, C_0 an M_0-good cut of G. A cut D_0 is *at least as M_0-rich as* C_0 (writen as $D_0 \succeq_{M_0} C_0$) if it satisfies the following properties:

(i) cut D_0 is M_0-good, and

(ii) for each positive odd integer j, $\mathcal{M}_j(C_0) \subseteq \mathcal{M}_{\leq j}(D_0)$.

In addition, if, for some positive odd integer j, $\mathcal{M}_j(C_0) \neq \mathcal{M}_j(D_0)$ then we say that D_0 is *strictly M_0-richer than* C_0 and we indicate that fact by writing $D_0 \succ_{M_0} C_0$. Cut D_0 is *maximal* with respect to relation $\succeq_{M_0}$ if, for no M_0-good cut D_1 we have that $D_1 \succ_{M_0} D_0$. Whenever M_0 is understood, we omit the subscript and simply write $\succeq$ and $\succ$.

Lemma 7.1 *Let M_0 denote a perfect matching of a brick G and let C be an M_0-good cut of G. Let $\mathcal{D}$ denote a collection of k nontrivial odd cuts of G such that for every perfect matching M of G*

$$\sum_{D \in \mathcal{D}} |M \cap D| = |M \cap C| + k - 1. \tag{11}$$

If $k > 1$ then C is not maximal with respect to $\succeq_{M_0}$.

Proof: We first show that each D in $\mathcal{D}$ is separating in G. For this, let e denote any edge of G. Observe that C, a good cut, is separating, therefore there exists a perfect matching M_e of G that contains e and just one edge in C. From equation (11) it follows that M_e has just one edge in each cut in $\mathcal{D}$. Since this conclusion holds for each edge e in G and each cut in $\mathcal{D}$, it follows that each cut in $\mathcal{D}$ is separating in G.

We now show that at least one of the cuts in $\mathcal{D}$ is M_0-good. For this, assume, to the contrary, that each cut in $\mathcal{D}$ contains just one edge in M_0. From Equation (11) it follows that C also has just one edge in M_0, a contradiction. Thus, at least one cut in $\mathcal{D}$, D_0, say, has more than one edge in M_0. Since every cut of $\mathcal{D}$ is separating in G, it follows that D_0 is M_0-good.

The desired conclusion is that C not maximal with respect to $\succeq_{M_0}$. This would follow if we can show that $D_0 \succ C$. We now proceed to show this.

Let j be any positive odd integer, let M denote any perfect matching in $\mathcal{M}_j(C)$. From Equation (11) it follows that each cut in $\mathcal{D}$ has at most j edges in M. Since this conclusion holds for each M in $\mathcal{M}_j(C)$, it follows that $\mathcal{M}_j(C) \subseteq \mathcal{M}_{\leq j}(D_0)$. This conclusion holds for each positive odd integer j, therefore $D_0 \succeq C$.

To complete the proof, recall that $k > 1$ and that each cut in $\mathcal{D}$ is nontrivial in G. Let D_1 denote a cut in $\mathcal{D}$ distinct from D_0. Since G is a brick, there exists a perfect matching M_1 of G that contains more than one edge in D_1. Let $j := |M_1 \cap D_0|$. From equation (11) it follows that $|M_1 \cap C| > j$. Therefore, $D_0 \succ C$. As asserted, C is not maximal with respect to $\succeq$. ■

Lemma 7.2 *Let G be a brick. Let $C := \nabla(X)$ denote an M_0-good cut of G that is maximal with respect to $\succeq_{M_0}$. Then $G_1 := G\{X; \overline{x}\}$ is a near-brick.*

Proof: We prove the assertion by induction on X.

We begin the proof by showing that G_1 has no 2-separations. For this, assume, to the contrary, that G_1 has a 2-separation, say, $\{u, v\}$. Since G is a brick, $\overline{x}$ is one of u and v. Adjust notation so that $\overline{x} = u$. Let K be an even component of $G_1 - \{u, v\}$, let $D_1 := \nabla(V(K) \cup \{u\})$ and $D_2 := \nabla(V(K) \cup \{v\})$ denote two cuts associated with the 2-separation. For any perfect matching M of G,

$$|M \cap D_1| + |M \cap D_2| = |M \cap \nabla(\overline{x})| + |M \cap \nabla(v)|.$$

Since $\nabla(\overline{x}) = C$ and v is a vertex of G,

$$|M \cap D_1| + |M \cap D_2| = |M \cap C| + 1.$$

By Lemma 7.1, the maximality of C is contradicted. Indeed, G_1 is free of 2-separations.

If G_1 is free of nontrivial barriers then G_1 is a brick and the assertion holds trivially. So suppose that G_1 has a nontrivial barrier, say B. We now proceed to show that B is special, that is, precisely one component of $G_1 - B$ is nontrivial. Graph G is a brick, therefore free of nontrivial barriers. Consequently, $\overline{x} \in B$. Let $\mathcal{D}$ denote the set of cuts associated with nontrivial components of $G_1 - B$. Let k denote $|\mathcal{D}|$. Cut C is not tight, therefore G_1 is not bipartite, whence $k \geq 1$. Let M denote any perfect matching of G. A simple counting argument shows that

$$\sum_{D \in \mathcal{D}} |\, M \cap D\,| + (|B| - k) = |\, M \cap C\,| + (|B| - 1),$$

whence equation (11) in the assertion of Lemma 7.1 holds. The maximality of C thus implies that $k = 1$.

Denote by Y the vertex set of the only nontrivial component of $G_1 - B$, by D the cut $\nabla(Y)$. Then $|\, M \cap C\,| = |\, M \cap D\,|$ for each perfect matching M of G. Therefore D is also maximal with respect to $\succeq_{M_0}$. But Y is a proper subset of X. By induction hypothesis, $G\{Y; \overline{y}\}$ is a near-brick. Since Y is the vertex set of the only nontrivial component of $G_1 - B$, G_1 is also a near-brick. ■

Corollary 7.3 *Let G be a brick, C an M_0-good cut of G that is maximal with respect to $\succeq_{M_0}$. Then C is M_0-robust.*

For any good cut C in a brick G, let M_0 denote a perfect matching of G such that $|\, M_0 \cap C\,| = \lambda(C)$, let D be a cut that is at least as M_0-rich as C and, subject to these conditions, maximal with respect to $\succeq_{M_0}$. We then say that D is *a cut M_0-induced by C*.

Theorem 7.4 *Let G be a brick, D a cut M_0-induced by a good cut C such that $|\, M_0 \cap C\,| = \lambda(C)$. Then D is M_0-robust and $\lambda(D) \leq \lambda(C)$.*

Proof: We note that the cut D, by definition, is maximal with respect to $\succeq_{M_0}$. Therefore, by Corollary 7.3 D is M_0-robust. Moreover, $\mathcal{M}_{\lambda(C)}(C) \subseteq \mathcal{M}_{\leq \lambda(C)}(D)$. In particular, $|\, M_0 \cap D\,| \leq \lambda(C)$. Therefore, $\lambda(D) \leq \lambda(C)$. ■

Corollary 7.5 *Every nonsolid brick G has a robust cut of characteristic $\lambda(G)$.*

7.2 The Case $b(G - e) = 2$

Let G be a matching covered graph, and let $C = \nabla(X)$ and $D = \nabla(Y)$ be two tight cuts of G. Then, C and D are said to be a *barrier cut pair* of G if there exists a barrier B of G such that $G - B$ has precisely two nontrivial components K and L, and $C = \nabla(V(K))$ and $D = \nabla(V(L))$. And, C and D are said to be an *essentially a 2-separation pair* of G if they cross and the graphs $G\{\, X \cap Y\,\}$ and $G\{\, \overline{X} \cap \overline{Y}\,\}$ are both bipartite (where the notation has been adjusted so that both $X \cap Y$ and $\overline{X} \cap \overline{Y}$ are odd).

Theorem 7.6 *Let G be a brick, let e be a removable edge of G such that $b(G - e) = 2$, and let C be a strictly separating cut of G. If $C - e$ is tight in $G - e$ then there exists a strictly separating cut D of G such that $D - e$ is tight in $G - e$, and $C - e$ and $D - e$ are either a barrier cut pair of $G - e$ or essentially a 2-separation pair of $G - e$.*

<u>Proof:</u> By Theorem 6.4, graph G has two good cuts D_1 and D_2 that are not matching-equivalent and such that both $D_1 - e$ and $D_2 - e$ are tight in $G - e$. Since D_1 and D_2 are not matching-equivalent then either C and D_1 are not matching-equivalent or C and D_2 are not matching-equivalent. Let D be one of D_1 and D_2 such that C and D are not matching-equivalent.

By Lemma 2.16, cut D is strictly separating in G. Thus, cuts C and D have the same properties: both C and D are strictly separating in G and both $C - e$ and $D - e$ are tight in $G - e$.

Adjust notation, by interchanging X with $\overline{X}$ if necessary, such that both $X \cap Y$ and $\overline{X} \cap \overline{Y}$ are odd. Let

- $G_1 := G\{X; \overline{x}\}$,
- $G_2 := G\{\overline{X}; x\}$,
- $H_1 := G\{Y; \overline{y}\}$, and
- $H_2 := G\{\overline{Y}; y\}$.

Observe that each of $C - e$ and $D - e$ is strictly separating in $G - e$, by Lemma 2.18. Also, observe that each of $G_1 - e$, $G_2 - e$, $H_1 - e$ and $H_2 - e$ is a near-brick, because G is a brick, $b(G - e) = 2$ and each of $C - e$ and $D - e$ is strictly separating and tight in $G - e$.

Lemma 7.7 *If cuts C and D do not cross then $C - e$ and $D - e$ are a barrier cut pair of $G - e$.*

<u>Proof:</u> Adjust notation, by interchanging X with $\overline{X}$ and Y with $\overline{Y}$, if necessary, so that D is a cut of G_1. Thus, $Y \subset X$. By hypothesis, cuts $C - e$ and $D - e$ are tight in $G - e$. Thus, cut $D - e$ is tight in $G_1 - e$. Cut D is nontrivial and distinct from C, therefore D is nontrivial in G_1. We conclude that cut $D - e$ is nontrivial and tight in $G - e$. We have seen that $G_1 - e$ is a near-brick. Thus, one of the $(D - e)$-contractions of $G_1 - e$ is bipartite. Also, as $D - e$ is strongly separating in $G - e$, $(G_1 - e)\{Y\} = (G - e)\{Y\}$ is not bipartite. Thus, $L := (G_1 - e)\{(X \cap \overline{Y}) \cup \{\overline{x}\}; y\}$ is bipartite. Let B denote the part of the bipartition of L that does <u>not</u> contain vertex y.

If vertex $\overline{x}$ lies in B then cuts C and D are matching-equivalent, a contradiction. We conclude that neither vertex $\overline{x}$ nor y lies in B, whence B is a barrier of $G - e$ such that both $C - e$ and $D - e$ are (barrier) cuts associated with B. ■

Lemma 7.8 *If cuts C and D cross then $C - e$ and $D - e$ are essentially a 2-separation pair of $G - e$.*

<u>Proof:</u> Let

- $I := \nabla(X \cap Y)$,
- $U := \nabla(\overline{X} \cap \overline{Y})$,
- $G_{11} := G\{X \cap Y\,; v_{11}\}$,

- $G_{22} := G\{\overline{X} \cap \overline{Y}\,; v_{22}\}$,
- $G_{12} := G_1\{X \cap \overline{Y} \cup \{\overline{x}\}; v_{12}\}$, $H_{21} := H_2\{X \cap \overline{Y} \cup \{y\}; w_{21}\}$,
- $G_{21} := G_2\{\overline{X} \cap Y \cup \{x\}; v_{21}\}$, and $H_{12} := H_1\{\overline{X} \cap Y \cup \{\overline{y}\}; w_{12}\}$.

See Figure 9 for an illustration.

Observe that since each of $C-e$ and $D-e$ is tight in $G-e$, the cuts $I-e$ and $U-e$ are tight in $G-e$. Moreover, no edge of $G-e$ joins a vertex of $X \cap \overline{Y}$ to a vertex of $\overline{X} \cap Y$. It is now easy to see that, up to multiple edges, graphs $G_{12}-e$ and $H_{21}-e$ are equal. Similarly, up to multiple edges, graphs $G_{21}-e$ and $H_{12}-e$ are equal.

In order to prove that $C-e$ and $D-e$ are essentially 2-separation cuts of $G-e$, it suffices to show that $G_{11}-e$ and $G_{22}-e$ are both bipartite.

For this, assume the contrary. If $G_{11}-e$ is not bipartite, then $G_{12}-e$ is bipartite, because those two graphs are the $(I-e)$-contractions of G_1-e, cut $I-e$ is tight in G_1-e and G_1-e is a near-brick. The same reasoning my be repeated, replacing G_{12} by G_{21} and G_1 by G_2, thereby showing that $G_{21}-e$ is also bipartite. Likewise, if $G_{22}-e$ is not bipartite then both $G_{12}-e$ and $G_{21}-e$ are bipartite.

We conclude that the hypothesis that one of $G_{11}-e$ and $G_{22}-e$ is not bipartite implies that both $G_{12}-e$ and $G_{21}-e$ are bipartite.

Observe now that clearly vertices $\overline{x}$ and v_{12} are adjacent in $G_{12}-e$. Therefore they lie in distinct parts of the bipartition of $G_{12}-e$. If edge e has both ends in the same part that contains vertex $\overline{x}$, then, for every perfect matching M of G that contains edge e, $|M \cap C| = 1 + 2|M \cap I| \geq 3$. Therefore, C is not separating in G. Likewise, if both ends of e lie in the same part that contains v_{12}, then, recalling that up to multiple edges $G_{12}-e$ is the graph $H_{21}-e$, it follows that, for every perfect matching M of G that contains edge e, $|M \cap D| = 1 + 2|M \cap U| \geq 3$, whence D is not separating in G. We conclude that graph G_{12} is bipartite. Since vertices $\overline{x}$ and v_{12} lie in distinct parts of the bipartition of G_{12}, the cuts C and I are matching-equivalent in G. (Also, the cuts D and U are matching-equivalent.)

Since graph $G_{21}-e$ is bipartite, a similar reasoning leads to the conclusion that G_{21} is bipartite and cuts I and D are matching-equivalent in G. (Also cuts C and U are matching-equivalent in G.)

Since C and I are matching-equivalent, and I and D are matching-equivalent, we conclude that cuts C and D are matching-equivalent, a contradiction. Thus, indeed, both $G_{11}-e$ and $G_{22}-e$ are bipartite. This completes the proof of Lemma 7.8.

The proof of Theorem 7.6 now follows from Lemma 7.8. ■

8 Proof of the Main Theorem

Theorem 8.1 (The Main Theorem) *Let G be any brick different from K_4 and $\overline{C_6}$. Then G has a $(b+p)$-removable edge. Furthermore, if G is nonsolid, and its characteristic is greater than three, then the underlying simple graph of G is the Petersen graph.*

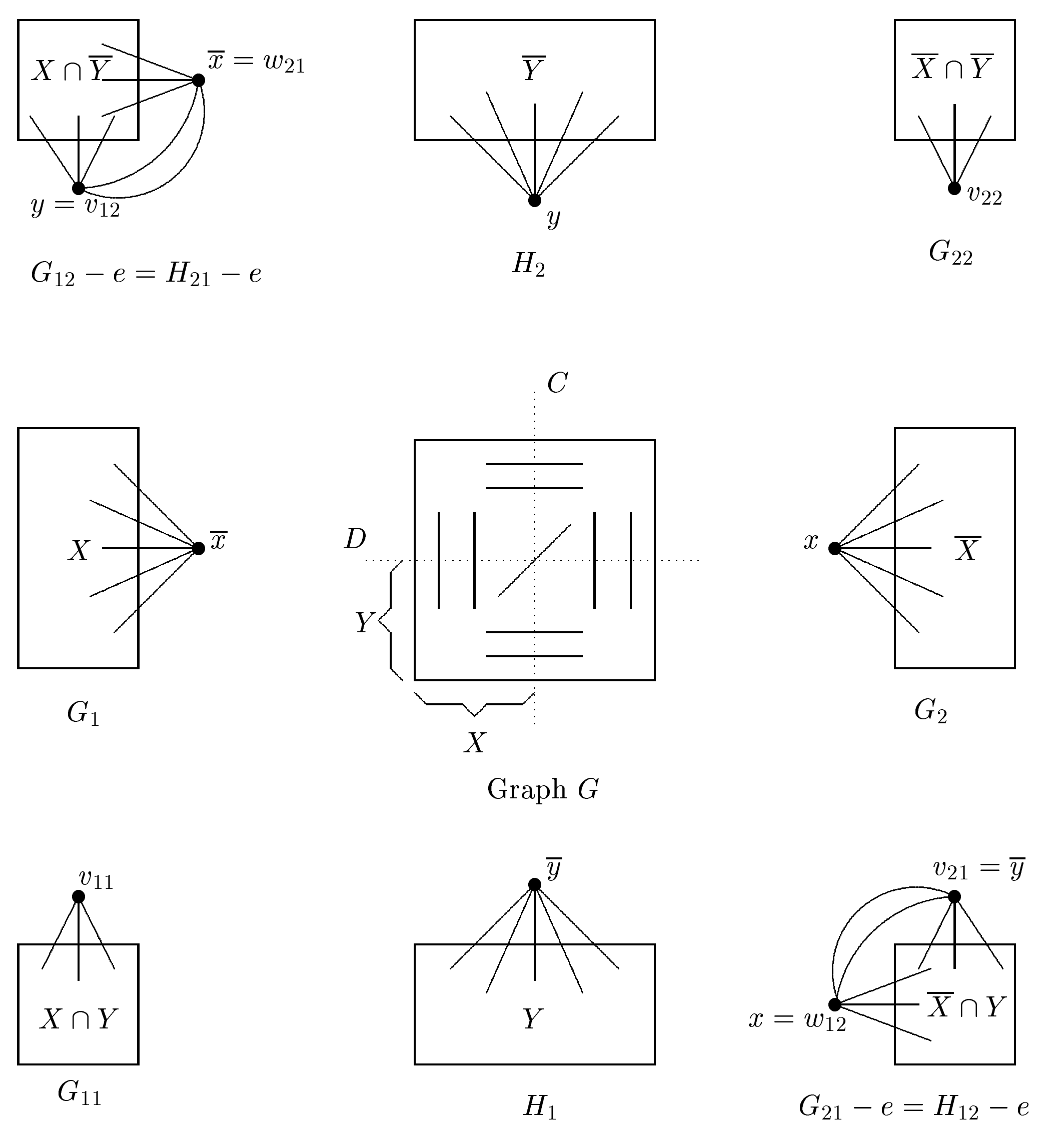


Figure 9: Illustration for the case in which $C - e$ and $D - e$ are 2-separation pair

Proof: The proof is by induction on $|V(G)| + |E(G)|$. We begin with two simple cases.

Case: G has multiple edges: Let e denote a multiple edge of G. Clearly, edge e is removable. Moreover, $G - e$ is a brick and its underlying simple graph is the same as that of G. Therefore, $b(G) = b(G - e) = 1$, $p(G - e) = p(G)$ and $\lambda(G - e) = \lambda(G)$. Thus, edge e is $(b + p)$-removable.

If $3 < \lambda(G) < \infty$, then $3 < \lambda(G - e) < \infty$; in this case, by induction hypothesis, the underlying simple graph of $G - e$ is the Petersen graph, whence so too is the underlying simple graph of G. We conclude that the assertion holds, by induction, if G has multiple edges.

Case: G is solid: Assume that G is solid. By Theorem 3.5, graph G has a removable edge, say e. By Corollary 6.7, it follows that e is b-removable. Now, by Corollary 5.7, it follows that e is also $(b + p)$-removable. Therefore the assertion holds for solid bricks.

We may thus assume that brick G is simple and nonsolid. The proof would be complete if we can show that **either** $\lambda(G) = 3$ and G has a $(b + p)$-removable edge, **or** G is the Petersen graph (see Theorem 3.4). But this follows from the Main Lemma proved in the next section. ■

8.1 The Main Lemma

Lemma 8.2 (The Main Lemma) *Let G be any nonsolid simple brick different from $\overline{C}_6$. Suppose that G satisfies the following conditions:*

(I) If G' is any brick with $|V(G')|+|E(G')| < |V(G)|+|E(G)|$, then G' satisfies the statement of the Theorem 8.1, and

(II) Either $\lambda(G) > 3$ or G has no b-removable edges.

Then G is the Petersen graph.

Proof: Let G be a brick satisfying the hypotheses of the Lemma. As a first step towards unraveling the structure of G, we shall prove the following.

Lemma 8.3 *The brick G has a robust cut $C = \nabla(X)$ such that the C-contraction $G_1 = G\{X, \overline{x}\}$ of G is an odd wheel with $\overline{x}$ as its hub, except possibly for multiple edges incident with the hub $\overline{x}$.*

Since G is nonsolid, observe that, by Corollary 7.5, G has robust cuts of characteristic $\lambda(G)$. We shall first establish several lemmas involving arbitrary $\lambda(G)$-robust cuts of G. Then we shall use these lemmas in proving Lemma 8.3 as well as in our further analysis of the structure of G.

8.2 Preliminary Lemmas

Let G be a nonsolid brick, and let $C := \nabla(X)$ be any $\lambda(G)$-robust cut of G. Let $G_1 := G\{X; \overline{x}\}$ and $G_2 := G\{\overline{X}; x\}$. Let M_0 denote a perfect matching of G such that $|M_0 \cap C| = \lambda(G)$. We shall adopt the convention that if an edge e of G does not lie in G_1 (respectively, in G_2) then e is b-removable in G_1 (G_2). The following lemmas illustrate some of the basic ideas used in the proof of the main theorem.

8.2.1 The nonexistence of a certifying robust cut

Lemma 8.4 *Let $C := \nabla(X)$ be any $\lambda(G)$-robust cut of G, and let $G_1 := G\{X; \overline{x}\}$ and $G_2 := G\{\overline{X}; x\}$ be the two C-contractions of G. If e is an edge that is b-removable in both G_1 and G_2, then the cut $C - e$ is tight in $G - e$ (equivalently, $C - e$ cannot be a good cut of $G - e$).*

Proof: By hypothesis, e is b-removable in both G_1 and G_2. Thus, both $G_1 - e$ and $G_2 - e$ are matching covered, and $b(G_1 - e) = 1$ and $b(G_2 - e) = 1$. Also, it follows that $G - e$ is matching covered and $C - e$ is separating in $G - e$. Suppose that $C - e$ is not tight in $G - e$. Then, $C - e$ is good in $G - e$. By the subadditivity of function b, it follows that $G - e$ is a near-brick. This, in turn, implies that the edge e is b-removable in G.

Also, if $C - e$ is good in $G - e$, then $G - e$ is not solid. By the induction hypothesis, either $\lambda(G - e) = 3$ or the underlying simple graph of the brick of $G - e$ is the Petersen graph.

Consider first the case in which $\lambda(G - e) = 3$. By the monotonicity of the function λ, it follows that $\lambda(G) = 3$. This, together with the fact that e is b-removable in G, contradicts the second hypothesis.

Now consider the case in which the underlying simple graph of the brick of $G - e$ is the Petersen graph. Since G is simple, it is not the Petersen graph. Therefore, edge e is b-removable, but not $(b + p)$-removable in G. By Theorem 5.4, it follows that $\lambda(G) = 3$. Thus, again, we have a contradiction. ■

Corollary 8.5 *If C is a robust cut of G as in the above Lemma, and e is any edge of G that is b-removable in both G_1 and G_2, then e belongs to every perfect matching of G which intersects C in more than one edge.*

8.2.2 The shore size implication lemmas

Most of the remaining lemmas in this section establish analogous statements applicable to G_1 and G_2, respectively. These lemmas include two separate statements, one for G_1 and one for G_2, but we prove only the statement that pertains to G_1. Similar arguments can be used to prove the statements that apply to G_2.

Lemma 8.6 *Let e be a b-removable edge of G_1 (respectively G_2) that does not lie in C. If cut C is tight in $G - e$ then G has a 3-robust cut such that one of its shores has fewer vertices than the shore X (shore $\overline{X}$) of C.*

Proof: Consider G_1. Suppose that e is a b-removable edge of G_1 that does not lie in C and that C is a tight cut of $G-e$. Then, Clearly, edge e is removable in each of G_1 and G_2. Therefore it is removable in G. As C is tight in $G-e$, we have $b(G-e) = b(G_1-e)+b(G_2) = 2$. By Theorem 7.6, there exists a strictly separating cut D of G such that $D-e$ is tight in $G-e$ and cuts $C-e$ and $D-e$ are either a barrier cut pair of $G-e$ or essentially a 2-separation pair of $G-e$.

Suppose that $C = C-e$ and $D-e$ are a barrier cut pair of $G-e$. Let B denote the corresponding barrier of $G-e$. Since e has both ends in X, it follows that $G[\overline{X}]$ is one of the nontrivial components of $G-e-B$. Thus, D is a nontrivial cut of G_1. Moreover, D is a 3-robust cut of G. One of its shores is a proper subset of X.

Suppose now that $C-e$ and $D-e$ are essentially a 2-separation pair of $G-e$. Let Y be a shore of D such that $I := \nabla(X \cap Y)$ and $U := \nabla(\overline{X} \cap \overline{Y})$ are both odd cuts. Let $G_{11} := G\{X \cap Y\}$, and let $G_{22} := G\{\overline{X} \cap \overline{Y}\}$. Since e has both ends in X, modularity relates C, D, I and U. Therefore, $I-e$ and $U-e$ are both tight in $G-e$. Moreover, $G_{22}-e$ is bipartite and equal to G_{22}. Therefore, U is tight in G. Since G is a brick, set $\overline{X} \cap \overline{Y}$ is a singleton. On the other hand, for every perfect matching M of G, $|M \cap I| \leq 3$, with equality only if edge e lies in M. Therefore, both C and D, which are strictly separating cuts in G, have characteristic three. Since $b(G-e) = 2$, cut D is robust. Since $\overline{X} \cap \overline{Y}$ is a singleton, it follows that $X \cap Y$ cannot be a singleton, else G would have a 2-separation pair. Therefore, $\left|\overline{Y}\right| < |X|$, and D is a 3-robust cut one of whose shores is a proper subset of X. ■

On tight cuts of G_1 and G_2

Lemma 8.7 *Let D denote a nontrivial tight cut of G_1, and let Y denote the shore of D in G_1 that does not contain the vertex $\overline{x}$. Then, the graph $H_1 = G_1\{\overline{Y}; y\}$ is bipartite, and $\overline{x}$ and y lie in distinct parts of H_1.*

(Similarly, if D is a nontrivial tight cut of G_2, and Y is the shore of D in G_2 that does not contain the vertex x, then $H_1 = G_2\{\overline{Y}; y\}$ is bipartite, and x and y lie in distinct parts of H_1.)

Proof: Let $H_2 := G_1\{Y; \overline{y}\}$ denote the other D-contraction of G_1. Since G_1 is a near-brick, precisely one of H_1 and H_2 is bipartite. Graph H_2 cannot be bipartite, else the part of H_2 not containing vertex $\overline{y}$ is a nontrivial barrier of G. Thus H_1 is bipartite. All vertices of $V(H_1) \setminus \{\overline{x}, y\}$ are vertices of G. Thus $\overline{x}$ and y do not lie in the same part of H_1, else the other part would be a nontrivial barrier of G. As asserted, $\overline{x}$ and y lie in distinct parts of H_1.

■

Corollary 8.8 *If $D = \nabla(Y)$ is a nontrivial tight cut of G_1 with $\overline{x} \notin Y$, then $\nabla(Y)$ is a robust cut of G with $|Y| < |X|$.*

(Similarly, if $D = \nabla(Y)$ is a nontrivial tight cut of G_2 with $x \notin Y$, then $\nabla(Y)$ is a robust cut of G with $|Y| < |\overline{X}|$.)

Proof: By the above Lemma (8.7), $\overline{x}$ and y lie in distinct parts of $H_1 = G_1\{\overline{Y}; y\}$. It follows that C and D are matching-equivalent. Since C is a robust cut, we conclude that D is also a robust cut. ■

The next two preliminary lemmas involve the relation of dependence on the edge set of a matching covered graph introduced by Carvalho and Lucchesi [2], and further explored by Carvalho, Lucchesi, and Murty in [3]. We review the relevant properties of this relation.

8.2.3 A dependence relation

Let G be a matching covered graph, and let e and f be any two edges of G. Then we say e *depends* on f, or e *implies* f, if every perfect matching that contains e also contains f. (Equivalently, e depends on f if e is not admissible in $G - f$.)

We write $e \Rightarrow f$ to indicate that e depends on f. Clearly, $\Rightarrow$ is reflexive and transitive. It is convenient to visualize $\Rightarrow$ in terms of the digraph it defines on the set of edges of G.

We say that two edges e and f are *mutually dependent* if $e \Rightarrow f$, and $f \Rightarrow e$. In this case we write $e \Leftrightarrow f$. Clearly $\Leftrightarrow$ is an equivalence relation on $E(G)$. By identifying the vertices in the equivalence classes in the digraph representing dependence relation ($\Rightarrow$) on $E(G)$, we obtain the digraph $D(G)$ representing the dependence relation ($\Rightarrow$) on the set of equivalence classes. This digraph is clearly acyclic. The sources in this digraph are called *minimal classes*. (See Figure 10 for an illustration. We refer to the graph used in this illustration as R_8; it is a member of an interesting family of cubic matching covered graphs.) Let e be an edge of G, and let $[e]$ denote the equivalence class containing the edge e. Then a minimal class in the component of $D(G)$ to which $[e]$ belongs is said to be *induced* by e.

If Q is a minimal class, then every edge not in Q is admissible in $G - Q$. Thus, if $G - Q$ is connected, then $G - Q$ is matching covered. The equivalence classes in a brick have the following attractive properties:

Lemma 8.9 (See [3]) *Let G be a brick and let Q be an equivalence class of G. Then, $|Q| \leq 2$, and $G - Q$ is matching covered. Moreover, if $|Q| = 2$, say $Q = \{e, f\}$, then $G - e - f$ is bipartite, with both parts of the bipartition having equal cardinality; and both ends of e lie in one part of the bipartition, and both ends of f lie in the other part of the bipartition.* ■

With the aid of the above discussion, we can now state and prove the remaining two lemmas of this section.

8.2.4 The removable doubleton lemma

Lemma 8.10 *Suppose that G_1 is a brick, and that G_1 has a removable doubleton $Q = \{e, f\}$. Then $\lambda(G) = 3$, one of the edges of Q, say e, does not lie in $M_0 \cup C$, the other, say f, lies in M_0. Moreover, if f is b-removable in G_2, then e is a b-removable edge of G.*

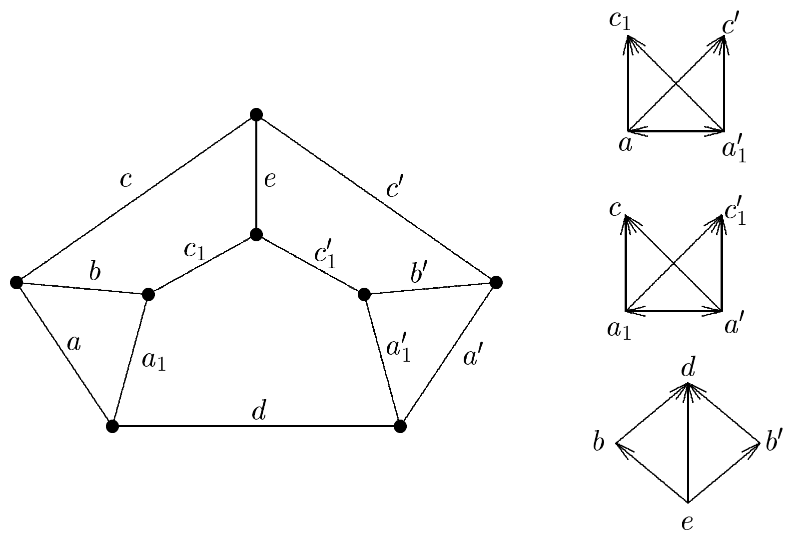


Figure 10: Dependence Digraph

(Similarly, suppose that G_2 is a brick, and that G_2 has a removable doubleton $Q = \{e, f\}$. Then $\lambda(G) = 3$, one of the edges of Q, say e, does not lie in $M_0 \cup C$, the other, say f, lies in M_0. Moreover, if f is b-removable in G_1, then e is a b-removable edge of G.)

Proof: First, let us consider G_1. Suppose that G_1 is a brick and that Q is a removable doubleton in G_1. Then $G_1 - Q$ has a bipartition, say (A, B). Adjust notation so that vertex $\overline{x}$ lies in B. Let e denote the edge of Q having both ends in A, let f denote the other edge of Q, that has both ends in B. A simple counting argument shows that

$$|\,M_0 \cap \nabla(\overline{x})\,| = 3,\ f \in M_0,\ e \notin M_0.$$

So the first part of the assertion of the Lemma holds.

Now suppose that f is b-removable in G_2. As the doubleton Q is removable in G_1, we have that graphs $G_1 - Q$ and $G_2 - Q$ are both matching covered. Thus, $G - Q$ is also matching covered. Moreover, $G_1 - Q$ is bipartite, whence $C - f$ is tight in $G - Q$. Since f is b-removable in G_2, we conclude that $b(G - Q) = b(G_2 - f) = 1$.

Observe that graph $G - e$ is matching covered, because $G - Q$ is matching covered and M_0 is a perfect matching of G that contains edge f but not edge e. By the monotonicity of b, we have that

$$1 = b(G - Q) \geq b(G - e) \geq b(G) = 1.$$

Therefore, edge e is b-removable in G.

Analogous arguments show that if G_2 is a brick, and Q is a removable doubleton of G_2, then $\lambda(G) = 3$, one of the edges of Q, say, e, does not lie in $M_0 \cup C$, the other, say, f, lies in M_0; and furthermore, if f is b-removable in G_1, then G has a b-removable edge. ∎

Remark: If G_1 (G_2) is a brick, and G_1 (G_2) has a removable doubleton Q disjoint from C, then $\lambda(G) = 3$, and G has a b-removable edge.

8.2.5 The Lemma on odd wheels

The following simple lemma will play an important role in the proof of the Lemma 8.3, and finally in showing that G is the Petersen graph.

Lemma 8.11 (on odd wheels) *Let G be a simple brick, and let v be a specified vertex of G. Then at least one of the properties holds:*

(i) G is an odd wheel, or

(ii) there is a minimal class Q of G such that $Q \cap \nabla(v) = \emptyset$, or

(iii) G has characteristic 3.

Proof: Consider the graph $G - v$. Since G is a brick, $G - v$ is critical.

Case (i): $G - v$ is an odd circuit. In this case, G is an odd wheel.

If $G - v$ is not an odd circuit, by a theorem of Lovász (Theorem 5.5.1, page 196, [5]), we can write $G - v$ as

$$G - v = H + W,$$

where H is a critical subgraph of $G - v$ and W is an odd path with its ends in H but internally disjoint from H.

Case (ii): $|E(W)| = 1$. Say $E(W) = \{e\}$. In this case, $H = (G - v) - e$. Since H is critical, any edge of $\nabla(v)$ is in a perfect matching in $G - e$. Therefore no edge of $\nabla(v)$ implies e in G. Let Q denote a minimal class induced by e. Then, $Q \cap \nabla(v) = \emptyset$, and $G - Q$ is matching covered.

Case (iii): $|E(W)| > 1$. Say $W = (u_0, u_1, ..., u_{2k}, u_{2k+1})$, where u_0 and u_{2k+1} are in H, and $u_1, ..., u_{2k}$ are not in H. In this case $u_1, ..., u_{2k}$ have degree two in $G - v$, so, in G, they must all be adjacent to v. Let

$$X = \{v\} \cup \{u_1, ..., u_{2k}\}$$

Then, the underlying simple graph of $G\{X; \overline{x}\}$ is an odd wheel with v as its hub. So, it is matching covered. Also, $G\{\overline{X}; x\}$ is matching covered. (In fact, $G\{\overline{X}; x\}$ bicritical. To see this, first observe that $G[\overline{X}] = H$, which is critical. Suppose that $G\{\overline{X}; x\}$ has a nontrivial barrier B. The vertex x must be in B; otherwise B would be a nontrivial barrier in G itself. Now, if we set $S = B \backslash \{x\}$, then S is a nonempty subset of $\overline{X}$ such that $C_{odd}(H - S) = |S| + 1$. This is impossible because H is critical.) Therefore $\nabla(X)$ is a strictly separating cut of G. The characteristic of this cut must be three because $\{u_0, u_{2k+1}\} \cup \{v\}$ is a 3-vertex cut of G, and every edge of $\nabla(X)$ is incident with a vertex in this cut. ■

8.3 Proof of the Lemma 8.3

We shall now present a proof of Lemma 8.3. We shall in fact show that any $\lambda(G)$-robust cut $C := \nabla(X)$, with $|X|$ is as small as possible, has the required property. So, let $C := \nabla(X)$ be any such cut. Let $G_1 := G\{X; \overline{x}\}$, $G_2 := G\{\overline{X}; x\}$, and let M_0 denote a perfect matching of G such that $|M_0 \cap C| = \lambda(G)$.

8.3.1 Graph G_1 is a solid brick

Firstly let us note that G_1 must be a brick. This follows from the fact that if G_1 has a nontrivial tight cut, then, by Corollary 8.8, there exists a robust cut $\nabla(Y)$ of G with $|Y| < |X|$ which contradicts the minimality of $|X|$. We now prove that $\lambda(G_1) > 3$.

Lemma 8.12 *Suppose that G_1 has a 3-good cut. Then there exists a 3-robust cut of G with a shore of cardinality smaller than that of X.*

Proof: Let D_1 be a 3-good cut of G_1. Let M_1' denote any perfect matching of G_1 containing precisely three edges in D_1. Let M_1'' denote a perfect matching of G_2 that contains the edge of M_1' in C. Let $M_1 := M_1' \cup M_1''$. Then M_1 is a perfect matching of G that contains precisely three edges in D_1 and just one edge in C.

We first observe that C and D_1 are "synchronous" separating cuts of G, in the following sense: for every edge e of G there exists a perfect matching M_e of G that contains edge e and just one edge in each of C and D_1. To see this, consider first the case in which e lies in G_1; in that case, G_1 has a perfect matching, M_e', that contains e and just one edge in D_1; let M_e'' denote a perfect matching of G_2 that contains the edge of M_e' in C. Then $M_e' \cup M_e''$ is a perfect matching of G that contains edge e and just one edge in each of C and D_1. Now consider the case in which edge e does not lie in G_1. Let M_e'' denote a perfect matching of G_2 that contains edge e. Let M_e' denote a perfect matching of G_1 that contains the edge of M_e'' in C and just one edge in D_1. Again, $M_e' \cup M_e''$ is a perfect matching of G that contains edge e and just one edge in each of C and D_1.

Let D denote a 3-robust cut of G M_1-induced by D_1. Since M_1 has more than one edge in D, but has only one edge in C, it follows that D is not matching-equivalent to C. In particular, $D \neq C$.

Let Y denote a shore of D such that both $I := \nabla(X \cap Y)$ and $U := \nabla(\overline{X} \cap \overline{Y})$ are odd cuts. For any edge e of G, matching M_e, defined above, contains edge e and just one edge in each of C and D_1. Since D is induced by D_1, it follows M_e also has just one edge in D. Therefore, each of I and U also have just one edge in M_e. We conclude that no edge of G joins a vertex of $X \setminus Y$ to a vertex of $Y \setminus X$. Thus modularity applies to the pair C, D, that is, for every perfect matching M of G,

$$|M \cap C| + |M \cap D| = |M \cap I| + |M \cap U|.$$

We assert that U is a tight cut in G_2. For this, assume, to the contrary, that there exists a perfect matching M_2'' of G_2 that contains more than one edge in U. Let M_2' denote a perfect matching of G_1 that contains the edge of M_2'' in C and just one edge in D_1. Then $M_2 := M_2' \cup M_2''$ constitutes a perfect matching of G that contains one edge in each of C

and D_1, but more than one edge in U. Simple counting shows that M_2 contains more than one edge in D. Thus M_2 has just one edge in D_1 but more than one edge in D. This contradicts the definition of D. As asserted, U is tight in G_2.

Consider first the case in which $\overline{X} \cap \overline{Y}$ is a singleton. In that case, $X \cap Y$ cannot be a singleton, otherwise both C and D would be tight cuts associated with a 2-separation of G. Thus, $\left|\overline{Y}\right| < |X|$. Moreover, $\lambda(D) = 3$.

Now consider the case in which either U is not trivial in G_2 or $U = C$. If U is not trivial then, by Lemma 8.7, cuts U and C are matching-equivalent. The same conclusion holds if $U = C$. In that case, by modularity, I and D are also matching-equivalent, whence I is a 3-robust cut of G. Moreover, I and C are distinct, because M_1 has just one edge in C and three edges in I. Thus, $X \cap Y$ is a proper subset of X.

In both cases, we obtained a 3-robust cut of G having a shore of cardinality strictly less than that of X. ■

We now observe that brick G_1 is not the Petersen graph, even allowing multiple edges incident to $\overline{x}$. For this, assume the contrary. Let X' be the set of vertices of any pentagon of G_1 not containing vertex $\overline{x}$. It is not difficult to see that $\nabla(X')$ is a 3-robust cut of G.

To summarize, the characteristic of G_1 is greater than three but G_1 is not the Petersen graph, even allowing multiple edges. By the first hypothesis about G, it follows that G_1 is a solid brick.

8.3.2 Graph G_1 is an odd wheel with $\overline{x}$ as its hub

Assume that G_1 is not an odd wheel, up to multiple edges incident to $\overline{x}$. Since G_1 is solid, it follows from the Lemma 8.11 on odd wheels that G_1 has a removable class disjoint from $\nabla(\overline{x})$.

Consider first the case in which the removable class is a doubleton. In this case, no edge of the doubleton lies in C, therefore the hypothesis of Lemma 8.10 is applicable. We conclude that $\lambda(G) = 3$ and G has a b-removable edge. This is a contradiction.

We may thus assume that G_1 has a removable edge e not incident to $\overline{x}$. Thus edge e does not lie in C. Since G_1 is a solid brick, edge e is b-removable in G_1. Also, since e is not in $C = C - e$, it is also b-removable in G_2. By Lemma 8.4, the cut $C - e$ is tight in $G - e$. But, by Lemma 8.6, this implies that G has a 3-robust cut D such that one of the shores of D has fewer vertices than X. This is not possible by the choice of X. ■

8.4 Choosing X

Now consider the family of $\lambda(G)$-robust cuts of G such that, for any member $C = \nabla(X)$ in the family, the C-contraction $G_1 = G\{X; \overline{x}\}$ is an odd wheel, up to multiple edges incident to $\overline{x}$. Among all such cuts choose a cut $C = \nabla(X)$ with $|X|$ as large as possible. As before, let $G_2 := G\{\overline{X}; x\}$, and let M_0 denote a perfect matching of G such that $|M_0 \cap C| = \lambda(G)$.

8.5 Two simple facts about Cut C

The next two assertions establish simple but important properties of the edges of cut C.

Lemma 8.13 *Every edge of $C \setminus M_0$ is b-removable in G_1. If $|X| > 3$ then every edge of C is b-removable in G_1.*

Lemma 8.14 *If $|X| = 3$, then every edge f of C which is b-removable in G_2 must be a multiple edge of G_1.*

Proof: Graph G is simple, and $|X| = 3$, therefore G_1 is K_4, up to multiple edges incident to $\overline{x}$. If the edge f is not a multiple edge, then the removable class that contains edge f is a doubleton of G_1. By hypothesis, edge f is b-removable in G_2. Then, by Lemma 8.10, G has a b-removable edge. Thus, we have that $\lambda(G) = 3$, and that G has a b-removable edge. This contradicts the second hypothesis about G. ■

8.6 Graph G_2 is a Brick different from K_4, $\overline{C}_6$, and the Petersen graph.

In this section we show that we may assume that G_2 is a simple brick distinct from the three special bricks K_4, $\overline{C_6}$ and the Petersen graph. Let us first show that G_2 must be simple. To see this, observe that if G_2 has a pair of parallel edges e and f then, because G itself is simple, they lie in C, and at least one of them, say e, is not in M_0. Then, being a multiple edge of G_2, e is b-removable in G_2, and since e is in $C \setminus M_0$, it is b-removable in G_1 by Lemma 8.13. This is not possible by Corollary 8.5.

For proving that G_2 is a brick, we require the following simple lemma concerning removable edges in braces.

Lemma 8.15 *Let H be a brace distinct from K_2, denote by δ the minimum vertex degree of H. If $|V(H)| \neq 4$ then each edge of H is removable. If $|V(H)| = 4$ and $\delta \geq 3$ then there exists a perfect matching M of H such that every edge of $E(H) \setminus M$ is removable.*

Proof: For proof of the case in which $|V(H)| \neq 4$, see [3]. The other case is straightforward. ■

Now, let us proceed to show that G_2 is a brick. Towards this end, suppose that G_2 is not a brick. Then G_2 has nontrivial tight cuts. If $D = \nabla(Y)$ is any tight cut of G_2, such that the shore Y contains x, then by Lemma 8.7, the graph $H_1 := G_2\{Y; \overline{y}\}$ is bipartite, and vertices x and $\overline{y}$ lie in distinct parts of H_1. Choose a nontrivial tight cut $D = \nabla(Y)$ of G_2, with x in Y, so that Y is minimal.

We assert that H_1 is a brace with at least four vertices. It certainly has at least four vertices, for D is nontrivial in G_2. To prove that it is a brace, assume, to the contrary, that there exists a nontrivial tight cut D' in H_1. Let Y' be the shore of D' in H_1 that contains vertex x. Thus, x is a vertex of the bipartite graph $H_1[Y']$. If $\overline{y}$ is also in $H_1[Y']$, then the major part of the bipartition of $H_1[\overline{Y'}]$ would be a nontrivial barrier of the brick G. So, $\overline{y}$ is not in Y'. Also, if x is in the minor part of $H_1[Y']$, then the major part of $H_1[Y']$ would be a nontrivial barrier of the brick G. Therefore, x is in the major part $H_1[Y']$, and $\overline{y}$ is in the major part of $H_1[\overline{Y'}]$. It follows that D' is a nontrivial tight cut of G_2 and Y' is a proper subset of Y, a contradiction.

As asserted, H_1 is a brace with at least four vertices. By Lemma 8.15, there exists a perfect matching M of H_1 such that every edge of $E(H_1) \setminus M$ is removable in H_1. Let v denote a vertex distinct from $\overline{y}$ in the part of the bipartition of H_1 that does not contain x. Clearly, vertex v has degree at least three in H_1 and exactly one edge of each of M and M_0 is incident to v. Thus there exists a removable edge of H_1, say e, that does not lie in M_0 and is incident to v. Clearly, edge e is b-removable in both G_1 and G_2. This is impossible by Corollary 8.5. Thus, G_2 is a brick.

We would now like to use the induction hypothesis on G_2. But, in order to do that, it is necessary for us to show that G_2 cannot be either K_4, $\overline{C}_6$, or the Petersen graph. Handling these special cases is very simple. Observe that these three special graphs are cubic.

If $|C| > 3$, then clearly there exists a pair of parallel edges of C in G_2, a case which has been dealt with. We may thus assume that $|C| = 3$, whence G_1 is K_4, without multiple edges. If G_2 is also K_4 then G is $\overline{C_6}$. If G_2 is $\overline{C_6}$ then G_2 has a removable doubleton that is not incident to x. In this case, $\lambda(G) = 3$ and G has a b-removable edge, by Lemma 8.10. This is not possible by the second hypothesis. (Incidentally, in this case G is R_8.) If G_2 is the Petersen graph, then the cut associated with any pentagon not containing x is a 3-robust cut of G and the corresponding G_1 is an odd wheel of order 5, a better choice than the current one. Indeed, G_2 is a simple, nonspecial brick.

8.7 Choosing the Edge e

Since G_2 is a brick different from K_4, $\overline{C}_6$, and the Petersen graph, and clearly has fewer vertices than G, by induction hypothesis, G_2 has b-removable edges. We select a b-removable edge of G_2 according to a rule to be specified. Before stating that rule, we note that such an edge e of G_2 may be assumed to satisfy the following properties:

(i) **The edge e is b-removable in $\mathbf{G_1}$**. (Let us first consider the case in which e is in C. If $|X| > 3$, then e is b-removable in G_1 by Lemma 8.13. On the other hand, if $|X| = 3$, e must be a multiple edge of G_1 by Lemma 8.14, and thus is clearly b-removable in G_1. Now consider the case in which e is not in C. In this case, e is not an edge of G_1, and thus is b-removable in G_1.)

(ii) **Cut $\mathbf{C - e}$ is tight in $\mathbf{G - e}$**. (By Lemma 8.4.) Note that this, in particular, implies that edge e lies in M_0 (Corollary 8.5), and that $b(G - e) = b(G_1 - e) + b(G_2 - e) = 2$.

(ii) $\mathbf{|C - e| \geq 3}$. (If $|C - e| < 3$, then $|C| = |X| = 3$, which implies that G_1 is K_4 without multiple edges, whence e is not b-removable in G_1. We have already seen that this may be assumed not to be the case.)

We choose a b-removable edge e of G_2 using the following rule: **If there are b-removable edges of $\mathbf{G_2}$ incident with x, that is, belonging to C, then e is a b-removable edge of $\mathbf{G_2}$ incident with x. Otherwise, e is any b-removable edge of $\mathbf{G_2}$.**

Suppose that a b-removable edge e of G_2 has been chosen according to the aforementioned rule. We have seen that we may assume that the following properties hold:

- Edge e is b-removable in G_1.
- Cut $C - e$ is tight in $G - e$.
- Edge e lies in M_0.
- $b(G - e) = 2$.
- $|C - e| \geq 3$.

By Theorem 7.6, there exists a strictly separating cut D of G such that $D - e$ is tight in $G - e$ and cuts $C - e$ and $D - e$ constitute either a barrier cut pair or essentially a 2-separation pair. Let M_1 denote any perfect matching of G that contains more than one edge in D. Since $D - e$ is tight in $G - e$, it follows that edge e lies in M_1.

The two cases mentioned above require separate treatment. We shall first consider the case in which $C - e$ and $D - e$ are a barrier cut pair. For dealing with this case, we need the following lemma concerning removable edges in bipartite graphs.

Lemma 8.16 *Let G be a brick and let e be a removable edge of G such that $G - e$ has a special barrier B. Let K denote the unique nontrivial component of $G - e - B$, let $C_K := \nabla_G(V(K))$. Let H denote the $(C_K - e)$-contraction $(G - e)\{\overline{V(K)}; v_K\}$ of $G - e$. Then every edge of $\nabla_H(v_K)$ is removable in the bipartite graph H.*

Proof: Since G is matching covered and e is removable in G, graph $G - e$ is matching covered. Moreover, cut $C_K - e$ is tight in $G - e$. Therefore, graph H is matching covered.

Let f denote any edge of $\nabla_H(v_K)$. Assume, to the contrary, that edge f is not removable in H. Let A denote set $V(H) \setminus B$. By Theorem 2.3, there exists a partition (A', A'') of A and a partition (B', B'') of B such that $|A'| = |B'|$ and edge f is the only edge of H having one end in A', the other in B''.

Set A'' is a barrier of G. Since G is a brick, set A'' is a singleton. Let v'' denote the vertex of B'' (and also the end of f in B). The neighbors of v'' in G are: the vertex of A'' and the end of f distinct from v''. Therefore those two vertices constitute a nontrivial barrier of G, a contradiction, because G is bicritical. We conclude that each edge in $\nabla_H(V(K))$ is removable in H. ∎

8.8 The Case in which $C - e$ and $D - e$ are a Barrier Cut Pair

Let B denote a barrier of $G - e$ such that $G - e - B$ has precisely two nontrivial components, K_C and K_D, where $C = \nabla_G(V(K_C))$ and $D = \nabla_G(V(K_D))$. Then perfect matching M_0 of G contains edge e, precisely three edges in C and just one edge in D. Similarly, M_1 has precisely three edges in D, contains edge e and has just one edge in C.

Since $C = \nabla_G(V(K_C))$, $V(K_C)$ is either X or $\overline{X}$. We note that $V(K_C)$ must in fact be X. For, suppose that $V(K_C) = \overline{X}$. Then, the barrier B of $G - e$ is also a barrier of $G_1 - e$. Note that G_1 is an odd wheel, and $G_1 - e$ is matching covered. Clearly e must be an edge of G_1; otherwise $G_1 - e$ has no nontrivial barriers. So, it follows that e is incident with $\overline{x}$.

Let v be the other end of e in G_1, and let u and w be the neighbours of v on the rim of G_1. Then $B = \{u, w\}$, for $\{u, w\}$ is the only nontrivial barrier of $G_1 - e$. But now, $G_1 - e - B$ has two barriers, and one of them is trivial. This implies that the companion D of C is trivial. A contradiction.

Consider the D-contraction H of $G_2 - e$ defined as follows:

$$H := (G_2 - e)\{\overline{X} \cup \{x\} \setminus V(K_D); v_D\}.$$

Note that H is the bipartite graph obtained from $G - e$ by contracting X to a single vertex x, to obtain $G_2 - e$, and then contracting $V(K_D)$ to a single vertex v_D. One of the parts of the bipartition of H is B. Let A denote the other part of that bipartition; it contains vertices x and v_D.

Lemma 8.17 *At most one edge of $C - e$ is not removable in H.*

Proof: Let f denote any edge of $C - e$ that is not removable in H. Then there is a partition (A', A'') of A and a partition (B', B'') of B such that $|A'| = |B'|$, $|A''| = |B''|$ and f is the only edge joining a vertex of A' to a vertex of B''. (See Figure 11.)

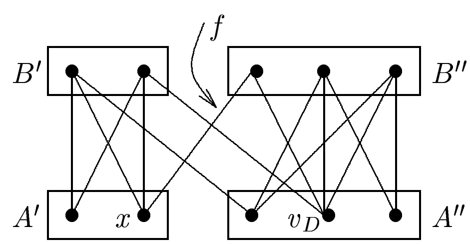


Figure 11: Cuts $C - e$ and $D - e$ are a barrier cut pair

We now prove that v_D lies in A''. For this, assume that v_D lies in A'. In that case, each vertex of A'' is a vertex of G and therefore A'' is a barrier of G. Since G is a brick, A'' is a singleton. The vertex of B'' is thus joined to the vertex of A'' by at least two multiple edges, a contradiction, since G is simple.

Observe now that B' is a special barrier of $G - e$, where the cut associated with the nontrivial component of $G - e - B'$ contains $C - \{e, f\}$. Moreover, B' is nontrivial. For this, recall that $|C - e| \geq 3$, therefore at least two edges of C join vertex x to vertices of B'. Those vertices are original vertices of G_2, a simple brick. Therefore, B' is a nontrivial special barrier of $G - e$.

By Lemma 8.16, each edge of the cut associated with the nontrivial component of $G - e - B'$ is removable in $H\{B' \cup (A' - x)\}$. In the other contraction of H, the edges of $C - \{e, f\}$ are multiple edges, because $|C - e| \geq 3$. Indeed, each edge of $C - f$ is removable in H. ■

Lemma 8.18 *Every edge of $(C-e)\setminus M_1$ that is removable in H is b-removable in G_2.*

Proof: Let h be an edge of $(C-e)\setminus M_1$ that is removable in H. Since $C-e$ and $D-e$ are disjoint, edge h does not lie in D. Therefore G_2-e-h is a near-brick. Edge e lies in M_1 and edge h does not, whence M_1 is a perfect matching of G_2-h. Therefore, G_2-h is matching covered. Moreover, $b(G_2-h)\leq b(G_2-e-h)=1$, whence G_2-h is a near-brick. ■

We assert that edge e lies in C. To see this, observe that by Lemma 8.17, at least $|C-e|-1$ edges of $C-e$ are removable in H. Since M_1 has just one edge in C, it follows that at least $|C-e|-2$ edges of $C-e$ are b-removable in G_2, by Lemma 8.18. But $|C-e|\geq 3$, whence at least one edge of $C-e$ is b-removable in G_2. By the choice of e, edge e lies in C.

Let us now redo the above analysis, taking into account that edge e lies in $M_0\cap M_1\cap C$. By Lemma 8.17, at least $|C-e|-1$ edges of $C-e$ are removable in H. Since M_1 has just one edge in C, namely, e, it follows that at least $|C-e|-1$ edges of $C-e$ are b-removable in G_2, by Lemma 8.18. If one of those edges does not lie in M_0, then that edge is b-removable in G. We may thus assume that each of the $|C-e|-1$ edges of $C-e$ that are b-removable in G_2 lies in M_0. But M_0 has just three edges in C, one of which is e. Therefore, $|C-e|-1\leq 2$. We have seen that $|C-e|\geq 3$. Therefore, $|C|=4$, and hence $|X|=3$. This implies that at least one edge of M_0-e is not a multiple edge in G_1 and is b-removable in G_2. This is impossible by Lemma 8.14.

We have shown that $\lambda(G)=3$ and G has a b-removable edge, if $C-e$ is a barrier cut in $G-e$. This is a contradiction. We may thus assume that cuts $C-e$ and $D-e$ are essentially a 2-separation pair of $G-e$.

8.9 The Case in which $C-e$ and $D-e$ are Essentially a 2-Separation Pair

Let Y be a shore of D such that $I:=\nabla(X\cap Y)$ and $U:=\nabla(\overline{X}\cap\overline{Y})$ are both odd cuts. By hypothesis, each of $C-e$ and $D-e$ is tight in $G-e$. Therefore, both $I-e$ and $U-e$ are tight in $G-e$. Thus, no edge of $G-e$ joins a vertex of $X\cap\overline{Y}$ to a vertex of $\overline{X}\cap Y$. (See Figure 12.) Let

$$\begin{aligned} G_{11} &:= G\{X\cap Y;v_{11}\}, & G_{12} &:= G_1\{X\cap\overline{Y}\cup\{\overline{x}\};v_{12}\},\\ G_{22} &:= G\{\overline{X}\cap\overline{Y};v_{22}\}, & G_{21} &:= G_2\{\overline{X}\cap Y\cup\{x\};v_{21}\}.\end{aligned}$$

By hypothesis, each of $G_{11}-e$ and $G_{22}-e$ is bipartite. Let (A_1,B_1) denote a bipartition of $G_{11}-e$ such that $v_{11}\in A_1$. Likewise, let (A_2,B_2) denote a bipartition of $G_{22}-e$ such that $v_{22}\in A_2$. Cut $I-e$ is tight in $G-e$ and $G_{11}-e$ is an $(I-e)$-contraction of $G-e$. Therefore, set B_1 is a special barrier of $G-e$. Likewise, set B_2 is a special barrier of $G-e$.

We need a trivial result about odd wheels:

Lemma 8.19 *Let W denote an odd wheel of order greater than three. Let e be a spoke of W, v its end in the rim of W. Let u and w denote the two neighbors of v in the rim. Then $\nabla(\{u,v,w\})-e$ is the only nontrivial tight cut of $W-e$.*

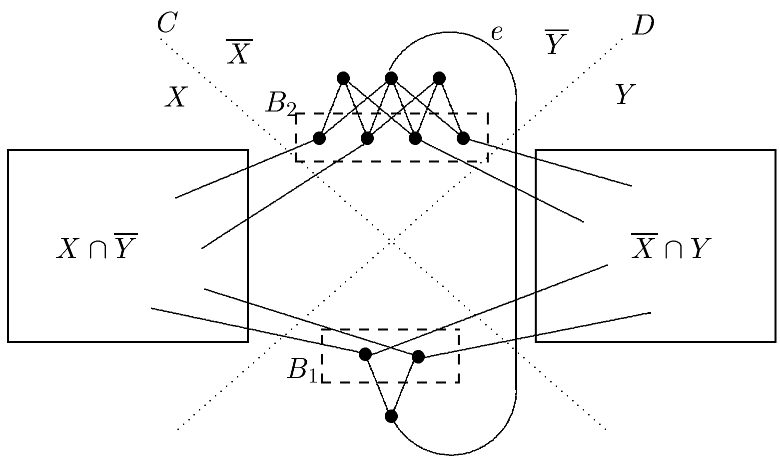


Figure 12: Cuts $C-e$ and $D-e$ are 2-separation pair

Lemma 8.20 *If $X\cap Y$ is not a singleton then it induces in G a path of length 2 and the internal vertex of that path is an end of e.*

Proof: The cut $I-e$ is a tight cut of G_1-e. If, in addition, $X\cap Y$ is not a singleton, then I is nontrivial in G_1-e. Since G_1 is an odd wheel, the assertion follows immediately from Lemma 8.19. ■

Let T denote the set of edges of G that join vertices in $\overline{X}\cap\overline{Y}$ to vertices in $X\cap\overline{Y}$.

Lemma 8.21 *Set T contains at least two edges.*

Proof: Since G_1 is an odd wheel of hub $\overline{x}$, at least $\left|X\cap\overline{Y}\right|$ edges of G join vertices of $X\cap\overline{Y}$ to vertices of $\overline{X}$. Moreover, no edge of $G-e$ joins a vertex of $X\cap\overline{Y}$ to a vertex of $\overline{X}\cap Y$. Consequently, if $\left|X\cap\overline{Y}\right|>2$ or if edge e does not join a vertex of $X\cap\overline{Y}$ to a vertex of $\overline{X}\cap Y$, then the assertion holds immediately.

We may thus assume that edge e joins a vertex of $X\cap\overline{Y}$ to a vertex of $\overline{X}\cap Y$ and also that $\left|X\cap\overline{Y}\right|=2$.

Edge e therefore has no end in $X\cap Y$. In this case B_1 is a barrier of the brick G, and so must be trivial. We conclude that $|X|=3$. By Lemma 8.14, edge e is a member of a pair of multiple edges in G_1. The other edge of that pair joins a vertex of $\overline{X}\cap\overline{Y}$ to the end of e in $X\cap\overline{Y}$. The other vertex of $X\cap\overline{Y}$ is also joined to some vertex of $\overline{X}\cap\overline{Y}$. ■

Lemma 8.22 *Set $\overline{X}\cap\overline{Y}$ is not a singleton and edge e has at least one end in $\overline{X}\cap\overline{Y}$. Moreover, modularity applies to cuts C, D, I and U. That is, for every perfect matching M of G,*

$$|M\cap C|+|M\cap D|=|M\cap I|+|M\cap U|.$$

Proof: By Lemma 8.21, $|T| \geq 2$. Graph G_2 is simple. Therefore, $\overline{X} \cap \overline{Y}$ is not a singleton. That is, cut U is nontrivial.

The $(U-e)$-contraction $G_{22}-e$ of $G-e$ is bipartite and cut $U-e$ is tight in $G-e$. Therefore, edge e must have at least one end in $\overline{X} \cap \overline{Y}$, otherwise B_2 would be a nontrivial barrier of G.

No edge of $G-e$ joins a vertex of $X \cap \overline{Y}$ to a vertex of $\overline{X} \cap Y$. Neither does edge e. Therefore, modularity applies to the four cuts mentioned in the assertion. ■

Lemma 8.23 *Every edge of $T \setminus M_1$ is b-removable in G_2.*

Proof: Note that $G_{21}-e$ and $G_{22}-e$ are the two $(U-e)$-contractions of G_2-e. Recall that cut $U-e$ is tight in $G-e$.

In G_{21}, each edge of T joins vertex v_{21} to vertex x. By Lemma 8.21, there are at least two such edges. They are all parallel to each other, and therefore are b-removable in G_{21}.

On the other hand, B_2 is a special barrier of $G-e$. By Lemma 8.22, B_2 is nontrivial. Note that, in $G-e$, the edges of $T-e$ all join vertices of B_2 to vertices of the only nontrivial component of $G-e-B_2$. By Lemma 8.16, each edge of $T-e$ is removable in $G_{22}-e$. We conclude that each edge of $T-e$ is b-removable in G_2-e.

Let f be any edge of $T-e$ that does not lie in M_1. Then G_2-f is matching covered, because $G_2-\{e,f\}$ is matching covered and M_1 is a perfect matching of G that contains edge e but does not contain edge f. Moreover, that $b(G_2-f) \leq b(G_2-\{e,f\}) = b(G_2-e) = 1$. Therefore, f is b-removable in G_2. This conclusion holds for every edge f of $T-e$ that does not lie in M_1. Moreover, edge e lies in M_1. Indeed, every edge of $T \setminus M_1$ is b-removable in G_2. ■

Lemma 8.24 *The cut I cannot be trivial.*

Proof: Assume that the cut I is trivial. Since $G_{22}-e$ is bipartite, and $U-e$ is tight in $G-e$, and so, for every perfect matching M of G, we have that $|M \cap U| \leq 3$, with equality if, and only if, $e \in M$. By modularity, M_0 has precisely three edges in C and just one edge in D, whereas M_1 has just one edge in C and precisely three edges in D. In particular, the equality $|M_0 \cap C| = 3$ implies that $\lambda(G) = 3$.

We assert that edge e lies in C. For this, observe that set T has at least two edges, by Lemma 8.21. Since M_1 has just one edge in C, set T contains at least one edge that is b-removable in G_2. By the criterion used to choose edge e, it follows that edge e lies in C.

Consider first the case in which T contains an edge, say, f, that does not lie in M_0. Since the edge of M_1 in C is e, an edge of M_0, it follows that edge f lies in $T \setminus (M_0 \cup M_1)$. Thus, edge f lies in $T \setminus M_1$, whence, by Lemma 8.23, edge f is b-removable in G_2. Edge f also lies in $C \setminus M_0$, therefore edge f is b-removable in G_1, by Lemma 8.13. Now, since edge f does not lie in M_0, we have a contradiction (Corollary 8.5).

We may thus assume that $T \subseteq M_0$. This implies that no edge of T is a multiple edge in G_1. It also implies that $\left|X \cap \overline{Y}\right| = 2$. Therefore, $|X| = 3$. By Lemma 8.14, edge e is a multiple edge in G_1. This implies that edge e does not lie in T. Thus, $|X| = 3$ and T

contains precisely two edges, neither of which is a multiple edge in G_1. By Lemma 8.14, G has a b-removable edge. This is a contradiction. ■

Lemma 8.25 *G is the Petersen graph.*

Proof: By the previous Lemma, $X \cap Y$ is a not a singleton. By 8.20, the subgraph of G_1 induced by $X \cap Y$ is a path of length 2, the internal vertex of that path is an end of e.

Cut $I - e$ is tight in $G - e$ and $G_{11} - e$ is bipartite. Therefore, for every perfect matching M of G, $|M \cap I| \leq 3$, with equality if, and only if, $e \in M$, otherwise nontrivial cut I would be tight in G.

Likewise, cut $U - e$ is tight in $G - e$, cut U is nontrivial and $G_{22} - e$ is bipartite. Therefore, for every perfect matching M of G, $|M \cap U| \leq 3$, with equality if, and only if, $e \in M$.

By modularity we conclude that, for every perfect matching M of G,

$$|M \cap C| + |M \cap D| = 2 + 4|\{e\} \cap M|.$$

Consider first the case in which $\lambda(G) = 3$. Then, by modularity, M_0 has precisely 3 edges in each of C and D. In that case, it is convenient to take $M_1 := M_0$.

We assert that set $T \setminus M_0$ is nonempty. For this, assume the contrary. Perfect matching M_0 has exactly 3 edges in C. One of these edges is e, an edge not in T. Therefore $|T| = 2$. This implies that $\left|X \cap \overline{Y}\right| = 2$; and moreover, the two vertices of $X \cap \overline{Y}$ are incident to the two edges of T. On the other hand, graph $G[X \cap Y]$ is a path P of length two, its internal vertex is incident to edge e. It follows that the extremal vertices of P are matched by M_0 with vertices of $\overline{X}$, whence M_0 has five edges in C, a contradiction. As asserted, $T \setminus M_0$ is nonempty.

Let f denote an edge of T that does not lie in M_0. But $M_1 = M_0$, therefore edge f does not lie in either M_0 or M_1. By Lemmas 8.13 and 8.23, edge f is b-removable in each of G_2 and G_1; moreover, cut $C - f$ is separating in $G - f$ and has precisely three edges in M_0. This is impossible by Corollary 8.5.

Finally, consider the case in which $\lambda(G) > 3$. By modularity, perfect matching M_0 has precisely five edges in C and exactly one edge in D. Thus, $\lambda(G) = 5$. Hence $\lambda(G_2) \geq 5$. But if $\lambda(G_2) = 5$, by the induction hypothesis, G_2 is the Petersen graph, which we know is not the case. Therefore G_2 is solid, and, every removable edge of G_2 is b-removable in G_2. Suppose that there is a removable edge f of G_2 not incident with x. Then, by Lemma 8.4, cut $C - f$ is tight in $G - f$. In this case, by Lemma 8.6, G has a 3-robust cut which is not possible because $\lambda(G) = 5$. Thus, G_2 has no removable edges not incident with x. By Lemma 8.10, no doubleton disjoint with C is removable in G_2. Now, by the Lemma 8.11 on odd wheels, recalling that G_2 is simple, it follows that G_2 is an odd wheel.

Note that every edge of C is b-removable in G_1, because G_1 is an odd wheel of hub $\overline{x}$ and order at least five. Likewise, every edge of C is b-removable in G_2. By Corollary 8.5 every edge of C lies in M_0. Thus, graphs G_1 and G_2 are both simple odd wheels of order five. We conclude that $C = M_0$.

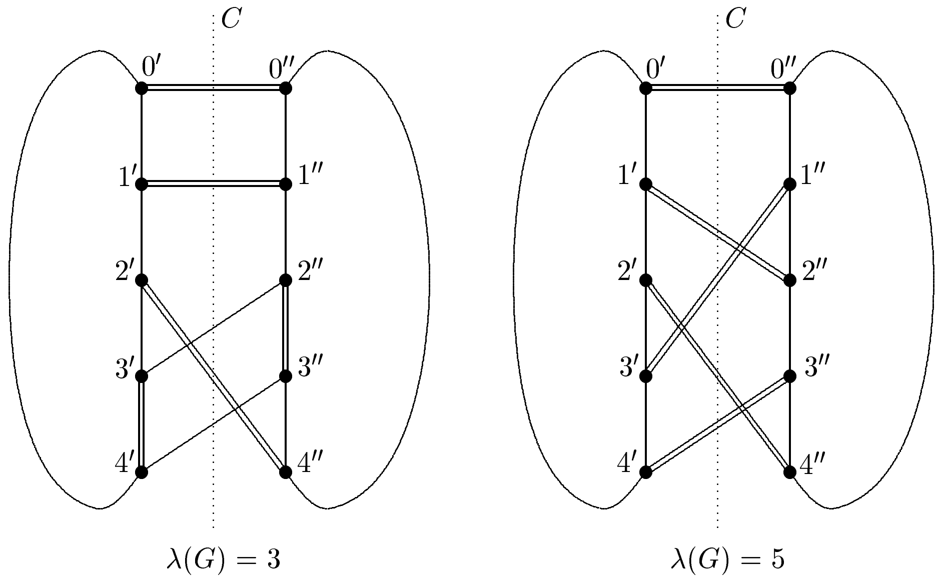


Figure 13: The Petersen graph is the only possibilty

We now show that G is the Petersen graph. For this, number the vertices of the rim of G_1 cyclically in sequence: $(0', 1', 2', 3', 4')$. Likewise, number the vertices of the rim of G_2 cyclically as: $(0'', 1'', 2'', 3'', 4'')$.

No two edges of C have their ends in G_1 adjacent in G_1 and their ends in G_2 adjacent in G_2. To see this, assume the contrary and adjust notation so that $0'$ and $0''$ are adjacent and $1'$ and $1''$ are also adjacent. At least one of vertices $2'$ and $4'$ would be adjacent to one of the vertices $2''$ and $4''$. The edge connecting these two vertices plus edges $(0', 0'')$ and $(1', 1'')$ could then be extended to a perfect matching of G containing just three edges in C. This would mean that $\lambda(G) = 3$, a contradiction. (See Figure 13(a).)

We may thus adjust notation so that $0'$ is adjacent to $0''$. Vertex $1'$ must then be adjacent to one of $2''$ and $3''$. Adjust notation, reversing the orientation of the numbering of the rim of G_2, so that vertex $1'$ be adjacent to $2''$. Then vertex $2'$ must be adjacent to vertex $4''$. In turn, vertex $3'$ must be adjacent to vertex $1''$. Finally, $4'$ must be adjacent to $3''$. (See Figure 13(b).) That is, G is the Petersen graph.

This completes the proof of the Main Lemma.

■

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