Bases for the Matching Lattice of Matching Covered Graphs

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Abstract

This article contains a brief introduction to the theory of matching covered graphs: ear-decompositions, the matching lattice and the most important results of the theory. Its last section contains rather recent results and a conjecture relating the minimum number of double ears of any ear-decomposition of a matching covered graph and the number of bricks and bricks isomorphic to the Petersen graph in any brick decomposition of the same graph.

1   Ear-Decomposition

We consider simple graphs, i.e., finite graphs without loops and multiple edges. We denote respectively by $E(G)$ and $V(G)$ the set of edges and vertices of a graph G. A *matching* is a set of edges no two of which have a vertex in common. A matching is *perfect* if its edges match up all vertices. Recall the basic theorem of Tutte [8] which asserts that a graph G has a perfect matching if and only if $c_1(G - X) \leq |X|$, for each set X of vertices of G; here, $c_1(H)$ denotes the number of odd components of a graph H. If equality holds in the above inequality then X is called a *barrier*. If G has a perfect matching then clearly each vertex of G constitutes a barrier; the empty set is also a barrier. Those barriers, containing at most one vertex, are said to be *trivial*.

A connected graph is *matching covered* if each of its edges lies in some perfect matching and *bicritical* if deletion of any two of its vertices yields a graph having a perfect matching. It is easy to see that a connected graph with a perfect matching is (i) matching covered if and only if no barrier spans an edge and (ii) bicritical if and only if it has only trivial barriers.

A 3-connected bicritical graph is called a *brick*. Three bricks play a special role in the theory of matching covered graphs: K_4, the complete graph on 4 vertices, $\overline{C_6}$, the triangular prism, and the Petersen graph.

Let H be a subgraph of graph G. A path P in $G - E(H)$ is an *ear* of H if (i) both ends of P lie in H and (ii) P is internally disjoint from H. An ear is *odd* if it has odd length. Henceforth, by an 'ear' we shall mean an 'odd ear'. An *ear-decomposition* of a matching covered graph G is a sequence $K_2 = G_0 \subset G_1 \subset \cdots \subset G_k = G$ of matching covered subgraphs of G, where for $0 \leq i < k$, G_{i+1} is the union of G_i and one or two

vertex-disjoint ears of G_i, called *single* and *double* ears, respectively. A subgraph H of matching covered graph G is *nice* if $G - H$ has a perfect matching. Clearly, each term G_i of an ear-decomposition of matching covered graph G is nice. The following theorem was proved by Lovász and Plummer [5].

Theorem 1 *Every matching covered graph has an ear-decomposition.*   □

One of the open questions about ear-decomposition of matching covered graphs is the minimization of the number of double ears. In section 3 we solve this question for bricks, and propose a conjecture for the general case.

Ear-decomposition constitutes a nice way of finding independent incidence vectors of perfect matchings. If the number of double ears is sufficiently small then it is possible to get a basis for the matching lattice of a matching covered graph formed only by incidence vectors of perfect matchings.

2   The Matching Lattice

This section constitutes a brief summary on the subject, based on an article by Lovász [4], and a report by Murty [7]. For a better understanding of that article and a more complete explanation of the theory of matching covered graphs we suggest the report by Murty [7].

From the early years of Linear Programming, a fruitful interaction between combinatorics and polyhedral theory began to develop, resulting in the branch of combinatorics which we now call Combinatorial Optimization. This interaction has been especially helpful for the advancement of matching theory. Edmonds [1] led the way in these developments by characterizing the polytope generated by the incidence vectors of perfect matchings of a graph. Lovász [4], building on the theory of the matching covered graphs, gave a complete characterization of the matching lattice. In this section we will describe the main results concerning the theorem of Lovász that characterizes the matching lattice.

Let G be a graph. We consider the space Q^E af all rational valued functions on the edge set E of G. Each member of Q^E may be viewed as a labeling of the edges of G by rational numbers. For a subset A of E, the *incidence vectors* χ^A of A is defined by $\chi^A(e) = 1$, if $e \in A$, and $\chi^A(e) = 0$, otherwise. If w is any function in Q^E, and C is any subset of E, the $\sum_{e \in C} w(e)$ is denoted simply by $w(C)$.

We are interested in studying various subsets of Q^E generated by the incidence vectors of perfect matchings of a graph G. If e is an edge that is in no perfect matching of G, then all the vectors of interest would have a zero corresponding to e. For this reason, we restrict ourselves to graphs in which every edge is in some perfect matching. Without loss of generality we also require the graph to be connected. A connected graph in which every edge lies in some perfect matching is called *matching covered*. The set of all perfect matchings of a graph is denoted by $\mathcal{M}$.

The lattice generated by the incidence vectors of perfect matchings of G, called the *matching lattice* of G and denoted by $Lat(\mathcal{M})$, is defined by

$$Lat(\mathcal{M}) := \{w \in Z^E : w = \sum_{M \in \mathcal{M}} \alpha_M \chi^M, \alpha_M \in Z\}.$$

Given a graph G and $w \in Z^E$, we would like to be able to decide whether or not w is in $Lat(\mathcal{M})$. There is an obvious necessary condition, which requires the notion of cut. For subset S of $V(G)$, a *cut* $\nabla(S)$ is the set of edges having one end in S, the other in $V(G) \backslash S$. Set S is called a *shore* of $\nabla(S)$. Clearly, for any perfect matching M and any vertex v, $\chi^M(\nabla(v)) = 1$. Therefore, if $w = \sum_{M \in \mathcal{M}} \alpha_M \chi^M$, then, for any vertex v, $w(\nabla(v)) = \sum_{M \in \mathcal{M}} \alpha_M$. Thus we have:

Lemma 2 *A necessary condition for vector w to belong to $Lat(\mathcal{M})$ is that $w(\nabla(u)) = w(\nabla(v))$, for all u, v in V.* □

Each edge-weighting of G induces a natural vertex-weighting of G, where the weight of a vertex v is simply the sum of the weights of the edges incident with v. The above lemma says that an edge-weighting w of G is in $Lat(\mathcal{M})$ only if, in the vertex-weighting induced by w, all the vertex weights are the same. For bipartite graphs, this condition is also sufficient.

Theorem 3 *Let G be a bipartite matching covered graph, and let $w \in Z^E$. Then w is in $Lat(\mathcal{M})$ if, and only if, $w(\nabla(u)) = w(\nabla(v))$, for all u, v in V.* □

However, in general, the condition of Lemma 2 is not sufficient for a vector to belong to $Lat(\mathcal{M})$. Let $\mathbf{1}$ denote the vector whose coordinates are all equal to 1.

Lemma 4 *For the Petersen graph, the vector $\mathbf{1}$ satisfies the condition of Lemma 2, but it is not in the matching lattice.*

Proof. Let C be a fixed pentagon in the Petersen graph. It is easy to see that every perfect matching of the graph meets C in zero or two edges. Thus, $\chi^M(C) \equiv 0 \pmod 2$, for any $M \in \mathcal{M}$. Hence, if w is any integral linear combination of χ^M, $w(C) \equiv 0 \pmod 2$. However, $\mathbf{1}(C) = 5 \equiv 1 \pmod 2$. Therefore, $\mathbf{1}$ cannot be in the matching lattice for the Petersen graph. □

To describe the matching lattice of a general matching covered graph, we require the notion of a tight cut. A cut $\nabla(S)$ is *tight* if $|M \cap \nabla(S)| = 1$ for every perfect matching M of G. For each vertex v of G, $\nabla(\{v\})$ is tight; these tight cuts are called *trivial*.

Let $\nabla(S)$ be a non-trivial tight cut of G. The two graphs G_1 and G_2 obtained from G by contracting the two shores S and $G - S$, respectively, are called the *cut-contractions* of G with respect to $\nabla(S)$. If a matching covered graph has a non-trivial tight cut then it is possible to obtain smaller matching covered graphs by contracting the shores of the cut.

Lemma 5 *Let G be a matching covered graph, $\nabla(S)$ a non-trivial tight cut of G. Then the cut-contractions of G are matching covered.* □

If either G_1 or G_2 has a non-trivial tight cut, we can take its cut-contractions, in the same manner as above, and obtain smaller matching covered graphs. Thus, given any matching covered graph G, by repeatedly applying cut-contractions, we can obtain a list of graphs which do not have non-trivial tight cuts (we shall refer to such graphs as *undecomposable graphs*). This procedure is known as the *tight cut decomposition* procedure. It is useful for determining the matching lattice of G because $Lat(\mathcal{M})$ can be expressed in terms of the matching lattices of the resulting smaller graphs.

Theorem 6 *Let G be a matching covered graph, and let G_1 and G_2 be the two cut contractions of G with respect to a non-trivial tight cut. Let w be a vector in Z^E, and let w_1 and w_2 be the restrictions of w to $E(G_1)$ and $E(G_2)$, respectively. Then w is in $Lat(\mathcal{M})$ if, and only if, w_1 and w_2 are in $Lat(\mathcal{M}_1)$ and $Lat(\mathcal{M}_2)$, respectively.* $\square$

Thus, the matching lattice of a matching covered graph may be expressed in terms of the matching lattices of the graphs in the list of undecomposable graphs resulting from an application of the tight cut decomposition procedure. Different applications of the tight cut decomposition procedure on the same matching covered graph G may yield different lists of undecomposable graphs, but the lists cannot differ from each other significantly, as shown by Lovász [4].

Theorem 7 *The results of any two applications of the tight cut decomposition procedure on a matching covered graph are the same list of graphs except possibly for the multiplicities of edges.* $\square$

Recall that a brick is a 3-connected graph G such that, for any two vertices u and v of G, the graph $G - \{u, v\}$ has a perfect matching. Bricks play an important role in this theory.

Since the matching lattice of bipartite graphs are well understood, it is convenient to distinguish undecomposable graphs which are bipartite from those which are not. Bipartite undecomposable graphs are called *braces*, and non-bipartite undecomposable graphs turn out to be precisely the bricks, as shown by Edmonds *et al.* [2].

Theorem 8 *A non-bipartite matching covered graph has no non-trivial tight cuts if and only if it is a brick.* $\square$

In view of the above theorem, bricks and braces can be regarded as the building blocks of all matching covered graphs.

A vector w is called *matching integral* if $w(M)$ is an integer for each $M \in \mathcal{M}$. For example, in the Petersen graph, let C be a fixed pentagon. Let w be a vector which takes the value $1/2$ on the edges of C and the value 0 on all other edges. Each perfect matching in this graph has zero or two edges from C, that is, $w(M)$ is 0 or 1 for each perfect matching M. Hence, w is a matching integral vector.

For graph G, a *vertex labeling* is a function $\phi : V(G) \longrightarrow Q$. An *edge labeling induced by a vertex labeling* ϕ is a function $\varphi : E(G) \longrightarrow Q$ such that, if $e := (u, v)$ is an edge then $\varphi(e) := \phi(u) + \phi(v)$. A vector w is called *matching orthogonal* if $w(M) = 0$ for each $M \in \mathcal{M}$. Clearly, every matching orthogonal vector is also matching integral. Let ϕ be a vertex labeling which satisfies the condition that $\sum_{v \in V} \phi(v) = 0$. Then the edge labeling induced by ϕ is clearly a matching orthogonal vector. A matching orthogonal vector which can be obtained in this manner is said to be *node-induced*.

The following Lemma plays a fundamental role in the theory [4].

Lemma 9 *Let G be a brick different from the Petersen graph. Then every matching integral vector on G can be written as the sum of a node-induced matching orthogonal vector and an integral vector.* $\square$

We have seen that the obvious necessary condition (Lemma 2) for a vector to belong to the matching lattice is not sufficient in the case of the Petersen graph. The following Theorem shows that, in this regard, the Petersen graph is the only exception among bricks [4].

Theorem 10 *If G is a brick different from the Petersen graph then a vector $w \in Z^E$ belongs to $Lat(\mathcal{M})$ if and only if $w(\nabla(u)) = w(\nabla(v))$, for all u, v in V.* $\square$

One of the consequences of this theory, for example, is that for any 2-connected cubic graph without the Petersen graph as a brick, the vector **1** is in its matching lattice. This supports Tutte's conjecture which says that "If a cubic graph without cut edges does not contain the Petersen graph as a minor, then it is 3-edge-colourable".

3   Bases for the Matching Lattice

In this section we consider a matching covered graph G and an ear-decomposition $K_2 = G_0 \subset G_1 \subset \cdots \subset G_{r-1} = G$ of G. Among the r ears of that decomposition, let d denote the number of double ears. Thus, $r - d$ is the number of single ears in the decomposition. We denote, respectively, the number of edges, vertices and bricks of G by m, n and b. Among the b bricks, we denote by p the number of those which are isomorphic to the Petersen graph.

Lemma 11 $r + d = m - n + 2$. $\square$

The *linear hull* of perfect matchings of G is defined by

$$Lin(\mathcal{M}) := \{x \in R^E : x = \sum_{M \in \mathcal{M}} \alpha_M \chi^M, \alpha_M \in R\}.$$

The following Theorem is proved in Murty [7].

Theorem 12 *The dimension of $Lin(\mathcal{M})$ over $GF(2)$ is $m - n + 2 - p - b$.* $\square$

The next result was proved by Carvalho and Lucchesi.

Theorem 13 *If G is a matching covered graph then every ear-decompositon of G requires at least $p + b$ double ears.*

Proof. Let α_0 denote the edge of G_0, the first nice subgraph of G in the ear-decomposition, isomorphic to K_2. Let M_0 be a perfect matching of G containing α_0. For each i ($0 \le i < r - 1$), let M_{i+1} be a perfect matching of G formed by a perfect matching of G_{i+1} containing an edge of $E(G_{i+1}) \backslash (E(G_i) \cup M_0)$ plus the edges of $M_0 \backslash E(G_{i+1})$. Clearly, the sequence $\chi^{M_0}, \chi^{M_1} \ldots \chi^{M_{r-1}}$ of incidence vectors of perfect matchings of G is linearly independent over $GF(2)$, because each matching contains an edge which does not occur in any previous matching in the sequence. By Lemma 11 and Theorem 12, we have that

$$r + d - p - b = m - n + 2 - p - b \geq r,$$

which proves the Theorem.   □

Theorem 13 suggests the following minimax equality:

Conjecture[Carvalho-Lucchesi] *Every matching covered graph admits an ear-decomposition that uses precisely $p + b$ double ears.*   □

We now introduce three important results:

R1. (Lovász) Let G be a brick different from the Petersen graph. Then every matching integral vector on G can be written as a sum of a node-induced matching orthogonal vector and an integral vector.

R2. (Carvalho-Lucchesi) Every brick different from the Petersen graph admits an ear-decomposition that uses a unique double ear.

R3. (Vempala-Lovász) Let G be a brick different from the Petersen graph, K_4 and $\overline{C_6}$. Then there exist an edge $e \in E(G)$ such that $G - e$ is matching covered and its brick decomposition has exactly one brick.

Assertion R2 is a proof of the Conjecture for bricks. In view of Theorem 12, one consequence of Assertion R2 is that, in the case of bricks, it is possible to find a basis for $Lin(\mathcal{M})$ formed only by perfect matchings. Note that this is also a basis for $Lat(\mathcal{M})$.

If the Conjecture is true in general, then for every matching covered graph it is possible to find a basis for $Lin(\mathcal{M})$ and $Lat(\mathcal{M})$ formed only by perfect matchings.

R1 is known as the 'Main Lemma' in the article of Lovász [4]. R3 is a theorem proved recently by Lovász and Santosh Vempala [6]. R2 is a new result. It is possible to prove that R2 $\iff$ R3 and R2 $\Rightarrow$ R1. These results are part of the doctoral thesis of Marcelo H. Carvalho, written under supervision of C. L. Lucchesi.

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