Matching Covered Graphs and Subdivisions of K_4 and $\overline{C_6}$

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Abstract

We give a very simple proof that every non-bipartite matching covered graph contains a nice subgraph that is an odd subdivision of K_4 or $\overline{C_6}$. It follows immediately that every brick different from K_4 and $\overline{C_6}$ has an edge whose removal preserves the matching covered property. These are classical and very useful results due to Lovász.

1 Introduction

We consider simple graphs, i.e., finite graphs without loops and multiple edges. We denote respectively by $E(G)$ and $V(G)$ the set of edges and vertices of a graph G. A *matching* is a set of edges no two of which have a vertex in common. A matching is *perfect* if its edges match up all vertices. Recall the following fundamental result of Tutte [7], where $c_1(H)$ denotes the number of odd components of a graph H:

Theorem 1 *A graph G has a perfect matching if and only if $c_1(G - X) \leq |X|$, for each set X of vertices of G.* □

If equality holds in the inequality stated in Theorem 1 for some set X of vertices then X is called a *barrier*. If G has a perfect matching then clearly each vertex of G constitutes a barrier; the empty set is also a barrier. Those barriers, containing at most one vertex, are said to be *trivial*.

A connected graph is *matching covered* if each of its edges lies in some perfect matching and *bicritical* if deletion of any two of its vertices yields a graph having a perfect matching. It is easy to see that a connected graph with a perfect matching is (i) matching covered if and only if no barrier spans an edge and (ii) bicritical if and only if it has only trivial barriers.

A 3-connected bicritical graph is called a *brick*. Three bricks play a special role in the theory of matching covered graphs: K_4, the complete graph on 4 vertices, $\overline{C_6}$, the triangular prism, and the Petersen graph.

Let H be a subgraph of G. A path P in $G - E(H)$ is an *ear* of H if (i) both ends of P lie in H and (ii) P is internally disjoint from H. An ear is *odd* if it has odd length. Henceforth, by an 'ear' we shall mean an 'odd ear'. An *ear-decomposition* of a matching covered graph G is a sequence $K_2 = G_0 \subset G_1 \subset \cdots \subset G_k = G$ of matching covered subgraphs of G, where for $0 \leq i < k$, G_{i+1} is the union of G_i and one or two vertex-disjoint ears of G_i.

A subgraph H of matching covered graph G is *nice* if $G - H$ has a perfect matching. Clearly, each term G_i of an ear-decomposition of matching covered graph G is nice. The following theorem was proved by Lovász and Plummer [5].

Theorem 2 *Every matching covered graph has an ear-decomposition* □

There is an alternative way of defining an ear-decomposition, in which one might think of starting with just any nice matching covered subgraph H of G, not necessarily K_2. In view of Theorem 2, one could then prefix that ear-decomposition with an ear-decomposition of H starting with K_2, thereby obtaining an ear-decomposition of G in the original sense. With that in mind, the following generalization of Theorem 2 was proved by Lovász and Plummer [5].

Theorem 3 *Every matching covered graph has an ear-decomposition starting with any nice matching covered subgraph.* □

The following theorem, proved by Little [2] (see also [5, page 177]), plays an important role in the proof given herein.

Theorem 4 *Any two edges of a matching covered graph lie in a nice circuit.* □

To complete this section, we require the notion of a tight cut. For subset S of $V(G)$, a *cut* $\nabla(S)$ is the set of edges having one end in S, the other in $V(G) \setminus S$. A cut $\nabla(S)$ is *tight* if $|M \cap \nabla(S)| = 1$ for every perfect matching M of G. For each vertex v of G, $\nabla(\{v\})$ is tight: these tight cuts are called *trivial*.

A bipartite graph is called a *brace* if deletion of any four vertices, two from each color class, yields a graph having a perfect matching. The following theorem was proved partly by Edmonds *et al.* [1], partly by Lovász [4].

Theorem 5 *A matching covered graph has no non trivial tight cut if and only if it is either a brick or a brace.* □

2 Odd Subdivisions of K_4 and $\overline{C_6}$

An *odd subdivision* of a graph G is a graph obtained from G by subdividing each edge in an odd number of edges.

We now give a concise proof of a classical result due to Lovász [3]. The best known proof of this theorem appears in Lovász and Plummer's book [5]. It is important to note that in their book they call these *even* subdivisions.

Theorem 6 *Every non-bipartite matching covered graph G contains a nice subgraph that is an odd subdivision of K_4 or $\overline{C_6}$.*

Proof. By induction on $|V(G)| + |A(G)|$.

Case 1 *G has a proper subgraph H that is non bipartite, matching covered and nice.*

By induction hypothesis, H has a nice subgraph K that is an odd subdivision of K_4 or $\overline{C_6}$. Since K is a nice subgraph of H and H is a nice subgraph of G, K is a nice subgraph of G.

Case 2 *G has a vertex u with degree two.*

Let v and w be the adjacent vertices of u along the edges α_v and α_w, respectively. Let H be a graph obtained from G by contracting $\{u, v, w\}$ to a single vertex x. It is clear that H is non bipartite. For every perfect matching M of G, precisely one of α_v and α_w belongs to M, and so $M \backslash \{\alpha_v, \alpha_w\}$ is a perfect matching of H. Thus, H is matching covered. By induction hypothesis, H has a nice subgraph H' that is an odd subdivision of K_4 or $\overline{C_6}$.

Consider $G'' := G[E(H')]$, the subgraph of G induced by the edges of H'. If at most one of v and w is a vertex of G'', then clearly G'' is nice (relative to G) and isomorphic to H'. We may thus assume that both v and w are vertices of G''. Since H' is a subdivision of K_4 or $\overline{C_6}$, no vertex of H' has degree greater than three. Thus one of v or w has degree one in G''. In that case, addition of α_v and α_w to G'' yields an odd subdivision of H' which is nice relative to G.

Case 3 *None of the previous cases apply.*

We show that G is K_4 or $\overline{C_6}$ by the following strategy: it is sufficient to get a nice subgraph L of G that is an odd subdivision of K_4 or $\overline{C_6}$. In fact, since K_4 and $\overline{C_6}$ are non bipartite matching covered graphs, so is L. Since Case 1 does not apply, $G = L$. Since Case 2 does not apply, $L = K_4$ or $L = \overline{C_6}$.

To get L observe initially that G, being matching covered, has an ear decomposition $G_0, G_1, \ldots, G_n$, where $G_0 = K_2$. Graph G is not bipartite, so $n > 0$. Graph G_{n-1} is nice, matching covered, and since Case 1 does not apply, it is bipartite. Let (A, B) be a bipartition of G_{n-1}. Since $G_n = G$ is not bipartite, the last ear is a 2-ear (or double ear) with one ear having both ends in A and the other having both ends in B. But Case 2 does not apply, so both ears are simply edges. Let us denote them by $\alpha := (a_1, a_2)$ and $\beta := (b_1, b_2)$ so that

$$\{a_1, a_2\} \subseteq A \quad \text{and} \quad \{b_1, b_2\} \subseteq B.$$

Since G_{n-1} is matching covered, by Theorem 4 any two edges of G_{n-1} lie in a nice circuit. In particular, if we take two edges of G_{n-1} with ends in a_1 and a_2, respectively, G_{n-1} has a nice circuit having α as a chord.

Among all nice circuits of G_{n-1} having α or β as a chord, choose one, C, with minimum length. Let M_1 be a perfect matching of G_{n-1} so that C is M_1-alternating.

Proposition 7 *If α (or β) is a chord of C then it crosses every chord γ of C different from α and β.*

Proof. Let C' and C'' be the two subpaths of C having ends in the two ends of γ. Edge γ, being different from α and β, belongs to G_{n-1}, a graph with bipartition (A, B). Circuit C is M_1-alternating and belongs to G_{n-1}. So the circuits $D' := C' \cup \{\gamma\}$ and $D'' := C'' \cup \{\gamma\}$ are both in G_{n-1}, one of them is M_1-alternating and the other is $(C \backslash M_1)$-alternating. Thus, both circuits are nice in G_{n-1}.

Suppose that α is a chord of C that does not cross γ. In this case both ends of α belong to C' or both belong to C'', say to C'. So D' is a nice circuit of G_{n-1} having α as a chord. But $|D'| < |C|$, in contradiction to the definition of C. A similar conclusion holds if β is a chord of C that does not cross γ. □

Let M be a perfect matching in G containing α (and β). Let D be the M_1-alternating circuit of $M \oplus M_1$ containing α. $|D|$ is even and all the edges of $D \backslash \{\alpha, \beta\}$ lie in G_{n-1}, so $|D|$ contains β. The subgraph $G' := G[C \cup D]$ of G, generated by $C \cup D$, is certainly:

- nice and matching covered, because C and D are both M_1-alternating circuits.
- non bipartite, because one of α or β is a chord of C that lies in D and has both ends in one of A and B.

Since Case 1 does not apply, $G = G[C \cup D]$. So, all vertices of G have degree 2 or 3. But Case 2 does not apply. So G is cubic and its set of edges has a partition in three perfect matchings M_1, M_2, M_3, where $M_1 = C \cap D$, $M_2 = C \backslash M_1$ and $M_3 = D \backslash M_1$. Note that M_3 is the set of all chords of C and it contains both α and β.

To complete the proof that G is either K_4 or $\overline{C_6}$, we consider separately two cases, depending on whether chords α and β cross or not. Consider first the case in which α and β cross. That is, the vertices a_1, b_1, a_2, b_2 are in C in this cyclic order. In this case, the graph $L := G[C \cup \{\alpha, \beta\}]$ is an odd subdivision of K_4. Moreover, $V(L) = V(C) = V(G)$. Therefore, L is nice. Thus $G = L = K_4$.

Let us now consider the remaining case, in which α and β do not cross. Let C_α be the segment of C between the ends a_1 and a_2 of α that does not contain the ends b_1 and b_2 of β. Analogously, C_β is the segment of C between b_1 and b_2 that does not contain a_1 and a_2. The ends a_1 and a_2 of C_α belong to A. But C_α is a path in G_{n-1}, so it contains (at least) one vertex b in B. Let γ be the edge of G that is incident with b (and is a chord of C). The other end a of γ belongs to A. By Proposition 7, a is in C_β. So, $L := G[C \cup \{\alpha, \beta, \gamma\}]$ is an odd subdivision of $\overline{C_6}$. Moreover, L is nice because $V(L) = V(C) = V(G)$. Thus $G = L = \overline{C_6}$. □

Lemma 8 *Let G be a brick, $\{e, f\} \subseteq E(G)$ such that $G - e - f$ is matching covered and any perfect matching that contains one of these edges also contains the other. Then $G - e - f$ is bipartite.*

Proof. By hypothesis, if we remove both ends of e in $G - f$, the resulting graph has no perfect matching. But graph $G - e - f$ has a perfect matching. Thus graph $G - f$ has a barrier B containing both ends of e. Moreover, $G - e - f$ is matching covered. Thus e is the only edge of $G - f$ having both ends in B.

Suppose $G - e - f$ is not bipartite. Then some component H of $G - e - f - B$ is non trivial. Since G is a brick, by Theorem 5 the non trivial odd cut $\nabla(H)$ is not tight. So there is a perfect matching M so that $|M \cap \nabla(H)| \geq 3$. A simple counting argument shows that this happens only if:

1. $|M \cap \nabla(H)| = 3$;
2. The ends of f are in distinct components of $G - e - f - B$;
3. f is in M and e is not. But this is a contradiction.

Therefore $G - e - f$ is bipartite. □

Theorem 9 *Let G be a brick different from K_4 and $\overline{C_6}$. Then G has an edge e such that $G - e$ is matching covered.*

Proof. By Theorems 6 and 3, G has an ear-decomposition $G_0, G_1, \ldots, G_k = G$ in which the first non bipartite graph is an odd subdivision of K_4 or $\overline{C_6}$. Since G is 3-connected, the last ear consists of single edges. We now show that the last ear is simple, thereby proving the Theorem. Since G is different from K_4 and $\overline{C_6}$, then G_{k-1} cannot be bipartite. By Lemma 8, G_k arises from G_{k-1} by the adjunction of a single ear. □

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