Cryptographic hash functions

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Introduction

Objectives:

- Study properties and constructions for cryptographic hashing.

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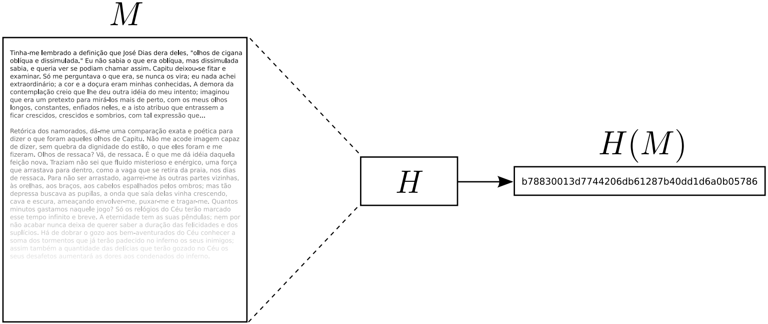
Hidden intentions:

- Simultaneously infer the limitations of cryptographic hash functions.

Cryptographic hash functions

Informal definition

Cryptographic hash functions are employed to produce a short descriptor of a message. Informally, this descriptor is analogous to a fingerprint for human identification.



Cryptographic hash functions

Formal definition

A **cryptographic hash function** maps messages from a set $\mathcal{X}$ to hash values or authenticators in a set $\mathcal{Y}$. In this first case, it is denoted by $h : \mathcal{X} \rightarrow \mathcal{Y}$. In the second, it is parameterized by a key $K \in \mathcal{K}$ and represented by $h_K : \mathcal{X} \rightarrow \mathcal{Y}$. If $\mathcal{X}$ is finite h is also called a **compression function**.

Many different applications:

- Password storage (store $h(s)$ instead of s).
- Key derivation ($k = h(g^{xy} \bmod p)$, $k_i = h(k_{i-1})$).
- Integrity verification ($y = h(x)$).
- Digital signatures (sign $h(m)$ instead of just m).
- Message Authentication Codes (MACs) ($y = h_K(x)$).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Properties of hash functions

- **Preimage resistance**: Given hash y, it should be computationally infeasible to find x such that $y = h(x)$.

- **Second preimage resistance**: Given hash y and a message x such that $y = h(x)$, it should be computationally infeasible find $x' \neq x$ such that $h(x') = h(x) = y$.

- **Collision resistance**: It should be computationally infeasible to find x, x' such that $h(x) = h(x')$.

Important: Each property implies the previous one (in the first case, conditionally).

Properties of hash functions

Collision from second preimage

1. Choose random x.
2. Compute $y = H(x)$.
3. Obtain second preimage $x' \neq x$ such that $H(x') = H(x) = y$.
4. Return collision (x, x').

Second preimage from first preimage

1. Compute $y = H(x)$.
2. Invert $x' = H^{-1}(y)$ until you obtain $x' \neq x$.
3. Return collision (x, x').

Important: If $|\mathcal{X}| \geq 2|\mathcal{Y}|$, not possible to obtain collision resistance if S is not resistant to both first and second preimages!

Properties of hash functions

From the reductions:

- Collision resistance implies second preimage resistance.
- If $|\mathcal{X}| \geq 2|\mathcal{Y}|$, collision resistance implies preimage resistance.
- Finding collisions has no impact to first and second preimages.
- Not possible to find first or second preimages without affecting collision resistance.

Hash functions design

- **Merkle-Damgård paradigm**: MD4, MD5, SHA-1, SHA-2.
- Block cipher-based: Matyas-Meyer-Oseas, David-Meyer.
- New paradigms: Sponge (SHA3/Keccak).
- Number theory: VHS (integer factoring), ECOH (elliptic curves).

Random Oracle Model (ROM)

Definition

The **Random Oracle Model** is a mathematical model of an *ideal* hash function: the function is chosen randomly from all such functions $f : \mathcal{X} \rightarrow \mathcal{Y}$ and represented by an oracle. Because the formula or algorithm are unknown, the only way to compute the hash function is to sample the oracle.

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Advantages: Models the security requirements of hash functions and allows reducing the security of protocols to oracle properties.

Disadvantages: Real hash functions are not ideal!

Birthday paradox

It is a classic problem that demonstrates how counter-intuitive results in probability can be to the human brain.

Definition

What is the minimum value k such that the probability of two persons in a room with k people share their birthdays is higher than 50%?

Birthday paradox

Let $p'(n), n \leq 365$ the probability that all birthdays are different:

$$p'(n) = 1 \cdot \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdot \ldots \cdot \left(1 - \frac{n-1}{365}\right) = \frac{365!}{(365-n)!365^n}$$

We have that $p(n) = 1 - p'(n)$. Thus, $p(n) > 0.5$ if $n \geq 23$ and $p(n) = 1$ if $n \geq 100$.

Important: With only $k = 23$ people, the probability that two of them share birthdays is already over 50%!

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Important: Do not confuse with the much probability of another person in the room share a fixed birthday $q(n) = 1 - \left(\frac{364}{365}\right)^n$.

Birthday attack

Generalizing to hash functions where $|\mathcal{Y}| = M$, the probability of finding collisions after n random samples is:

$$p(n) = 1 - \left(1 - \frac{1}{M}\right) \cdot \left(1 - \frac{2}{M}\right) \cdot \ldots \cdot \left(1 - \frac{n-1}{M}\right) \approx 1 - e^{-n(n-1)/(2M)}$$

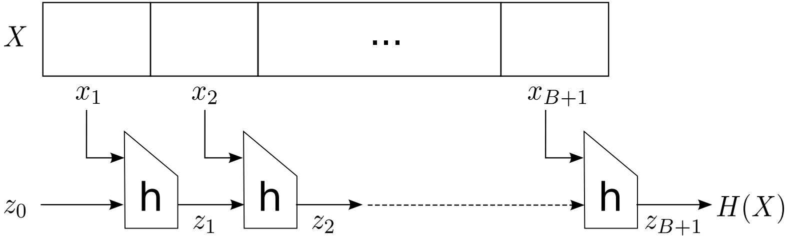
Replacing $p(n) = \frac{1}{2}$ and solving for n, we have that $n \approx 1.17\sqrt{M}$. In other words, sampling more than $\sqrt{M}$ elements should produce a collision with probability of 50%.

Important: That is why hash functions with output length of m *bits* offer security of only $\frac{m}{2}$ *bits*!

Iterated hash functions (Merkle-Damgård)

Definition

It is a technique that allows constructing a hash function with infinite domain $H : \{0,1\}^* \rightarrow \{0,1\}^m$ through consecutive applications of a **compression function** $h : \{0,1\}^{m+t} \rightarrow \{0,1\}^m$. **Padding** is needed for adding block x_{B+1} to an input x with B blocks.



Important: Collision resistance for S is given by collision resistance for h.

SHA-1 hash function

Definition

It is a cryptographic hash function $H : \{0,1\}^{2^{64}} \rightarrow \{0,1\}^{160}$ following the Merkle-Damgård paradigm.

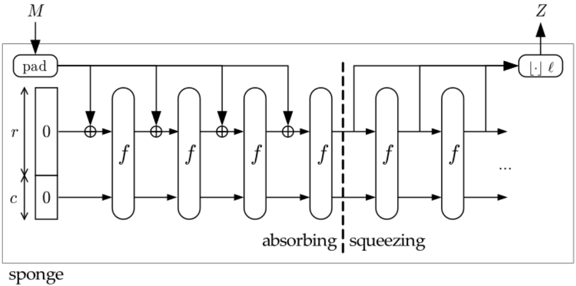
Brief history:

- Proposed by NIST in 1993.
- It is an improvement over SHA-0 (collision in 2^{61} operations).
- It is an 80-round iterated hash function with compression function $h : \{0,1\}^{512} \rightarrow \{0,1\}^{160}$.
- After attacks, SHA-2 and SHA-3 became standard.
- Security estimated in 60 *bits*.

Iterated hash functions (Sponge)

Definition

The **sponge construction** is a mode of operation based on a *fixed-length permutation* and a *padding rule*, which builds a function mapping variable-length input to variable-length output. A sponge function is a generalization of both hash functions, which have a fixed output length, and stream ciphers, which have a fixed input length.



Important: Collision resistance depends on **internal state size**!