Further Solution Concepts on Normal-Form Games

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Solution Concepts

Principles according to which we identify interesting subsets of the outcomes of a game.

- Most important solution concept is the **Nash Equilibrium**
- However, there are also a large number of others

Maxmin strategy

The ***maxmin strategy (1)*** of player *i* is a strategy that maximizes *i*'s worst-case payoff

The ***maxmin value (2)*** of the game for player *i* is that minimum amount of payoff guaranteed by a ***maxmin*** strategy

$$arg\,max_{s_i}\, min_{s_{-i}}\, u_i(s_i, s_{-i}) \qquad (1)$$

$$max_{s_i}\, min_{s_{-i}}\, u_i(s_i, s_{-i}) \qquad (2)$$

Minmax strategy

[2-palyer] The **minmax strategy** of the player *i* ***against*** *-i* is given by:

$$arg\,min_{s_i}\,max_{s_{-i}}\,u_i(s_i, s_{-i})$$

[n-player] If we consider that all other player can *"gang-up" and coordinate* **against** a player $j \neq i$ then:

$$arg\,min_{s_{-j}}\,max_{s_j}\,u_j(s_j, s_{-j})$$

Prisoner Dilemma

	Coop	Defect
Coop	-1 -1	0 -5
Defect	-5 0	-3 -3

League of Legends Role Selection

League of Legends Role Selection

League of Legends Role Selection

League of Legends Role Selection

	Mid	Bot
Mid	-1 -1	3 5
Bot	5 3	0 0

Minmax and Maxmin values (von Neumann, 1928)

Theorem. In any finite, two player, zero-sum game, in any **Nash equilibrium** each player receives a payoff that is *equals to both his maxmin value and his minmax value*.

Maxmin Theorem (von Neumann, 1928)

Proof. There is at least one Nash equilibrium, due to:

> *Every game with a finite number of players and action profiles has at least one Nash equilibrium. (Nash, 1951)*

Let's denote an arbitrary equilibrium as (s'_i, s'_{-i}), i's payoff to be v_i and maxmin value of i as $\overline{v_i}$.

We will show that $v_i = \overline{v_i}$.

We cannot have $\overline{v_i} > v_i$, otherwise i would be willing to change from his strategy s'_i to the maxmin, and hence we would not have an equilibrium.

Maxmin Theorem (von Neumann, 1928)

We still have to proof that $\overline{v_i}$ is not less than v_i.

Assume $\overline{v_i} < v_i$. Since it is an equilibrium, each player has played the best response to the strategy of the other.

Therefore,

$$v_{-i} = max_{s_{-i}} u_{-i}(s_i', s_{-i})$$

$$-v_{-i} = min_{s_{-i}} - u_{-i}(s_i', s_{-i})$$

Maxmin Theorem (von Neumann, 1928)

Since it is a *zero-sum* game, $v_i = -v_{-i}$ and $u_i = -u_{-i}$.

Thus,

$$v_i = min_{s_{-i}} u_i(s'_i, s_{-i})$$

But,

$$\overline{v_i} = max_{s_i} min_{s_{-i}} u_i(s_i, s_{-i}) \geq min_{s_{-i}} u_i(s'_i, s_{-i})$$

Which implies in the absurd $\overline{v_i} \geq v_i$, therefore:

$$v_i = \overline{v_i}$$

$\blacksquare$

Dominated Strategies

One strategy **dominates** another for a player i if the first strategy yields i a greater payoff than the second strategy, for any strategy profile of the remaining players.

Dominated Strategies

Let s_i and s'_i be two strategies of player i, and S_i the set of all strategy profiles of the remaining players. Then,

s_i strictly dominates s'_i if:

$$\forall s_{-i} \in S_{-i} \Rightarrow u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

s_i weakly dominates s'_i if:

$$\forall s_{-i} \in S_{-i} \Rightarrow u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$

and

$$\exists s_{-i} \in S_{-i} \Rightarrow u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

s_i very weakly dominates s'_i if:

$$\forall s_{-i} \in S_{-i} \Rightarrow u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$

Dominated Strategies

A strategy is strictly (weakly or very weakly) dominant **for an agent** if it strictly (weakly or very weakly) dominates **any other strategy** for that agent.

A strategy s_i is strictly (weakly; veryweakly) dominated **for an agent** i if **some other strategy** s_i' strictly (weakly; very weakly) dominates si.

Removal of Dominated Strategies

	L	C	R
U	3,1	0,1	0,0
M	1,1	1,1	5,0
D	0,1	4,1	0,0

$\Rightarrow$

	L	C
U	3,1	0,1
M	1,1	1,1
D	0,1	4,1

Removal of Dominated Strategies

	L	C
U	3,1	0,1
M	1,1	1,1
D	0,1	4,1

$\Rightarrow$

	L	C
U	3,1	0,1
D	0,1	4,1

Why is M dominated?

$\sigma = (\frac{1}{2}, 0, \frac{1}{2})$ $\quad \mathbb{E}[u_1(\sigma)] = \frac{4}{2}c + \frac{3}{2}l = \frac{3+c}{2}$

$\varphi = (0, 1, 0)$ $\quad \mathbb{E}[u_1(\varphi)] = 1$

$$\frac{3+c}{2} \geq 1.5 > 1$$

$$\mathbb{E}[u_1(\sigma)] > \mathbb{E}[u_1(\varphi)]$$

σ strictly dominates φ.

Maximally Reduced Game

	L	C
U	3,1	0,1
D	0,1	4,1

Rationalizability

A strategy is **rationalizable** if a *perfect rational player* could justifiably play it against one or more perfectly rational opponents.

Every player takes the other players to also be rational, thus accounting for their rationality, their knowledge of his rationality, their knowledge of his knowledge of their rationality and so on in an **infinite regress**.

Rationalizability

Let's define an infinite sequence of possible strategies

$S_i^0, S_i^1, S_i^2, \ldots$ $\qquad S_i^0 = S_i$ (all i' mixed strategies)

We define then S_i^k as all strategies $s_i \in S_i^{k-1}$ for which there exists some $s_{-i} \in \prod_{j \neq i} CH(S_j^{k-1})$ such that for all $s_i' \in S_i^{k-1}$

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

Correlated Equilibrium

It is a result of a concept that **generalizes** Nash equilibrium.

Believed to be the **most fundamental** solution concept of all.

In a standard game, each player produces his mixed strategy independently. However, it is possible to **correlate** players' strategies with a **random external variable**.

The standard game can be viewed as a **degenerate** case in which these variables are **probabilistically independent**.

Battle of Sexes

	VALOR	MYSTIC
VALOR	2,1	0,0
MYSTIC	0,0	1,2

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Dominated Strategy

Battle of Sexes

$$\sigma = \left(\frac{2}{3}, \frac{1}{3}\right) \qquad \varphi = \left(\frac{1}{3}, \frac{2}{3}\right)$$

$$\mathbb{E}[u_1(\sigma)] = \frac{2}{3} \qquad \mathbb{E}[u_2(\varphi)] = \frac{2}{3}$$

Battle of Sexes

$$\sigma'$$

Heads = Valor

Tails = Mystic

$$\mathbb{E}[u_i(\sigma')] = 0.5 * 2 + 0.5 * 1 = 1.5$$

$$\mathbb{E}[u_i(\sigma')] > \mathbb{E}[u_1(\sigma)]$$

$$\mathbb{E}[u_i(\sigma')] > \mathbb{E}[u_2(\varphi)]$$

Correlated Equilibrium

Both players can **observe an external event** with a commonly-known probability distribution.

Given an n-agent game G = (N, A, u), a correlated equilibrium is a tuple (v, π, ω) where $v = (v_1, \ldots, v_n)$ with respective domains $D = (D_1, \ldots, D_n)$, π is a joint distribution over v, $\omega = (\omega_1, \ldots, \omega_n)$ is a vector of mappings $\omega_i : D_i \mapsto A_i$, and for each agent i and every mapping $\omega'_i : D_i \mapsto A_i$ it is the case that

$$\sum_{d \in D} \pi(d) u_i(\omega_i(d_i), \omega_{-i}(d)) \geq \sum_{d \in D} \pi(d) u_i(\omega'_i(d_i), \omega_{-i}(d))$$

Battle of Sexes

$$v$$

$$D_1 = D_2$$

Heads = Valor

Tails = Mystic

$$\omega_1 = \omega_2$$

π Moedas não viciadas = 1/2

A_i = {valor, mystic}

Battle of Sexes

$\upsilon\,\pi$

$$\varphi_1 \rightarrow \omega_1 \begin{cases} \text{Heads = Valor} \\ \text{Tails = Mystic} \end{cases}$$

$$\varphi_2 \rightarrow \omega_2 \begin{cases} \text{Heads = Mystic} \\ \text{Tails = Valor} \end{cases}$$

$$\mathbb{E}[u_1(\varphi_1)] = \mathbb{E}[u_2(\varphi_2)] = 0$$

Correlated Equilibrium

Theorem. For every Nash equilibrium there exists a corresponding correlated equilibrium.

On the other hand, not every correlated equilibrium is equivalent to a Nash equilibrium.

ϵ-Nash equilibrium

Fix $\epsilon > 0$. A strategy profile $s = (s_1, \ldots, s_n)$ is a ϵ-Nash equilibrium if, for all agents i and for all strategies $s_i' \neq s_i$

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) - \epsilon$$

Every Nash equilibrium is surrounded by ϵ-Nash equilibrium region, but the reverse is not true.

A given ϵ-Nash equilibrium is **not necessarily close** to any Nash equilibrium.

ϵ-Nash equilibrium

This game has a unique Nash equilibrium at (D, R).

	L	R
U	1,1	0,0
D	$1+\frac{\epsilon}{2}, 1$	500,500

However, there is an ϵ-Nash equilibrium at (U, L).

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