Fractional Hedonic Games

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Abstract

Here we present a summary of the paper of [Aziz et al. 2014]. We give a formal definition of fractional hedonic games, and we show that for some classes of such games the core can be empty. We also present some classes for which the core is guaranteed to be non-empty.

1 Introduction

Here, we summarize the paper of [Aziz et al. 2014] about *fractional hedonic games*. *Hedonic games* are coalition formation games introduced by [Dreze and Greenberg 1980], where the outcomes are partitions of the agents over which they have preferences. Each agent only care about her coalition and is not influenced by how the others coalitions are formed. Hedonic games are subject of many studies, in particular, some classes that can be represented as graphs, we call these classes *graphical hedonic games*. *Fractional hedonic games* are a graphical hedonic games class introduced by [Aziz et al. 2014]. Below, we define formally the fractional hedonic games.

Let N be a set of players. For each $i \in N$, let $\mathcal{N}_i$ denote the set $\{S \subseteq N : i \in S\}$. A *hedonic game* is a pair $(N, \succsim)$, where $\succsim = (\succsim_1, \ldots, \succsim_n)$ is a profile of complete and transitive relations $\succsim_i$ modeling the agents preferences over the coalitions. Let $S, T \in \mathcal{N}_i$ be two coalitions that contains i, we use $S \succ_i T$ to denote that agent i stricly prefers coalition S over T, and we use $S \sim_i T$ to denote that agent i is indiferent between coalition S and T.

For all $i \in N$, let $v_i : N \to \mathbb{R}$ be a value function for agent i that determines how much agent i values each other agent. Unless stated otherwise, we define that $v_i(i) = 0$. We can extend a value function v_i to a value function over coalitions such that, for all $S \in \mathcal{N}_i$, $v_i(S) = \frac{\sum_{j \in S} v_i(j)}{|S|}$.

A hedonic game $(N, \succsim)$ is said to be a *fractional hedonic game* if for each player $i \in N$ there is a value function v_i such that for all coalition $S, T \in \mathcal{N}_i$, $S \succsim_i T$ if and only if $v_i(S) \geq v_i(T)$. A fractional hedonic game is said to be symmetric if $v_i(j) = v_j(i)$, and it is said to be simple if $v_i(j) \in \{0, 1\}$ for all agent $i, j \in N$.

Fractional hedonic games can be represented as weighted digraphs (V, A, w), where the vertice set $V = N$, and there is a arc $(i, j) \in A$ if and only if $v_i(j) \neq 0$, and $w(i, j) = v_i(j)$. A symmetric fractional hedonic game can be represented as a weighted graph (V, E, w), where $V = N$, and there is a edge $(i, j) \in E$ if and only if $v_i(j) \neq 0$, and $w(i, j) = v_i(j)$. Simple symmetric fractional hedonic games can be represented as a graph (V, E), where $V = N$ and there is a edge $(i, j) \in E$ if and only if $v_i(j) = 1$.

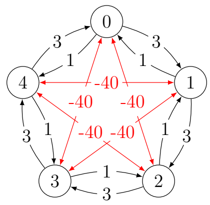

Figure 1: Example of a fractional hedonic game with an empty core.

The outcomes of hedonic games are agent partitions or coalitions structures. For a partition $\pi = \{\pi_1, \ldots, \pi_m\}$ of the agents, $\pi(i)$ denotes the coalition of which agent i is a member.

A solution concept formalizes desirable or optimal ways in which the players can be partitioned. We use solution concepts to analyze properties of outcomes of hedonic games. The *core* and the *strict core* are solution concepts that formalize the concept of stability of partitions. We say that a coalition $S \subseteq N$ *blocks* a partition π if for all $i \in S$, $S \succ_i \pi(i)$. A partition that is not blocked by any coalition is said to be in the core. We say that a coalition $S \subseteq N$ *weakly blocks* a partition π if each agent $i \in S$ weakly prefers S to her coalition in π, and there is at least an agent $j \in S$ such that $S \succ_j \pi(j)$. A partition that is not weakly blocked by any coalition is said to be in the strict core.

In the following section, we present some theorems from [Aziz et al. 2014].

2 Main Results

In this section, we present the main results of [Aziz et al. 2014] regarding the existence and computation of stable partitions. We begin by presenting some classes of fractional hedonic games for which the core can be empty.

Theorem 1. *For fractional hedonic games, the core can be empty.*

Proof. The game illustrated in Figure 1 is an example of fractional hedonic with an empty core. Note that any coalition with more than 2 agents has negative utility for at least one of the agents. If there is one agent alone, she can persuade one of her neighbors to join her. Then, it can be shown that no individually rational partition is core stable. □

Theorem 2. *Computing a core stable partition for fractional hedonic games with symmetric preferences is NP-hard. Moreover, even checking whether a partition is core stable is coNP-complete.*

Theorem 3. *In simple symmetric fractional hedonic games, the strict core can be empty.*

Now we present some classes of fractional hedonic games for which the core is not empty.

Theorem 4. *For simple symmetric fractional hedonic games represented by graphs of degree at most 2, the core is non-empty.*

Proof. We give an algorithm to compute a core stable partition. First, find all subgraphs isomorphs to K_3, for each one create a coalition and remove it from the graph. Let us call the set of vertices that are in some subgraph isomorph to K_3 by V_1. After that, take a maximum matching on the remaining graph, for each matching edge create a coalition with its extremes. Let us call the set of vertices that are covered by the maximum match by V_2. The set $V \setminus (V_1 \cup V_2)$ are the vertices leaved alone. Note that vertices in V_1 cannot be in a blocking coalition because each vertex in V_1 is in one of its most favored coalitions. Also, note that there is no blocking coalition formed solely by vertices from $V \setminus (V_1 \cup V_2)$. If this were the case, we had not computed a maximum matching of $(V \setminus V_1, E \setminus E_1)$. Now let us assume that there is a vertex $v_2 \in V_2$ which is in a blocking coalition. This coalition has the form $\{v_2, v_2', v_3\}$ where $v_2, v_2' \in V_2$ and $v_3 \in V \setminus (V_1 \cup V_2)$. If the utility of v_2 is greater than $1/2$, then the utility of v_2' is less than $1/2$, therefore $\{v_2, v_2', v_3\}$ is not a blocking coalition. □

Theorem 5. *For simple symmetric fractional hedonic games represented by undirected forests, the core is non-empty.*

Proof. We give an algorithm to compute a core stable partition. Let $G = (V, E)$ be the graph that represents the game. We can suppose that G is connected, otherwise, we can apply the same algorithm to each connected component. Pick an arbitrary vertex $v_0 \in V$ and run a breadth-first search on it. Let L_0 consist of v_0 and L_k consists of all vertices that are at a distance of k from v_0. Let L_l be the last layer of the tree. We construct a partition π that we claim to be core stable. Initialize π to the empty set. For each vertex v in the second last layer L_{l-1} that has a child in the last layer L_l, add the set $\{v\} \cup \{w : w \in L_l \text{ and } (v, w) \in E\}$ to π. Remove the sets of this form $\{v\} \cup \{w : w \in L_l \text{ and } (v, w) \in E\}$ from the tree and repeat the process until no more layers are left. Note that this procedure stops and returns a partition π. We now prove that π is core stable. For the base case, we prove that no blocking coalition can be formed only by vertices from layers L_{l-1} and L_l. For a vertex u from the second last layer, u would only be in a blocking coalition S if S contains u, all the children of u and the parent of u. But then, S is not a blocking coalition for the children of u. For a leaf v to be in a blocking coalition, it will need to be with its parent u but in a smaller coalition. But such coalition is not blocking for u. We remove all vertices from coalitions only containing vertices from the last and second last layer and repeat the argument inductively. □

Let $\Theta = \{\theta_1, \ldots, \theta_t\}$ be the set of types that partitions the set N of players, where $|\Theta| = t$. Let $\theta(i)$ denote the type player i belongs to. A hedonic game $(N, \succsim)$ is called a *Bakers and Millers game* if the preferences of each player i are such that for all coalitions $S, T \in \mathcal{N}_i$, $S \succsim_i T$ if and only if $\frac{|S \cap \theta(i)|}{|S|} \leq \frac{|T \cap \theta(i)|}{|T|}$.

Theorem 6. *Let $(N, \succsim)$ be a Bakers and Millers game with type space $\Theta = \{\theta_1, \ldots, \theta_t\}$ and $\pi = \{S_1, \ldots, S_m\}$ a partition. Then, π is strict core stable if and only if for all types $\theta \in \Theta$ and all coalitions $S, S' \in \pi$, $\frac{|S \cap \theta|}{|S|} = \frac{|S' \cap \theta|}{|S'|}$.*

Lemma 1. *Let (V, E) be a graph with $|V| \geq 3$. Then, (V, E) has girth of at least five if and only if all $v, w \in V$ have at most one neighbor in common.*

Theorem 7. *For simple symmetric fractional hedonic games represented by graphs with girth at least five, the core is non-empty.*

Proof. Let $\mathcal{F}$ be a set of undirected graphs. An $\mathcal{F}$-packing of a graph G is a subgraph H of G such that each component of H is isomorphic to a member of $\mathcal{F}$. We will consider star-packings of graphs, that is, $\mathcal{F}$-packings with $\mathcal{F} = \{S_2, S_3, S_4, \ldots\}$ such that each S_i is a star with i vertices. Each star S_i with $i > 2$ has one center c and $i-1$ leaves $l_1, \ldots, l_{i-1}$. We assume S_2 to have two centers and no leaves.

With each star packing, denoted by π we associate an objective vector $\vec{x}(\pi) = (x_1, \ldots, x_{|V|})$ such that $x_i \leq x_j$ if $1 \leq i \leq j \leq |V|$, and there is a bijection $f : V \to \{1, \ldots, |V|\}$ with $u_v(\pi) = x_{f(v)}$. We assume these objective vector to be ordered lexicographically by $\geq$.

Let π be a star packing of a graph $G = (V, E)$ that maximizes the objective vector. Note that a vertex v has utility 0 under π if and only if v has no neighbors in G. Suppose by contradiction that π is not in the core, then there is coalition S that blocks π. Note that S contains no isolated vertex, therefore, S contains only centers or leaves of π. Note that, for any two leaves l, l' in π we have $\{l, l'\} \notin E$. For a contradiction assume the opposite. Then, l and l' are from different centers, otherwise G would have a triangle. Moreover, $\pi' = \{\{l, l'\}, \pi'_1, \ldots, \pi'_k\}$, where $\pi'_i = \pi_i \setminus \{l, l'\}$, is a star packing for which the objective vector is larger than the one for π.

We can divide our proof in three cases: (i) S contains no center; (ii) S contains one center; (iii) and S contains more than one center.

If (i), S only contains leaves, therefore S is not a blocking coalition because every leaf in S has utility 0.

If (ii), we show that $\vec{x}(\pi)$ is not optimal. Let S consist of a center c and m leaves $l_1, \ldots, l_m$ of π. Note that S induces a star, provided that there are no edges between leaves. Let l be one of these leaves, and c' be one center of π such that $c' \in \pi(l)$. Consider the partition π' such that

$$\pi'(k) = \begin{cases} \pi(c) \cup \{l_i\} & \text{if } k \in \pi(c) \cup \{l_i\}, \text{ and} \\ \pi(k) \setminus \{l_i\} & \text{otherwise} \end{cases}$$

Note that $\frac{\pi(c)-1}{|\pi(c)|} < \frac{|S|-1}{|S|}$ because $u_c(\pi) < u_c(S)$. Moreover, we have that $\frac{1}{|\pi(l)|} < \frac{1}{|S|}$ because $u_l(\pi) < u_l(S)$. Therefore, $|\pi(c)| < |S| < |\pi(l)|$. It follows that $|\pi'(l)| = |\pi(c) \cup \{l\}| \leq |S| < \pi(l)$ and thus $u_l(\pi') > u_l(\pi)$. Let k be such that $u_k(\pi') < u_k(\pi)$. Then, either $k = c'$ or $k \in \pi(c) \setminus \{c\}$. As c' is a center and l a leaf, in π', c' still is a center, then $u_{c'}(\pi') \geq \frac{1}{2}$ and $u_{c'} > u_l(\pi')$. If $k \in \pi(c) \setminus \{c\}$, then k and l are leaves of the same star of π' and we have that $u_k(\pi') = u_l(\pi')$. Therefore, $\vec{x}(\pi') > \vec{x}(\pi)$, contradicting the assumption of π being optimal.

If (iii), S contains at least two centers c and c' of π. Then, $u_c(\pi) \geq \frac{1}{2}$ and $u_{c'}(\pi) \geq \frac{1}{2}$. Either $|S| = 2k + 2$ or $|S| = 2k + 3$ for some $k \geq 1$. Provided that $u_c(S) > \frac{1}{2}$ and $u_{c'}(S) > \frac{1}{2}$, we have that $|\{i \in S : (c, i) \in E\}| \geq k + 2$ and $|\{i \in S : (c', i) \in E\}| \geq k + 2$. It follows that c and c' has at least 2 neighbors in common, which implies by Lemma 1 that G has a girth less than 5. □

References

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