—Research Proposal—

Game-Theoretic Analysis of Transportation Problems

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Abstract

Traffic congestion and CO_2 emissions are major issues in today's society, and it is mostly related with transportation system. As a class of resource allocation games, transportation games models those situations, and through them we can analyze how the selfish behavior of agents can impact on the social optimal outcome. We will consider some possible extensions of these games and will study the existence and properties of pure Nash equilibrium on the cases admitting it. Furthermore, we aim to give bounds on two measures of inefficiency of equilibrium, price of anarchy and price of stability respectively, for those extensions.

Keywords: Algorithmic Game Theory; Transportation Games; Existence and Computation of Equilibrium; Inefficiency of Equilibrium.

1 Introduction and Justification

John von Neumann and Oskar Morgenstern published in 1944 the book *Theory of Games and Economic Behavior* [1], which is considered the pioneer book in Game Theory, and since then this field has been developed by scientists ranging from economy to biology and more recently computer science. The first Nobel Prize given to a Game Theory researcher was awarded, in 1994, to John Harsanyi, John Nash, and Reinhard Selten "for their pioneering analysis of equilibria in the theory of non-cooperative games" [2]. Moreover, in the past years the number of nobel prizes has increased with the last one given to Jean Tirole in 2014 "for his analysis of market power and regulation" [2].

The studying of how rational agents behave when dealing with situations of conflict and cooperation is the main objective of Game Theory. Those agents, usually called players, want to maximize their utilities as much as possible, and in order to do that, they may act selfishly. A game is defined as the environment where those players choose one strategy from a set of strategies aiming to achieve their own interest. For example, we can imagine an auction of rare goods where the winner of one specific object will be the one which gives the higher bid. Here, each participant (player) can choose to bid a value (strategy), this value can be chosen from the set of positive real numbers (set of strategies), to try to get the object (interest).

It is not difficult to see that, as the number of players and the set of strategies grows, a game can have an exponential number of possibilities to be evaluated. Therefore, questions that commonly emerged are: is it possible to, given a game, calculate the best strategy for a player? If so, can it be done efficiently?

In this aspect, Computer Science brings tools to help the analysis of issues that arise in Game Theory. For instance, computational complexity can help to prove if a problem can be solved in polynomial time, and for problems that cannot be computed in polynomial time, it may provide ways of finding good solutions efficiently. Thus, analyzing problems of Game Theory from the point of view of Computer Science characterizes the field called Algorithmic Game Theory.

In this proposal, we aim to investigate the environment where players are competing against each other for the use of shared resources, commonly called as *Resource Allocation Games*. Ideally, it would be optimal if each player were assigned to a single resource, but this is often not the case because usually it has a cost associated and it can become very expensive to maintain the system working. More specifically, we are going to study a family of resource allocation games called *transportation games*.

This class of games were recently introduced by Fotakis et al. [3], and they model situations motivated by ride sharing systems like *Uber, Dial-a-ride* or *Blablacar*. Those systems are also important because of their direct impact on the environment in general as they can induce less pollutant gas emisson and the reduction of traffic congestion. Problems which are related with transportation commonly appear in the combinatorial optimization area such as the *Traveling Salesman Problem* [4] and the *Vehicle Routing Problem* [5] because of their practical applications and theoretical challenges.

Our research will focus on the existence of pure Nash equilibrium and also the properties of these equilibria if it exists. Moreover, for the analysis of the inefficiency of equilibrium we will use the concepts of the price of anarchy and the price of stability.

2 Literature Review

In this section, we present the general concepts from Game Theory, including the basic definitions as well as the class of problems of our interest. Most of these definitions are somehow related with the definitions given by Nisan et al. [6]. As we exhibit the concepts, we will give throughout this section some aspects of the bibliographic history.

2.1 Basic Definitions from Game Theory

Definition 1 *A **game** $\mathcal{G}$ is defined by a tuple $(N, \mathcal{S}, u)$, where N is the finite set of players, and $\mathcal{S} = \times_{i \in N} \mathcal{S}_i$ is the finite set of strategy profiles, with $\mathcal{S}_i$ being the strategy set of player i. $u = (u_i)_{i \in N}$ is a vector with $u_i : s_i \in \mathcal{S} \to \mathbb{R}$ being a cost function which maps into a real value a strategy chosen by a player i.*

We assume that the players are both rational and selfish. By this, we mean that players will always choose a strategy that maximizes their utilities or minimizes their costs according to the game regardless the outcome of the other players. Let us suppose we are dealing with a game where players want to maximize their utilities. If a player $i \in N$ chooses a strategy $s_i \in S_i$, then we use $u_i(s_i)$ to represent the value incurred to i when choosing strategy s_i. We also use s_{-i} to denote the vector representing the strategies played by others players excluding player i. Using this notation, we present the next definition.

Definition 2 *We say that a strategy $s \in S$ is a **dominant strategy**, if for every player $i \in N$, and each alternate strategy $s' \in S$, we have*

$$u_i(s_i, s'_{-i}) \geq u_i(s'_i, s'_{-i}).$$

According to [6], not all the games possess dominant strategies as it is a very hard requirement to satisfy. Because of this, we need a less stringent concept to help us in analyzing them. One important notion that captures the property of a stable solution is the Nash Equilibrium which will be defined next. Stable solutions are those where all players are playing their best strategy against the strategies chosen by the others.

Definition 3 *We say that a strategy $s \in S$ is a **(Nash) Equilibrium**, if for every player $i \in N$, and each alternate strategy $s' \in S$, we have*

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}).$$

Putting this in words, in an equilibrium, a player i does not want to change unilaterally from her strategy s_i to s'_i since it would not benefit her, supposing all the others players still playing their strategies in s_{-i}. Hence, all the players in an equilibrium are satisfied with their choices. On the one hand, if within a game an equilibrium is reached by the players choosing their strategy deterministically, then this equilibrium is called *Pure Nash Equilibrium - PNE.* On the other hand, if this outcome is achieved by players' choices in a randomized way over the set of theirs strategies, then this equilibrium is called *Mixed*

Nash Equilibrium - MNE. MNE is an important concept because of the following theorem proved by Nash [7], one of the most relevant result of Game Theory.

Theorem 2.1 *Every game with a finite set of players and strategies has a Mixed Nash Equilibrium.*

It is worth to note that both the assumptions in the previous theorem are important because games with an infinite set of players or games with a finite set of players having access to an infinite set of strategies, may not have a MNE.

While in an equilibrium we are only concerned about individual deviations, there are some refinements of equilibrium which deals with group deviations. One of them is the concept of *Strong Equilibrium (SE)* where given an outcome of a game, no group of players C can jointly deviate such that all players in C improve their cost. Another one is the *Super Strong Equilibrium (SSE)* which instead of all players in C have an improvement as in SE, in a SSE outcome it cannot exist a joint deviation of C in such a way that at least one player of C improves her utility while all others do not have a decrease in their utilities.

Next, we introduce the problem called *Selfish Load Balancing Games*, and we give an example adapted from Nisan et al. [6] of this game to show the application of some of those definitions seen until now.

Game 1 *A* ***Load Balancing Game*** $\mathcal{J}$ *is defined by a tuple* (N, M, w)*, where* $N = \{1, \cdots, n\}$ *is the set of tasks, and* $M = \{1, \cdots, m\}$ *is the set of machines. The vector* $w = (w_{ij})_{i \in N, j \in M}$ *represents the values of processing time of task* i *in machine* j*. Let us say each player is responsible for one of the tasks, and their goal is to have their tasks processed as fast as possible. Here, we have the vector of strategies meaning an attribution of tasks into machines,* $A : N \to M$*, and* $A(i)$ *shows the machine where task* i *will be processed. Let* A_j *be the set of tasks allocated in machine* j*. Then, the load of a machine* j*,* $l_j(A)$*, is calculated as* $l_j(A) = \sum_{i \in A_j} w_{ij}$*. The cost associated with each player* i *is the load of the machine* $A(i)$*. Finally, we have the social cost under attribution* A *as represented by the maximum load over all machines, also called makespan, denoted by* $c(A) = \max\{l_j(A) : j \in M\}$.

Example 2.1 *Consider an instance of the load balancing game with two identical machines and four tasks, with two of them have processing time of 2 and the others two have processing time 4. Figure 1 shows the only two attributions of this instance in equilibrium.*

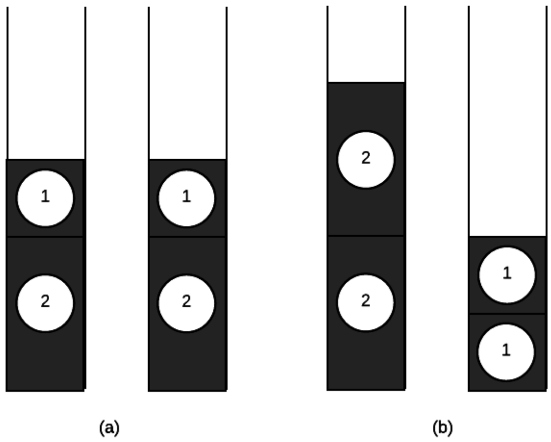

Figure 1: Two assignments in equilibrium for the instance in Example 2.1

In Figure 1-(a), it shows an optimal assignment A with $c(A) = 3$. It is clear here that A is an equilibrium since any task cannot improve her cost by changing to another machine (e.g., if one of the tasks with processing time 1 changes to another machine, it will have a cost worst than its current value, 4 instead of 3.). The other equilibrium of this instance is showed in Figure 1-(b) where here, under attribution A', it has a makespan of $c(A') = 4$. Note that these both assignments are SE and SSE since it does not exist a coalition of players that can benefit with they jointly deviate from their current strategies.

From Example 2.1, we can see that different equilibria can have different values. Because of it, we need tools for evaluating inefficiency of equilibria. Koutsoupias and Papadimitriou [8] introduced the term *Price of Anarchy*, which will be defined next, as being the the largest worst-ratio among all instances of a game between the worst equilibrium and the optimal social outcome of it.

Definition 4 *Given a function f representing the social function of a game $\mathcal{G}$. The **Price of Anarchy** (*PoA*) is defined as*

$$\mathrm{PoA}(f, \mathcal{G}) = \sup_{\mathcal{G}} \frac{\max_{A \in PNE(\mathcal{G})} f(A)}{\min_{A^* \in S} f(A^*)}.$$

Another relevant measure of inefficiency of equilibria is the *Price of Stability (*PoS*)*. It was proposed by Anshelevich et al. [9], and unlike PoA, it evaluates the largest best-ratio among all instances of a game between the best equilibrium and the optimal social outcome of it. As a consequence, we have that PoA $\geq$ PoS ≥ 1.

Definition 5 *Given a function f representing the social function of a game $\mathcal{G}$. The **Price of Stability (*PoS*)** is defined as*

$$\text{PoS}(f, \mathcal{G}) = \sup_{\mathcal{G}} \frac{\min_{A \in PNE(\mathcal{G})} f(A)}{\min_{A^* \in S} f(A^*)}.$$

Back to Example 2.1, it is clear to see that this instance presents PoS $= 1$ and PoA $= \frac{4}{3}$.

Even though every finite game has a MNE, there exists finite games without any PNE. Therefore, existence of PNE is an interesting issue when analyzing games. Indeed, an important tool for this purpose is the exact potential function because games possessing it, called *potential games*, has two main properties: they always have a PNE and converge to a PNE through better response dynamics. Theorem 2.2 was proved by Tardos and Wexler [10].

Definition 6 *An **exact potential function** Φ is a function which maps into a real value every strategy vector $s \in S$ for all players $i \in N$, such that:*

$$\Phi(s_i, s_{-i}) - \Phi(s'_i, s_{-i}) = c_i(s_i, s_{-i}) - c_i(s'_i, s_{-i}), \forall s'_i \in S_i. \qquad (1)$$

Theorem 2.2 *Finite potential games always converge to an equilibrium through better response dynamics.*

Proof. Let us consider a strategy vector s. If s is not in equilibrium, then there exists a player i which is not satisfied and desires to change her strategy s_i to another strategy s'_i. By definition 6, a potential game has an exact potential function Φ that satisfies equation (1). Because player i decreased her cost, we have that $\Phi(s_i, s_{-i}) > \Phi(s'_i, s_{-i})$ and hence the deviation done by player i has made the potential of the new strategy vector (s'_i) be strictly smaller than the previous one. In each iteration, an improving move is played and therefore a strategy vector is not evaluated more than once. As a result, since the set of strategies of the game is finite, this sequence of better response dynamics will reach an equilibrium eventually. $\square$

Moreover, using the method of providing potential functions for games has been used to show the existence of PNE in the literature, such as Congestion Games [11], Global Connection Games [9], Cost-Sharing Scheduling Games [12], and so forth. Another use of potential functions is that they can be used to give bounds on the PoS as showed in the next theorem [10].

Theorem 2.3 *If we have a finite potential game and assuming that for any outcome S, such that*

$$\frac{c(S)}{A} \leq \Phi(S) \leq B \cdot c(S), \tag{2}$$

for some constants $A, B > 0$. Then the PoS *of this game is at most AB.*

Proof. Let S^{min} be the strategy vector that minimizes the potential function Φ of this game. By Theorem 2.2, we have that S^{min} is an equilibrium, and then $\Phi(S^{min}) \leq \Phi(S^*)$ where S^* is the optimal social outcome of this game. By assumption, we have that $\frac{c(S^{min})}{A} \leq \Phi(S^{min})$. Now, following our assumption, the second inequality give us that $\Phi(S^*) \leq B \cdot c(S^*)$. Combining those inequalities we get that $c(S^{min}) \leq ABc(S^*)$. Hence, we have that PoS $\leq AB$. $\square$

2.2 Transportation Games

One example of transportation game, which will be the base of our research, was recently introduced by Fotakis et al. [3] and its model is given as follows. Given an undirected graph $G = (V, E)$ with a source node s and a destination node t where each edge $e \in E$ has a distance $d_e \in \mathbb{R}_+$. All players have as a goal to be transported from their location to t with lowest cost.

Game 2 *A* ***transportation game*** Γ *is a tuple* (N, M, G)*, where* N *is the set of* n *players with each of them located on a vertex of* G*.* M *is the set of* $m \geq 2$ *resources, also called buses, where each of them follows a path from* s *to* t *through some intermediate vertices in* V*. In order to determine the paths, we suppose each bus* $j \in M$ *has an algorithm* $\mathcal{A}_j$*, which, given* $V' \subseteq V$*, calculates its route which starts on* s*, goes through vertices of* V'*, and finishes its route on node* t*. In their paper, Fotakis et al. [3] consider that each algorithm* $\mathcal{A}_j$*, for* $j \in M$*, is just based on a permutation* $\pi_j : \{1, \ldots, n\} \to N$*, which is independent of any profile* $\sigma \in S$*. Also, those permutations* π_j *represents the reverse order in which the players are picked up. Moreover, it is assumed that a bus follows the shortest path between two vertices.*

The set of strategies $\mathcal{S}$ *of* Γ *is an assignment* $\mathcal{S} : N \to M$ *in which a player* i *chooses one bus* j *that will pick her up. Considering a strategy profile* $\sigma \in \mathcal{S}$*, player's cost under this profile* $c_i(\sigma)$ *is defined as the distance traveled by* σ_i*, the bus chosen by player* i *in profile* σ*, between the location of* i *and the destination* t*.*

Example 2.2 *Consider the instance depicted in Figure 2. In this instance we have* $N = \{1, \ldots, 5\}$ *and* $M = \{a, b\}$ *as the set of available resources. Let* π_j*, for* $j \in M$*, be the identity permutation, i.e.* $\pi_j = (1, 2, 3, 4, 5)$*. Clearly, both players* 1 *and* 2 *choose different buses because otherwise one of them would has a cost bigger than* 3*. Let us analyze the profile* $\sigma = (a, b, a, a, a)$*. Here, we have the following costs:* $c_1(\sigma) = 3$*,* $c_2(\sigma) = 3$*,* $c_3(\sigma) = 6$*,* $c_4(\sigma) = 8$*, and* $c_5(\sigma) = 12$*. Under* σ*, just player* 5 *is willing to deviate and does so. Now, with this changing, we have the profile* $\sigma' = (a, b, a, a, b)$*, and the improved cost of player* 5 *is* $c_5(\sigma') = 4$*. Since under profile* σ' *no one wants to do an unilateral deviation, this is an equilibrium.*

The main results from [3] are divided into two cases: (1) existence and computation of an equilibrium and (2) its quality measured by PoA and PoS. For the former, we have that a SE exists and can be computed in polynomial time if all the resources have the same permutation. Moreover, if distances are metric and there are only two resources, then better response dynamics converge to an equilibrium and it can be computed in

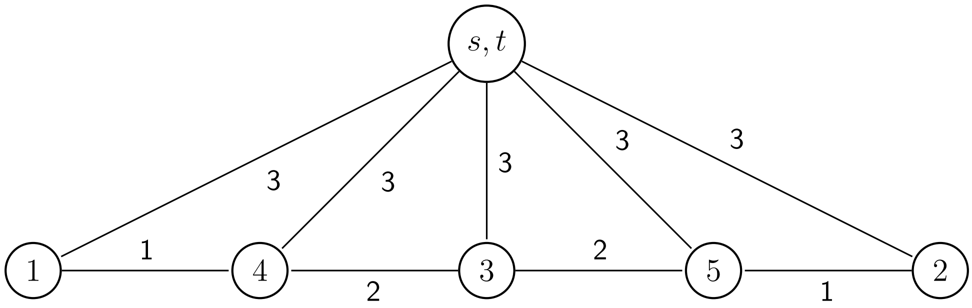

Figure 2: Instance with five players and their distances

polynomial time. Finally, for the special metric case where all distances are either 1 or 2, it is presented an algorithm that finds an equilibrium in *O(nm)*.

In order to analyse the PoA and PoS of transportation games, the authors considered two different social functions. The first one is described as *Vehicle Kilometers Travelled* which reflects the environmental impact of the game's outcome. It is defined as given a strategy profile σ and for $j \in M$, let $(j_1, \cdots, j_{n_j})$ be the ordering that players are picked up by bus j. Then,

$$D(\sigma) = \sum_{j \leq m} \sum_{i=1}^{n_j} d(j_i, j_{i+1}),$$

where $j_{n_j} + 1 = t$ for all buses. Indeed, function $D(\sigma)$ represents the total distance travelled by the buses when taking to destination t at least one player. *Egalitarian* cost $E(\sigma)$ is the second social function which is defined as

$$E(\sigma) = \max_{i \in N} c_i(\sigma).$$

This function represents the maximum distance travelled by a single bus, which is also called as the *makespan*. Both functions neglects the distance between s and the first client.

Next, we list the major results referring to the inefficiency of equilibrium with respect to social functions D and E. First, PoS is unbounded for D and E, for every $n \geq 3$, if the distance is not metric, even if all the permutations are identical. Second, considering function D and metric distances, we have that PoA $= \Theta(n)$ and there are instances of this game with $n \geq 2$ players and $m \geq 2$ resources where the PoS $= \Omega(n)$. Finally, with respect to function E, the PoS of the transportation game is $O(\frac{n}{m})$ and we have that PoA $= 2\lceil \frac{n}{m} \rceil - 1$ for $n > m$ and PoA $= 1$ if $n \leq m$.

Some open questions and possible extensions of transportation games are also left by the authors:

1. One of the results is that for $m = 2$ resources and metric distances, better response dynamics always converge to an equilibrium. Because of this result, the authors suggest the existence of an exact potential function for this particular case. Finding it and if Nash dynamics converge in polynomial time are open questions.

2. Defining different kinds of $c_i(\sigma)$ and analyzing the existence and properties of equilibrium, if it exists.

3. Proposing other way of how the routes are defined and how this modification impacts the PoA and PoS, under the assumption that the pure equilibrium exists.

4. Possible extensions: each bus having a capacity, its own speed and dedicated routes; different weights for the players.

3 Objectives

The main goal of this project is the investigation of theoretical aspects of transportation games. More specifically, the following objectives are aimed to be accomplished throughout the research:

- Investigation of transportation games' properties.

- Studies on the equilibrium convergence.

- Development or improvement of bounds on the inefficiency of equilibrium measured by PoA and PoS.

- Investigation of different social functions and their impact on equilibrium quality.

- Studies on the impact of cooperation in the game.

- Studies of others extensions of the problem such as each player having her own destination or each player being allowed to take more than one bus in order to reach her destination;
- Seeking for answering some of the open questions discussed on the previous section.

As we will be monitoring the literature about transportation games throughout this research, we will be open to adapt our research to new variations that can appear or we can even develop a new variant of transportation problem to be analyzed in the perspective of game theory.

4 Material and Methods

Initially, in order to learn the concepts of game theory, this research will be mainly conducted by consulting specific chapters of the book Algorithmic Game Theory [6]. Also, the student is taking the course of Game Theory in the Institute of Computing on this first semester of 2017.

Other sources of research will be articles related with transportation games and our objectives, specially articles in the line of coordination mechanisms.

5 Schedule

Table 1 shows the estimated time of each activity described bellow.

1. Studies of basic concepts of game theory;
2. Courses in the Institute of Computing;
3. Bibliographic review and bibliographic monitoring;
4. Working with the analysis of different social functions for the original version of the problem;
5. Working on different ways of computing the routes and its impact on equilibrium;

6. Working on one extension of the problem;

7. Thesis writing;

8. Thesis defense and review.

Table 1: Timetable

	First Year												Second Year											
Activities	1	2	3	4	5	6	7	8	9	10	11	12	1	2	3	4	5	6	7	8	9	10	11	12
1	✓	✓	✓	✓	✓																			
2	✓	✓	✓	✓	✓	✓	✓	•	•	•	•	•												
3					✓	✓	✓	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	
4										•	•	•	•											
5													•	•	•	•	•							
6															•	•	•	•	•	•	•	•		
7																				•	•	•	•	
8																								•

6 How the Results will be Analyzed

Since the nature of this research is theoretical, our discoveries will be reported in papers submitted to national/international conferences and/or journals. By doing this, we will have a peer review by the scientific community. Indeed, our mathematical rigor will ensure that our results are accurate in all the analysis done by our research.

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