Local conditions for edge-coloring *

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Abstract

In this note, we investigate three versions of the overfull property for graphs and their relation to the edge-coloring problem. Each of these properties implies that the graph cannot be edge-colored with Δ colors, where Δ is the maximum degree. The three versions are not equivalent for general graphs. However, we show that some equivalences hold for the classes of indifference graphs, split graphs, and complete multipartite graphs.

1 Introduction

Edge-coloring is an important problem in graph theory. It concerns coloring the edges of a graph so that incident edges get different colors. The goal is to do that using the minimum number of colors.

A celebrated theorem by Vizing [12, 8] states that this minimum is always Δ or $\Delta + 1$, where Δ is the maximum degree. To decide between these two possibilities is however NP-hard [7, 1]. More precisely, if we denote by C1 the class of graphs G that are edge-colorable with $\Delta(G)$ colors, and by C2 the complementary class, we have that C1 recognition $\in$ NP while C2 recognition $\in$ co-NP. This means that C1 graphs always have short certificates. Indeed, to convince someone that a graph is in C1 all we have to do is to exhibit a Δ-coloring. In contrast, to show that a graph is in C2 we must produce an argument that no Δ-coloring exists.

Nevertheless, for some graphs this argument can be very simple. For instance, the graph in Figure 1 is in C2. This graph is small enough to permit trying all possibilities, but the following argument is simpler. Notice that, in every valid coloring, each color corresponds to a matching and hence can be assigned to at most two edges in this particular case. Since the total number of edges is five, it becomes evident that $\Delta = 2$ colors do not suffice.

*This research was partially supported by CNPq/ProtemCC.

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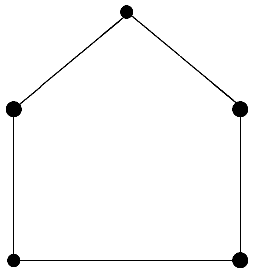

Figure 1: An overfull graph.

Graphs to which a similar argument can be applied have been termed **overfull** [9, 10, 5, 4]. We say that a graph G is **overfull** when the number of vertices n is odd and

$$\Delta\frac{n-1}{2} < m,$$

where m is the number of edges.

Notice that we are using the fact that a matching in an n-vertex graph has size at most $(n-1)/2$ when n is odd. The matching number of the graph could be used instead of the expression $(n-1)/2$, but we will concentrate in the traditional definition — the one given above — in this note.

In addition, observe that this argument will not work for a graph with even n. In this case, the following relation is always true:

$$\Delta\frac{n}{2} \geq m.$$

This is just another way of saying that the maximum degree is not smaller than the average degree, if we recall that $2m$ is equal to the sum of all degrees.

In our studies on edge-coloring [3], we found that two other definitions of overfullness are also of interest. We state them in the sequel.

A graph G is **subgraph-overfull** when there is an overfull subgraph H of G with $\Delta(H) = \Delta(G)$. Here, a subgraph is formed by taking some of the vertices and some of the edges of G; it does not need to be an induced subgraph. However, considering induced subgraphs leads to an equivalent definition. Notice that subgraph-overfull graphs are in C2, since we need at least $\Delta(G) + 1$ colors just for the edges in H.

If the overfull subgraph H happens to be a **neighborhood**, that is, is induced by a Δ-vertex and all its neighbors, then we say that G is **neighborhood-overfull**. Again, neighborhood-overfull graphs are in C2.

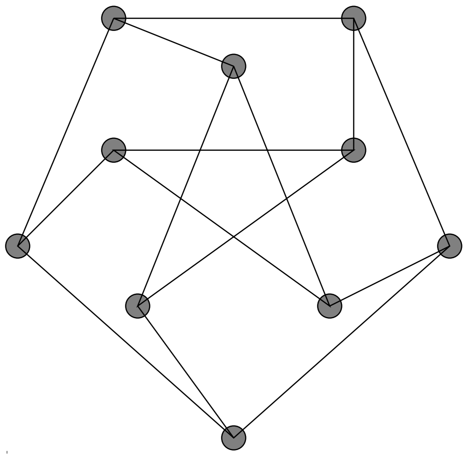

Figure 2: Petersen graph - C2 graph not subgraph-overfull.

In the rest of this note, we shall denote by O, SO, and NO the classes of overfull, subgraph-overfull, and neighborhood-overfull graphs, respectively. These classes are related as follows:

Theorem 1 *We have the following proper inclusions:*

$$O \subset SO \subset C2$$

and

$$NO \subset SO \subset C2.$$

In addition, O and NO are incomparable.

Proof: All inclusions in Theorem 1 are immediate from the definitions. The Petersen graph in Figure 2 is an example of a C2 graph which is not subgraph-overfull. Actually the Petersen graph with one vertex removed is a C2 graph that is not subgraph-overfull.

To get an example of a subgraph-overfull graph that is not overfull itself, consider the split graph F depicted in Figure 3. This graph can be partitioned into a clique of size 4 and a stable set of size 2. One vertex of the stable set sees three vertices of the clique and the other vertex of the stable set sees the fourth vertex of the clique. This graph is not overfull because it contains an even number of vertices. On the other hand, by removing its vertex of degree one, we obtain a copy of K_5 minus one edge, an overfull graph, with the same maximum degree as F. We note that the graph K_5 minus one edge is actually neighborhood-overfull because it contains a universal vertex. Thus, F is an example of a neighborhood-overfull graph that is not overfull.

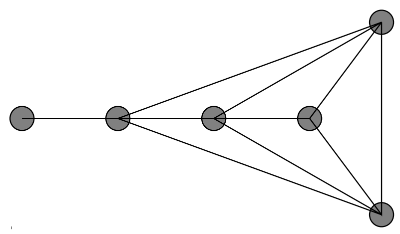

Figure 3: A neighborhood-overfull graph that is not overfull.

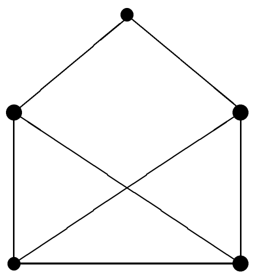

Figure 4: A subgraph-overfull graph that is not neighborhood-overfull.

Finally, Figure 4 shows an example of a graph that is overfull but not neighborhood-overfull. Thus this graph is a subgraph-overfull graph that is not neighborhood-overfull. We note that a graph with an odd maximum degree cannot be neighborhood-overfull. ■

The rest of this note is devoted to showing that some of these inclusions become equivalent if we restrict ourselves to special classes of graphs.

In Section 3, we show that O = SO for complete multipartite graphs. It is known that O = C2 for this class [6]. Although the equality O = C2 implies O = SO, we provide in Section 3 a simple counting argument for that fact. In Section 4, we show that NO = SO for split graphs. We conjecture that SO = C2 for this class. In Section 5, we show that NO = SO for indifference graphs. We conjecture that SO = C2 for this class. We conclude by conjecturing, in Section 6, that neighborhood-overfullness is the right condition for solving the edge-coloring problem for chordal graphs.

2 Definitions and notation

In this paper, G denotes a simple, undirected, finite, connected graph. $V(G)$ and $E(G)$ are the vertex and edge sets of G. A *stable set* is a set of vertices pairwise non-adjacent in G. A *clique* is a set of vertices pairwise adjacent in G. A *maximal clique* of G is a clique not properly contained in any other clique. A *subgraph* of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, we denote by $G[X]$ the *subgraph induced by* X, that is, $V(G[X]) = X$ and $E(G[X])$ consists of those edges of $E(G)$ having both ends in X.

For each vertex v of a graph G, $Adj(v)$ denotes the set of vertices which are adjacent to v. In addition, $N(v)$ denotes the *neighborhood* of v, that is, $N(v) = Adj(v) \cup \{v\}$. A subgraph that is induced by the neighborhood of a vertex is simply called a *neighborhood*. The *degree* of a vertex v is $\deg(v) = |\mathrm{Adj}(v)|$. The maximum degree of a graph G is then $\Delta(G) = \max_{v \in V(G)} \deg(v)$. A vertex u is *universal* if $\deg(u) = |V(G)| - 1$. A k-neighborhood is the neighborhood of a vertex of degree k. For us, K_n denotes the complete graph on $n \geq 1$ vertices.

A vertex v is *simplicial* if $N(v)$ is complete. A *perfect elimination order* of a graph G is a total order on its vertex set $v_1, v_2, \ldots, v_n$ such that for each i, $1 \leq i \leq n$, v_i is simplicial in $G[v_1, v_2, \ldots, v_i]$. A graph is *chordal* if it admits a perfect elimination order.

An *interval graph* is the intersection graph of a set of intervals of the real line. If unitary intervals can be taken, then the graph is called *unitary interval*, *proper interval* or *indifference graph*. We shall adopt the latter name, to be consistent with the terminology of indifference *orders*, defined below. Indifference graphs can be characterized by a linear order: their vertices can be linearly ordered so that the vertices contained in the same clique are consecutive [11]. We call such an order an *indifference order*. By definition, every indifference order is a perfect elimination order.

A *split graph* is a graph whose vertex set admits a partition into a stable set and a clique. Given such a partition for a split graph, a perfect elimination order can be defined by placing the vertices in the clique before the vertices in the stable set. Thus every split graph is a chordal graph.

Let $a_1 \leq a_2 \leq \ldots \leq a_p$ be positive integers. The *complete multipartite* graph $K(a_1, \ldots, a_p)$ is defined as follows. It has $n = a_1 + a_2 + \ldots + a_p$ vertices, partitioned into parts $A_1, A_2, \ldots, A_p$, where each A_i has cardinality a_i. If two vertices are in the same part, they are not adjacent, whereas if they are in different parts, they are adjacent.

3 Complete multipartite graphs

In this section we present a simple counting argument that shows that the particular structure of complete multipartite graphs forces subgraph-overfullness to be equivalent to overfullness in this case. Although this result is already known [6], this simple counting argument is useful for other classes of graphs as we show next.

Theorem 2 *Every complete multipartite graph that is subgraph-overfull is overfull.*

Proof: Suppose there exists a subgraph-overfull complete multipartite graph G that is not an overfull graph itself. Then, by definition, the graph G contains a proper induced subgraph H with same maximum degree as G such that H is an overfull graph.

Since H is an induced subgraph of G, it is also complete multipartite. Hence H has $n_H = h_1 + h_2 + \ldots + h_p$ vertices, partitioned into stable sets $H_1, H_2, \ldots, H_p$ such that $|H_i| = h_i$ and there is an edge between every two vertices that belong to distinct stable sets. Because H is assumed to be overfull, we have $p \geq 3$. We may assume that $h_1 \leq h_2 \leq \ldots \leq h_p$.

First, we note that we must have $h_1 < h_2$. For note that all vertices in H_1 are maximum degree vertices. We have $\Delta(H) = h_2 + \ldots + h_p$. Now because G and H have the same maximum degree every vertex of $G \setminus H$ is missed by all vertices of H_1. If we had $h_1 = h_2$, then we would have $\Delta(G) > \Delta(H)$.

We have $m_H = \sum_{i<j} h_i h_j$. On the other hand:

$$\Delta(H)(n_H - 1)/2 = (h_2 + \ldots + h_p)(h_1 + h_2 + \ldots + h_p - 1)/2.$$

The overfull condition on graph H says that $m_H > \Delta(H)(n_H - 1)/2$. This implies the following:

$$h_1h_2 + h_1h_3 + \ldots + h_1h_p > h_2h_2 + h_3h_3 + \ldots + h_ph_p - (h_2 + h_3 + \ldots + h_p) =$$
$$h_2(h_2 - 1) + h_3(h_3 - 1) + \ldots + h_p(h_p - 1).$$

Now because we have $h_1 < h_2 \leq \ldots \leq h_p$, we have:

$$h_2(h_2 - 1) + h_3(h_3 - 1) + \ldots + h_p(h_p - 1) \geq h_1h_2 + h_1h_3 + \ldots + h_1h_p.$$

This contradiction finishes the proof. ∎

4 Split graphs

In this section we prove that, for split graphs, being subgraph-overfull is equivalent to being neighborhood-overfull.

By definition, every neighborhood-overfull graph is also subgraph-overfull. We show in Corollary 1 below that, for split graphs, every subgraph-overfull graph is neighborhood-overfull. Because the class of split graphs is hereditary, it is enough to show that every overfull split graph is neighborhood-overfull.

We begin by showing that every overfull split graph must contain a universal vertex.

Lemma 1 *If G is split and overfull, then G always contains a universal vertex.*

Proof: Suppose that G is a split graph with vertex set partitioned into two sets A and B, such that set A is a clique and set B is a stable set. We may assume that A is a maximal clique.

We shall use the following notation: $|A| = a$ and $|B| = b$. Thus, $n = a + b$, $\Delta \geq a$, $m \leq a(a-1)/2 + a(\Delta - a + 1)$.

Let us write $\Delta = a + x$, where $x \geq 0$. Then, each vertex of A sees at most $x + 1$ vertices in B. Each Δ-vertex belongs to A and sees precisely $x + 1$ vertices in B.

We shall prove that every split graph that is overfull has $b = x + 1$. This means that each Δ-vertex belongs to set A and sees every vertex in B. Thus every Δ-vertex is universal.

We have the following condition for a graph to be overfull: $m > \Delta(n-1)/2$. As a consequence:

$$m > (a + x)(a + b - 1)/2 = a(a-1)/2 + ab/2 + x(a + b - 1)/2.$$

On the other hand, the definition of the partition gives:

$$m \leq a(a-1)/2 + a(\Delta - a + 1) = a(a-1)/2 + a(x + 1).$$

By cancelling $a(a-1)/2$ in these two inequalities we conclude that:

$$ab/2 + x(a + b - 1)/2 < a(x + 1).$$

And this in turn implies:

$$b(a + x) - x < ax + 2a.$$

Recall that we want to show that $b = x + 1$. Suppose for a moment that $b \geq x + 2$. This implies:

$$(x + 2)(a + x) - x \leq b(a + x) - x < ax + 2a.$$

But this contradicts $x \geq 0$. ■

Now Lemma 1 implies immediately corollary below.

Corollary 1 *If G is split and subgraph-overfull then G is neighborhood-overfull.*

5 Indifference graphs

In this section we prove that for indifference graphs being subgraph-overfull is equivalent to being neighborhood-overfull.

By definition, every neighborhood-overfull graph is also subgraph-overfull. We show in Theorem 4 below that for indifference graphs, every subgraph-overfull graph is neighborhood-overfull. Because the class of indifference graphs is hereditary, it is enough to show that every overfull indifference graph is neighborhood-overfull.

We begin by establishing a necessary structural condition for any indifference graph that is overfull. We show that every overfull indifference graph admits a partition of its vertex set into Δ-neighborhoods.

Consider an indifference graph $G = (V, E)$. An indifference order on V associates to each vertex a positive integer i, where $0 \leq i \leq n$.

Lemma 2 *If V is not the disjoint union of Δ-neighborhoods, then there exists a sequence of vertices $0 = w_0 < u_1 < v_1 < w_1 < ... < u_k < v_k < w_k, (k >= 0)$ satisfying the following properties:*

(i) *all u_i have degree Δ;*

(ii) *$w_i = v_i + 1$, for all $1 \leq i \leq k$;*

(iii) *vertices $w_0, u_1, \ldots, w_k$ in this given order induce a chordless path in G;*

(iv) *there is no vertex x of degree Δ such that $w_k < x$ and $x \in N(w_k) \setminus N(v_k)$.*

Proof: We argue by induction on $n = |V|$. We note that, for $n \leq 3$, G is trivially the disjoint union of Δ-neighborhoods, because G always has a universal vertex. So we may assume that $n \geq 4$.

Suppose that V is not the disjoint union of Δ-neighborhoods and that $|V| = 4$. Then, G has no universal vertex, which implies that $\Delta = 2$. In this case, G is isomorphic to a P_4, a path induced by four vertices whose vertices correspond to a sequence w_0, u_1, v_1, w_1, as required.

For $n > 4$, let u_1 be the rightmost neighbor of $w_0 = 0$. Because $n > 1$, we have $w_0 < u_1$. If $\deg(u_1) < \Delta$, then w_0 is the required sequence.

On the other hand, suppose that $\deg(u_1) = \Delta$. Note that $\deg(w_0) = \Delta$ implies that $V = N(w_0)$ is the disjoint union of Δ-neighborhoods. Thus, we have $\deg(u_1) > \deg(w_0)$. Let v_1 be the rightmost neighbor of u_1. We have $u_1 < v_1$. Again, because $V \neq N(u_1)$, there exists $w_1 = v_1 + 1$.

Note that the path induced by w_0, u_1, v_1, w_1 has no chords, by definition of u_1, v_1 as rightmost neighbors of w_0, u_1, respectively.

If there is no vertex x with $\deg(x) = \Delta$, and $x \in N(w_1) \setminus N(v_1)$, then $k = 1$ and w_0, u_1, v_1, w_1 is the required sequence.

On the other hand, in case there is such a vertex x, consider the graph G', induced by w_1 and its successors with respect to the indifference order. By definition, graph G' is itself indifference with $\Delta(G') = \Delta$. Now if V' is the disjoint union of Δ-neighborhoods, then $V = N(u_1) \cup V'$ and $N(u_1) \cap V' = \emptyset$ imply that V is the disjoint union of Δ-neighborhoods.

Thus by induction for G', we have a sequence $w_1 < u_2 < v_2 < w_2 < ... < u_k < v_k < w_k$ with the required properties. This together with the sequence $w_0 < u_1 < v_1 < w_1$ gives the required sequence for G. Indeed, properties 1, 2 and 4 are trivially satisfied. For property 3, if there was a chord in this path, connecting a vertex before w_1 to a vertex after w_1, we would have the chord $v_1 u_2$. This is not possible, because vertex u_2 has degree Δ in G' and therefore has no neighbors in $N(u_1)$. ∎

Lemma 3 *If an indifference graph admits a sequence* $0 = w_0 < u_1 < v_1 < w_1 < ... < u_k < v_k < w_k, (k >= 0)$ *defined as above, then it is not overfull.*

Proof: Let us define a function $c: V \to R$ as follows:

$$\begin{aligned} c(w_i) &= +1, 0 \le i \le k, \\ c(v_i) &= -1, 1 \le i \le k, \\ c(x) &= 0, x \neq w_i, v_i. \end{aligned}$$

Say that v is a *positive vertex* if $c(v) = +1$; *negative vertex* if $c(v) = -1$; *zero vertex* if $c(v) = 0$. This function satisfies that no neighborhood may contain two non-zero vertices with equal signs, as this gives a vertex of degree greater than Δ. Indeed, note that given w_i, u_{i+1}, v_{i+1}, by definition of this sequence, u_{i+1} is a vertex of degree Δ and its neighbors are precisely those vertices $y \neq u_{i+1}$, such that $w_i \le y \le v_{i+1}$.

Thus, there is at most one positive vertex or one negative vertex in each neighborhood $N(x)$, which gives:

$$\sum_{y \in N(x)} c(y) = \begin{cases} +1, & \text{if there exists one positive vertex and no negative vertex in } N(x); \\ -1, & \text{if there exists one negative vertex and no positive vertex in } N(x); \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we note that, for every $x \in V$:

$$\sum_{y \in N(x)} c(y) \le \Delta - \deg(x).$$

Indeed, we have $\Delta - \deg(x) \geq 0$. Suppose for a moment $\sum_{y \in N(x)} c(y) = +1$ and $\Delta = \deg(x)$. This means x sees a positive vertex and sees no negative vertex, i.e., x sees w_i but sees no v_j. Thus, $w_i < x$, with $i \neq k$. Hence, $i < k$ and $x < u_{i+1}$ as otherwise x sees v_{i+1}. But now the neighborhood of x lies between w_i and v_{i+1}, and does not contain v_{i+1}. Therefore it is properly contained in $N(u_{i+1})$, which contradicts $\deg(x) = \Delta$.

Now,

$$\sum_{x \in V} (1 - c(x))(\Delta - \deg(x) - \sum_{y \in N(x)} c(y)) \geq 0,$$

as it is a sum of positive terms. This in turn gives,

$$\sum_{x \in V} \Delta - \sum_{x \in V} \deg(x) - \sum_{x \in V} \sum_{y \in N(x)} c(y) - \sum_{x \in V} c(x)\Delta + \sum_{x \in V} c(x)\deg(x) + \sum_{x \in V} c(x) \sum_{y \in N(x)} c(y) \geq 0$$

Note that $\sum_{x \in V} c(x) = 1$, as w_0 is a positive vertex and all other w_i cancel with a v_i. Hence,

$$n\Delta - 2m - \sum_{x \in V} \sum_{y \in N(x)} c(y) - \Delta + \sum_{x \in V} c(x)\deg(x) + \sum_{x \in V} (c(x))^2 + 2 \sum_{xy \in E} c(x)c(y) \geq 0$$

i.e.,

$$n\Delta - 2m - \sum_{y \in V} (c(y)(1 + \deg(y))) - \Delta + \sum_{x \in V} c(x)\deg(x) + (2k+1) - 2k \geq 0$$

and finally,

$$n\Delta - 2m - \Delta \geq 0,$$

i.e., $(n-1)\Delta \geq 2m$, which implies that the graph is not overfull. ■

Theorem 3 *If G is indifference and overfull then $V(G)$ can be partitioned into Δ-neighborhoods. In particular, we have $n = k(\Delta + 1)$, for some integer $k \geq 1$.*

Proof: The result is a direct consequence of Lemma 2 and Lemma 3. ■

We are now ready to establish that, for indifference graphs, subgraph-overfullness is equivalent to neighborhood-overfullness. By definition, every neighborhood-overfull graph is subgraph-overfull. The following theorem proves the converse.

Theorem 4 *Every indifference graph that is subgraph-overfull is also neighborhood-overfull.*

Proof: It suffices to prove that if a graph G is indifference and overfull then G is neighborhood-overfull.

Let G be an indifference overfull graph. From Theorem 3, we know that $n = k(\Delta+1)$. Furthermore, since n must be odd, we have k odd and Δ even. We shall assume further that G is *not* neighborhood-overfull and arrive at a contradiction.

Consider the vertices of G in an indifference order. Theorem 3 allows us to divide the vertex set $V(G)$ into k consecutive blocks of $\Delta+1$ vertices each. Let $V_1, V_2, \ldots, V_k$ be these blocks.

Our goal is to count the edges and show that G cannot be overfull.

Let A_i be the set of edges whose left end point is in V_i. We claim that

$$|A_i| \leq \frac{\Delta(\Delta+1)}{2}, \text{ for } 1 \leq i \leq k-1, \tag{1}$$

and that

$$|A_k| \leq \frac{\Delta(\Delta+1)}{2} - \frac{\Delta}{2}. \tag{2}$$

Notice that Claim (2) above is immediate from our assumption that G is not neighborhood-overfull, since A_k is the edge set of the neighborhood $G[V_k]$. Let us now prove Claim (1).

There are two kinds of edges in A_i, for $1 \leq i \leq k-1$: those whose right end point is in V_i and those that go past V_i. The crucial fact here is that, for each edge going past V_i, there is an edge missing between vertices of V_i. Indeed, let $u \in V_i$ be the left end point of an edge going past V_i. Since $\deg(u) \leq \Delta$, for each vertex that u sees beyond V_i there must be a vertex in the beginning of V_i that u does not see. (This is because the closed neighborhood $N[u]$ contains at most $\Delta+1$ consecutive vertices.) Therefore, the total number of edges in A_i is bounded by the number of edges in a complete graph over V_i, that is, $\Delta(\Delta+1)/2$.

Now let us use (1) and (2). Notice that every edge of G must start somewhere, hence

$$\begin{aligned} m &= |E| \\ &= \sum_{i=1}^{k} |A_i| \\ &\leq \sum_{i=1}^{k-1} \frac{\Delta(\Delta+1)}{2} + \frac{\Delta(\Delta+1)}{2} - \frac{\Delta}{2} \\ &= k\frac{\Delta(\Delta+1)}{2} - \frac{\Delta}{2} \\ &= \frac{\Delta}{2}(n-1), \end{aligned}$$

showing that G cannot be overfull. This contradiction concludes the proof. ∎

6 Conclusions

We have the following conjecture about edge-coloring chordal graphs:

Conjecture 1 *Every C2 chordal graph is neighborhood-overfull.*

We are currently working on two necessary conditions for this conjecture. Firstly, we need all chordal graphs that are subgraph-overfull to be also neighborhood-overfull. In this paper, we have established this fact for two subclasses of chordal graphs: split graphs and indifference graphs.

Secondly, we need all odd maximum degree chordal graphs to be C1. We have established this fact for indifference graphs [3]. Recently, it was proved that all odd maximum degree split graphs are C1 [2].

Let us consider the class of complete multipartite graphs. It has been shown recently that any C2 complete multipartite graph is overfull [6]. This implies in particular that for complete multipartite graphs being overfull is equivalent to being subgraph-overfull. In this paper, we have shown that a simple counting argument provides an alternate proof for that fact. Now consider the complete multipartite graph with 9 vertices with parts: A_1, A_2, A_3, where each part has 3 vertices. This graph is overfull but not neighborhood-overfull. Now consider the complete multipartite graph with 7 vertices with parts: A_1, A_2, A_3, where $a_1 = a_2 = 2$ and $a_3 = 3$. This graph is overfull but not neighborhood-overfull, as its maximum degree is odd.

We have examples of odd maximum degree graphs that are overfull. All such examples are not chordal graphs. We conjecture that an odd maximum degree chordal graph cannot be overfull.

Finally we remark that one way to prove that an overfull graph is neighborhood-overfull is to exhibit a universal vertex. That is what we did for the case of split graphs.

For indifference graphs we had to use another argument to establish that overfull is also neighborhood-overfull. The following example shows an indifference graph that is overfull, neighborhood-overfull but contains no universal vertex. Consider the graph H obtained by removing an edge ab from K_7. This graph is neighborhood-overfull. Now consider the graph F obtained as follows. Take three copies of H, say H_1, H_2, H_3, where $H_i = K_7 \setminus a_i b_i$ respectively. Add edges $b_1 a_2$ and $b_2 a_3$.

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