Edge Coloring of Cycle Powers is Easy

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Abstract

In this note we solve the edge-coloring problem for cycle powers C_n^k. It is well-known that the edges o C_n can be colored with two colors if and only if n is even. We generalize this resut showing that the edges of C_n^k can be colored with $\Delta(C_n^k) = 2k$ colors if and only if n is even, for all k with $0 \leq k \leq n/2$. Coupled with the fact that C_n^k is overfull if n is odd, this solves the edge-coloring problem for this class of graphs.

1 Introduction

In this note we are concerned with optimal valid colorings for cycle powers (see Section 2 for precise definitions). In general, edge-coloring is a difficult problem. Although Vizing's theorem [4] guarantees that the optimal coloring of a simple graph G uses either Δ or $\Delta + 1$ colors, where Δ is the maximum degree, it is NP-complete to decide between these two possibilities. We call a graph *class 1* if its edges can be colored with Δ colors, and *class 2* otherwise. Cai and Ellis [1] reviewed the status of this problem for several classes of graphs.

Classical results in this area state that cycles and complete graphs are class 1 when they have an even number of vertices, and class 2 otherwise. Because cycles and complete graphs are special cases of cycle powers, it is natural to consider the question for cycle powers in general. This note extends the classical results to all cycle powers, that is, we prove that a cycle power is class 1 if and only if it has an even number of vertices.

The rest of this paper is organized as follows. Section 2 contains the precise definitions of concepts we use in the text. Section 3 treats the case of an odd number of vertices, and Section 4 solves the other case. Finally, our conclusions and considerations about future work appear in Section 5.

2 Definitions

We assume the reader is familiar with basic graph theory. Given a graph G, we denote by $V(G)$ its set of vertices, and by $E(G)$ its set of edges. We will deal exclusively with simple graphs here. For convenience, we consider *directed* graphs, so that we can talk about initial points of edges: and edge (u, v) has u

as its initial point. A directed graph is simple when its undirected counterpart is simple.

Given a nonempty subset $E \subseteq E(G)$, we denote by $G[E]$ the graph having E as edge set, and the vertices of G incident to some edge in E as vertex set.

A *coloring* of a graph G is a mapping $\kappa : E(G) \mapsto C$ of $E(G)$ onto some set C of *colors*. The coloring is *valid* when no two adjacent edges have the same image under κ. An *optimal valid coloring* is a valid coloring for which the cardinality $|C|$ is minimum. A celebrated theorem of Vizing [4] states that, for simple graphs, this minimum is either the maximum degree $\Delta(G)$ or $\Delta(G) + 1$. If there is a valid coloring of G with $\Delta(G)$ colors we say that G is *class 1*; otherwise, G is said to be *class 2*.

Let G be a graph with n vertices and m edges. If $m > \Delta(G)\lfloor n/2 \rfloor$, then G is said to be *overfull*. It is easy to see that overfull graphs are class 2, because each color can color at most $\lfloor n/2 \rfloor$ edges in a valid coloring.

A *complete* graph is a simple graph with edges between any pair of vertices.

For any integer $n \geq 3$ we define the *cycle* as being the graph C_n with

$$\begin{aligned} V(C_n) &= \{0, 1, 2, \cdots, n-1\}, \\ E(C_n) &= \{(0,1), (1,2), \cdots, (n-2, n-1), (n-1, 0)\}. \end{aligned}$$

(We exclude $n = 2$ because the graph would not be simple, but the main results are still valid for this case.) The *k-th power* of a directed graph G is defined recursively as follows. We first define the product $G \times H$ of two graphs G and H with the same vertex set V:

$$\begin{aligned} V(G \times H) &= V \\ E(G \times H) &= \{(u,v) \mid \exists w \in V((u,w) \in E(G) \text{ and } (w,v) \in E(H))\}. \end{aligned}$$

Then define $G^1 = G$ and $G^k = G \times G^{k-1}$ for $k \geq 2$.

When G is a cycle C_n, the k-th power is simple only when $1 \leq k < n/2$, so we will limit ourselves to these cases from here on. Observe that when n is odd and $k = (n-1)/2$, C_n^k is a complete graph. For even n, technically there is no k for which C_n^k is complete, because $C_n^{(n/2}$ is not simple, but we could have defined an undirected version of graph powers to remedy this. We prefer to keep the directed version, though, because it simplifies the arguments in Section 4.2.

Notice that $\Delta(C_n^k) = 2k$ for $1 \leq k < n/2$.

3 Powers of odd cycles are overfull

A cycle power C_n^k with odd n, like any regular graph of odd order, is overfull. To confirm this fact, observe that

$$m = \frac{n\Delta}{2} = \frac{n(2k)}{2} = nk,$$

for $0 \leq k < n/2$. On the other hand,

$$\Delta \left\lfloor \frac{n}{2} \right\rfloor = (2k)\frac{n-1}{2} = (n-1)k.$$

Therefore, $m > \Delta\lfloor n/2 \rfloor$ when $k > 0$. This shows that C_n^k is class 2 when n is odd.

4 Powers of even cycles

In this section we treat the case where n is even. We will see that in many cases grouping edges by size (to be defined later) will result in several groups that can be colored independently, using two colors for each group. In the remaining cases, some groups cannot be colored with two colors. We remedy this problem by joining edges with size l and size $l+1$, obtaining a four-colorable edge-set, which can again be colored independently.

4.1 Edges of size l are two-colorable when $n/\gcd(l,n)$ is even

Let n be an even number and k an integer with $1 \leq k < n/2$. Let S_l be the set of edges of size l, defined as

$$S_l = \{(x, x + l \bmod n) \mid 0 \leq x \leq n-1\}.$$

These sets form a partition of the edge-set of C_n^k, that is, $S_l \cap S_{l'} = \emptyset$ if $l \neq l'$ and

$$E(C_n^k) = \bigcup_{l=1}^{k} S_l.$$

The main result of this section is the following.

Theorem 1 *For $l \leq k$, the induced graph $C_n^k[S_l]$ has d connected components, each one being a cycle of length n/d, where $d = \gcd(l,n)$.*

Proof: Two vertices x and y are in the same component if and only if there is an integer r such that

$$y \equiv x + rl \pmod{n}.$$

This is equivalent to $d \mid (y - x)$. Hence, two vertices are in the same component if and only if they belong to the same residue class modulo d. The result follows because there are exactly d such residue classes, and all of them have the same size (see, for instance, the number theory book by Irelan and Rosen [3]). $\square$

The corollary below follows immediately from Theorem 1, since even cycles are two-colorable.

Corollary 1 *The graph $C_n^k[S_l]$ is two-colorable when $l \leq k < n/2$ and $n/\gcd(l,n)$ is even.*

4.2 Edges of sizes l and $l-1$ form a four-colorable subgraph when $n/\gcd(l,n)$ is odd

In this section we prove the following result.

Theorem 2 *The graph $C_n^k[S_l \cup S_{l-1}]$ is four-colorable when $l \leq k < n/2$ and $n/\gcd(l,n)$ is odd.*

Proof: Let d be $\gcd(l,n)$. If n/d is odd, then d and l are necessarily even. We know from the results of Section /refs:cycles that S_l is a disjoint union of odd cycles, whereas S_{l-1} is a disjoint union of even cycles. The idea is to exchange some edges between S_l and S_{l-1} in order to merge the odd cycles into even ones, without disturbing too much the coloring of S_{l-1}.

The set S_{l-1} can be colored with two colors. We select a special coloring of this edge set as follows. Paint with one color all edges with odd initial point, and paint with the other color the remaining edges, that is, the ones with even initial point. It is easy to see that this coloring is valid, because $l-1$ is odd.

Let C_1 be the following set of edges

$$\begin{aligned}C_1 = \{ & \\ & (1, l), \\ & (l+1, 2l), \\ & (3, l+2), \\ & (l+3, 2l+2), \\ & \vdots \\ & (d-1, d+l-2), \\ & (d+l-1, d+2l-2) \ \}.\end{aligned}$$

All these edges are of the form $(a, a+l-1)$ and therefore belong to S_{l-1}. Furthermore, the initial point a is odd for all these edges, which means that they will all be colored with the same color.

Now consider the set C_2 given below

$$\begin{aligned}C_2 = \{ & \\ & (1, l+1), \\ & (l, 2l), \\ & (3, l+3), \\ & (l+2, 2l+2), \\ & \vdots \\ & (d-1, d+l-1), \\ & (d+l-2, d+2l-2) \ \}.\end{aligned}$$

These edges are of the form $(b, b+l)$ and thus belong to S_l. An important observation is that they belong to distinct cycles in S_l. Indeed, two edges belong to the same cycle in S_l when their initial points a and b satisfy $d \mid (b-a)$. Let us divide C_2 in two parts A and B as follows.

$$\begin{array}{rcl} A & = & \{(a, a+l) \mid 1 \leq a \leq d-1 \text{ and } a \text{ is odd }\} \\ B & = & \{(b, b+l) \mid l \leq b \leq d+l-2 \text{ and } b \text{ is even }\} \end{array}$$

Two edges in A cannot be in the same cycle because their initial points belong to an interval of length less than d; the same holds for two edges in B. Now if we pick one edge from A and another from B, the difference between their initial points is odd and thus cannot be divisible by an even number such as d.

We exchange the edges in C_1 and C_2 between S_l and S_{l-1}. More precisely, consider

$$\begin{array}{rcl} S'_{l-1} & = & (S_{l-1} - C_1) \cup C_2, \\ S'_l & = & (S_l \cup C_1) - C_2. \end{array}$$

Notice that $S'_{l-1} \cup S'_l = S_{l-1} \cup S_l$, because we just exchanged edges between the two sets.

We claim that S'_{l-1} is two-colorable. In fact, C_1 is a matching because its edges were all of the same color in S_{l-1}. This means that the $2d$ endpoints are all distinct. But C_2 has the exact same set of $2d$ endpoints, so it is a matching as well. It is not difficult to see that we can give the color that C_1 had to C_2 and produce a valid coloring for S'_{l-1}.

We claim that S'_l is two-colorable as well. The reason is that, with the exchange, the d odd cycles in S_l are grouped in pairs (notice that d is even) and the two cycles in each pair are merged to form an even cycle. Take, for instance, the first two edges of C_1,

$$(1, l), (l+1, 2l)$$

and the first two edges of C_2,

$$(1, l+1), (l, 2l).$$

With the exchange, the cycles in S_l that contained edges $(1, l+1)$ and $(l, 2l)$ become a longer, even cycle (see Figure 1). A similar merge occurs with the other cycles in S_l. Given that there are d cycles and d exchanged edges with no two edges in the same cycle, all the cycles get merged by the exchange and S'_l is a disjoint union of even cycles. Thus, it is two-colorable. $\square$

4.3 Main theorem

Theorem 3 *The graph C_n^k is class 1 for even n.*

Proof: Induction on k.

If $k = 1$ then $C_n^k = C_n$ which is class 1 for even n.

If $k > 1$ we distinguish two cases according to the parity of n/d, where $d = \gcd(k, n)$. If n/d is even, $C_n^k[S_k]$ is two-colorable. By induction hypothesis

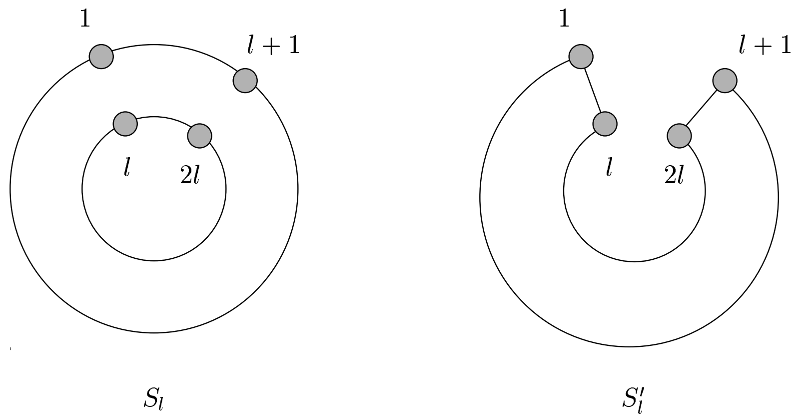

Figure 1: Effect of exchange on cycles in S_l.

C_n^{k-1} is class 1. But $E(C_n^k) = E_1 \cup E_2$, where $E_1 = E(C_n^{k-1})$ and $E_2 = S_k$. Because $\Delta(C_n^k[E_1]) = 2k-2$, $\Delta(C_n^k[E_2]) = 2$, and both induced graphs are class 1, we have that C_n^k is class 1.

If n/d is odd, we use the results of Section 4.2 and conclude that $C_n^k[S_k \cup s_{k-1}]$ is four-colorable. By induction hypothesis, C_n^{k-2} is class 1. But $E(C_n^k) = E_1 \cup E_2$, where $E_1 = E(C_n^{k-2})$ and $E_2 = S_k \cup S_{k-1}$. Because $\Delta(C_n^k[E_1]) = 2k-4$, $\Delta(C_n^k[E_2]) = 4$, and both induced graphs are class 1, we have that C_n^k is class 1. $\square$

5 Conclusions

We proved that powers of a cycle are class 1 if and only if the number of vertices is even, generalizing the known results for cycles and complete graphs. This new result can potentially lead to a Δ coloring of several other graphs by considering *pullback* functions. These functions were introduced by Figueiredo and colleagues [2] and are able to transport valid edge colorings from one graph to another. It would be interesting to characterize the graphs to which a Δ coloring of C_n^k can be transported.

Interesting questions are raised by considering nonsimple graphs. For odd n and $k > n/2$, the power C_n^k, although not simple, is still overfull, which implies that there is no Δ coloring. However, Vizing's theorem for general graphs bounds the minimum number of colors by $\Delta + \mu$, where μ is the maximum edge multiplicity. It would be interesting to color optimally these graphs. On the other hand, for even n, the Δ coloring of C_n^k shown here is valid for values of k greater than $n/2$.

References

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