On the Clique Operator

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Abstract. The clique operator K maps a graph G into its *clique graph*, which is the intersection graph of the (maximal) cliques of G. Among all the better studied graph operators, K seems to be the richest one and many questions regarding it remain open. In this note we are particularly interested in whether the image of K^2, $K^2(\mathcal{G})$ (here $\mathcal{G}$ is the class of all graphs), is the same as the image of K, $K(\mathcal{G})$, and we describe our progress toward answering this question. We obtain a necessary condition for a graph to be in the image of K in terms of the presence of certain subgraphs A and B. These graphs become natural candidates for being in the difference $K(\mathcal{G}) \setminus K^2(\mathcal{G})$. Since $A \in K^2(\mathcal{G})$, we concentrate on B. One way to prove that $B \notin K^2(\mathcal{G})$ is to show that every graph in $K^{-1}(B)$ is not in $K(\mathcal{G})$. However, $K^{-1}(B)$ is an infinite set, and, although every graph in $K^{-1}(B)$ can be obtained from a finite subclass of $K^{-1}(B)$ by addition of twin vertices, belonging or not to $K(\mathcal{G})$ is *not* a property that is maintained by addition of twins. We also present a result involving distances and a constructive characterization of $K^{-1}(G)$ for a fixed but generic G.

1 Introduction

In this note all graphs are simple, i.e., without loops or multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. A set C of vertices of G is *complete* when any two vertices of C are adjacent. A maximal complete subset of $V(G)$ is called a *clique*. We denote by $\mathcal{C}(G)$ the clique family of G.

Let $\mathcal{F} = (F_i)_{i \in I}$ be a finite family of finite sets. Its *dual family* $\mathcal{F}^*$ is the family $(F(x))_{x \in X}$ where $X = \bigcup_{i \in I} F_i$ and $F(x) = \{i \in I, x \in F_i\}$. We denote by $\Omega\mathcal{F}$ the *intersection graph* of $\mathcal{F}$, i.e., $V(\Omega\mathcal{F}) = I$ and two vertices i and j are adjacent if and only if $F_i \cap F_j \neq \emptyset$. We also say that $\mathcal{F}$ *represents* $\Omega\mathcal{F}$.

The *2-section* of $\mathcal{F}$, denoted by $\mathcal{F}_2$, is the graph with $V(\mathcal{F}_2) = \bigcup_{i \in I} F_i$ and two vertices x and y are adjacent if and only if there exists $i \in I$ such that $x, y \in F_i$.

It is easy to see that $\Omega\mathcal{F} = \mathcal{F}_2^*$ [1].

A family $\mathcal{F}$ of arbitrary sets satisfies the *Helly property*, or is *Helly*, when for every subfamily $J \subseteq \mathcal{F}$ such that any two sets $A, B \in J$ intersect, we have $\bigcap_{A \in J} A \neq \emptyset$. A graph is *Helly* when the family of its cliques is Helly. We denote by $\mathcal{H}$ the class of Helly graphs. A family $\mathcal{F}$ is *conformal* when the cliques of $\mathcal{F}_2$ are all members of $\mathcal{F}$. This amounts to saying that its dual family $\mathcal{F}^*$ is Helly [1]. A family $\mathcal{F}$ is *reduced* when none of its members is contained in another one.

The *clique operator* K transforms a graph G into a graph $K(G)$ having as vertices all the cliques of G, with two cliques being adjacent when they intersect. Thus, $K(G)$ is nothing else than the intersection graph of the family of all cliques of G. The graph $K(G)$ is called the *clique graph* of G.

In this note we will be interested in the image $K(\mathcal{G})$ of the operator K, where $\mathcal{G}$ is the class of all graphs. There are only two general results about $K(\mathcal{G})$ in the literature. The first result, due to Hamelink [3], says that $\mathcal{H}$ is properly contained in $K(\mathcal{G})$. In the second result, absed on the previous one, Roberts and Spencer [5] find the following characterization of $K(\mathcal{G})$:

Theorem 1 (Roberts and Spencer, 1971). *A graph G is in $K(\mathcal{G})$ if and only if there is a family $\mathcal{K}$ of complete sets in G such that:*

1. *$\mathcal{K}$ covers all the edges of G (i.e., if $xy \in E(G)$, then $\{x, y\}$ is contained in some element of $\mathcal{K}$).*
2. *$\mathcal{K}$ satisfies the Helly property.*

In spite of this characterization, the complexity of the recognition problem for $K(\mathcal{G})$ is still open.

In the proof of their theorem, Roberts and Spencer build, given a graph G satisfying the hypothesis, another graph H such that $K(H) = G$.

We call *RS family* of G a family of complete sets in G that fulfills the hypothesis of the Roberts and Spencer theorem. If in addition the family has a reduced dual, then it is an *ERS family* of G.

The contributions of this paper can be summarized as follows. In Section 2 we investigate the structure of graphs in $K(\mathcal{G})$ that are not Helly graphs, finding certain subgraphs that have to be present in this situation. Because $\mathcal{H} \subseteq K^2(\mathcal{G})$, these subgraphs become natural candidates to separate $K^2(\mathcal{G})$ from the rest of $K(\mathcal{G})$. We show that one of these subgraphs belongs to $K^2(\mathcal{G})$. We do not know if the other does.

In Section 3 we show that H is a graph in $K^{-1}(G)$ if and only if it is the intersection graph of an ERS family of G. We also study several properties of both RS and ERS families of G.

In Section 4 we obtain results that show that it is enough to study the recognition problem for $K(\mathcal{G})$ in graphs with diameter at most two.

The study of $K^2(\mathcal{G})$ is further complicated by the fact that being in $K(\mathcal{G})$ is *not* a property inherited from reduced graphs, as we show in Section 5. In fact, when $H \notin K(\mathcal{G})$ it is possible to get a graph in $K(\mathcal{G})$ by adding twin vertices to H. Of course, this addition will not modify $K(H)$. Finally, Section 6 contains our concluding remarks.

Some proofs are omitted for space limitations. All proofs appear in full in the extended version of this paper.

2 NonHelly Graphs in $K(\mathcal{G})$

We denote by F_2 the graph depicted in Figure 1. We say that a graph G *has* F_2 when G has three mutually adjacent vertices v_1, v_2, and v_3, and three other vertices v_4, v_5, and v_6 such that v_4 is adjacent to v_1 and v_2 but not to v_3, v_5 is adjacent to v_2 and v_3 but not to v_1, v_6 is adjacent to v_1 and v_3 but not to v_2. Notice that this is different from saying that G has F_2 as an induced subgraph, and it is also different from saying that G has a subgraph (not necessarily induced) isomorphic to F_2.

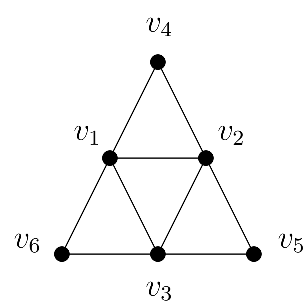

Fig. 1. The graph F_2.

However, this concept is important because of the following fact. Define a graph to be *Helly hereditary* when it is Helly and all its induced subgraphs are Helly as well. Prisner [4] showed that G is Helly hereditary if and only if G does not have F_2 in the sense defined above.

The following result tells us more about the structure of a graph in $K(\mathcal{G})$ that has F_2.

Theorem 2. *If $G \in K(\mathcal{G})$ and G has F_2 then G has a subgraph isomorphic to either A or B (see Figure 2).*

Proof. We sketch the main ideas of the proof here. The complete proof appears in the version.

To show that G has a subgraph isomorphic to A, we just need to find a vertex x adjacent to the three central vertices v_1, v_2, and v_3 of an F_2 plus at least one peripheral vertex (v_4, v_5, or v_6).

If $G \in K(\mathcal{G})$ there is a family of complete sets of G, $\mathcal{K}$, which holds the RS characterization. The proof is done basically by a case analysis. We ask ourselves the following questions:

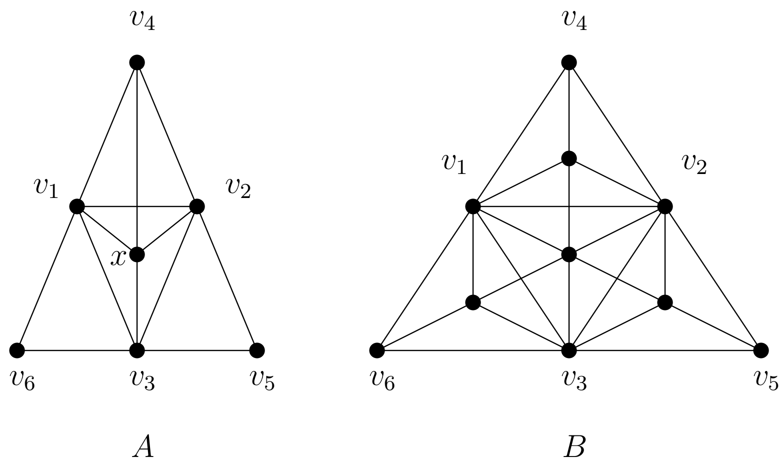

Fig. 2. Graphs A and B.

- Is there a set $L \in \mathcal{K}$ such that $\{v_1, v_2, v_3\} \subseteq L$?
- Is there a set $L \in \mathcal{K}$ such that $\{v_1, v_2, v_4\} \subseteq L$?
- Is there a set $L \in \mathcal{K}$ such that $\{v_1, v_3, v_6\} \subseteq L$?
- Is there a set $L \in \mathcal{K}$ such that $\{v_2, v_3, v_5\} \subseteq L$?

It turns out that if the answer is "yes" to any of these questions, then the graph G has a subgraph isomorphic to A. On the other hand, if the answer is "no" to all of them, then there is a vertex x adjacent to v_1, v_2, and v_3 simultaneously (see Figure 3).

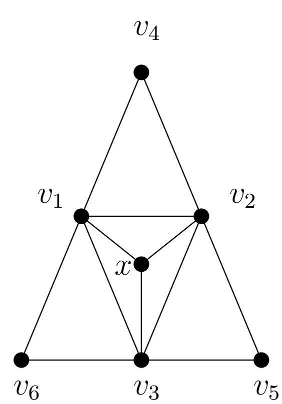

Fig. 3. The vertices v_1, v_2, and v_3 are not simultaneously contained in any set of the RS family $\mathcal{K}$. This implies the existence of x.

If x is adjacent to one of v_4, v_5, or v_6, we are done. Otherwise we can argue that G admits a subgraph isomorphic to B.

The following corollary easily follows.

Corollary 3. *If $G \in K(\mathcal{G})$, then G is Helly hereditary if and only if G does not have subgraphs isomorphic to either A or B.*

Proof. Since G is Helly hereditary if and only if G does not have subgraphs isomorphic to F_2 [4] the proof follows.

The graphs A and B are therefore the ones that "separate" the Helly hereditary ones inside $K(\mathcal{G})$. Hence it is natural to take A and B as natural candidates to be in $K(\mathcal{G})$ but not in $K^2(\mathcal{G})$. For the graph A, we have that it actually belongs to $K^2(\mathcal{G})$ (see Figure 4).

However, we do not know the status of B with respect to $K^2(\mathcal{G})$. We conjecture that $B \notin K^2(\mathcal{G})$.

3 Results for a fixed G

Given a graph G in $K(\mathcal{G})$, we characterize the class of graphs whose image under K is G. Before going to the characterization, let us recall a couple of results on intersection graphs.

Lemma 4. *[2] If $\mathcal{F}$ is a family of complete sets of G which covers all edges of G, then G is the intersection graph of family $\mathcal{F}^*$.*

Lemma 5. *[2] If G is the intersection graph of a family $\mathcal{F}$ then $\mathcal{F}^*$ is a family of complete sets of G wich covers all edges of G.*

The proofs are easy and will be omitted. The following is an immediate consequence.

Theorem 6. *Let G and H be two graphs. Then $K(H) = G$ if and only if H is the intersection graph of an ERS family of G.*

Proof. If $K(H) = G$, then G is the intersection graph of the family $\mathcal{C}(H)$. Hence, by Lemma 5, $\mathcal{C}(H)^*$ is a family of complete sets of G that covers the edges of G. In addition, $(\mathcal{C}(H)^*)^* = \mathcal{C}(H)$, which is conformal and reduced, since it is a family of cliques. Therefore $\mathcal{C}(H)^*$ is an ERS family of G that represents H.

Conversely, suppose H is the intersection graph of an ERS family $\mathcal{F}$ of G. Then H is the 2-section of $\mathcal{F}^*$. Notice that given that $\mathcal{F}$ is an ERS family, $\mathcal{F}^*$ is conformal and reduced. Therefore $\mathcal{C}(H) = \mathcal{F}^*$. But since $\mathcal{F}$ is also a family of complete sets that covers all the edges of G, we have by Lemma 4 that G is the intersection graph of $\mathcal{F}^*$. We conclude that $G = K(H)$.

This result shows that the properties of the RS and ERS families of $G \in K(\mathcal{G})$ is of great importance. We develop some of these properties in the sequel.

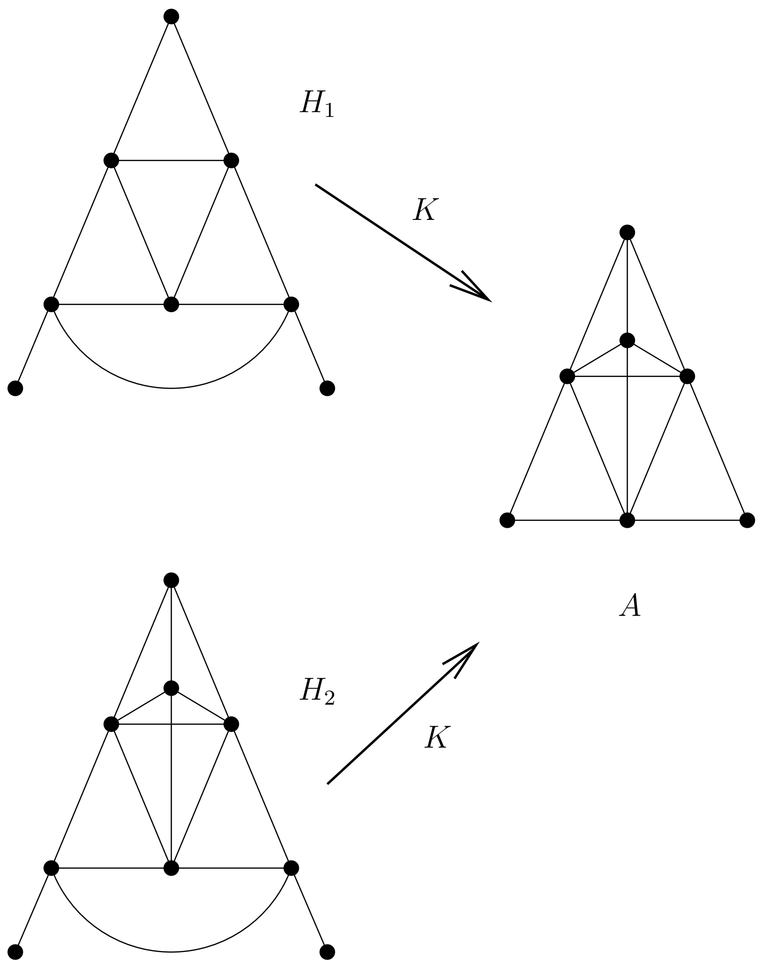

Fig. 4. Graph A and two graphs H_1 and H_2 in $K^{-1}(A)$. Notice that H_2 is in $K(\mathcal{G})$, showing that $A \in K^2(\mathcal{G})$.
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Theorem 7. *Let $\mathcal{F}$ be an RS family of G. If for all $v \in V(G)$ such that there is $u \in V(G)$ with $F(v) \subseteq F(u)$, we add $\{v\}$ to $\mathcal{F}$, the result is an ERS family of G.*

Theorem 8. *If $\mathcal{F}$ is an RS family of G then the family of maximal sets in $\mathcal{F}$ with respect to inclusion is also an RS family of G.*

Proof. Let $\mathcal{F}'$ be the family of maximal sets in $\mathcal{F}$. It is clear that $\mathcal{F}'$ covers all the edges of G. Since every subfamily of a Helly family is Helly , $\mathcal{F}'$ is an RS family of G.

Theorem 9. *Let $\mathcal{F}$ be a reduced RS family of G. An edge xy of G is in $\mathcal{F}$ if and only if xy is a clique of G.*

Proof. If $\{x, y\} \in \mathcal{F}$ and it is not a clique of G then there is $z \in V(G)$ such that $xz, yz \in E(G)$. Since $\mathcal{F}$ is a reduced RS family of G we have that there are two completes sets, $C, L \in \mathcal{F}$, such that $x, z \in C$ and $y, z \in L$. Thus C, L and $\{x, y\}$ are three pairwise intersecting elements in $\mathcal{F}$ then they have a nonempty intersection. But if x (resp. y) is in the intersection then $\{x, y\} \subset L$ (resp. $\{x, y\} \subset C$) and that is a contradiction because $\mathcal{F}$ is a reduced family.

Conversely, if $\{x, y\}$ is a clique of G then $\{x, y\} \in \mathcal{F}$ because it is the only complete set of G that covers this edge.

Theorem 10. *If $\{x, y, z\}$ is a clique of G, then $\{x, y, z\}$ belongs to every RS family of G.*

Proof. Let $\mathcal{F}$ be an RS family of G and let $C, L, T \in \mathcal{F}$ which cover edges xy, yz and xz respectively. By the Helly property there is a common element, h . Since $\{x, y, z\}$ is a clique of G then h is one of these vertices and thus $\{x, y, z\} \subseteq C$ or $\{x, y, z\} \subseteq L$ or $\{x, y, z\} \subseteq T$. But $\{x, y, z\}$ is a clique of G therefrom $\{x, y, z\}$ is one of these three complete sets

The converse is false. For instance, in the graph A (Figure 2) the triangle $\{v_2, v_4, x\}$ is in all RS families of A, but is not a clique of A.

Theorem 11. *If $\mathcal{F}$ is an RS family of G and F_1, F_2 are two members of $\mathcal{F}$ with nonempty intersection, then the family $\mathcal{F}' = \mathcal{F} \cup \{F_1 \cap F_2\}$ is also an RS familty of G. If moreover $\mathcal{F}$ is an ERS family of G, then $\mathcal{F}'$ is also and ERS family.*

Proof. Obviously $F_1 \cap F_2$ is a complete set of G and $\mathcal{F}'$ covers all the edges G. We will show that $\mathcal{F}'$ is Helly. Let $F_1 \cap F_2, F_3, F_4, ..., F_n$ a pairwise intersecting subfamily of $(\mathcal{F})'$ then $F_1, F_3, F_4, ..., F_n$ and $F_2, F_3, F_4, ..., F_n$ are pairwise intersecting subfamilies of $(\mathcal{F})$. Moreover,since $F_1 \cap F_2 \neq \emptyset$ then $F_1, F_2, F_3, F_4, ..., F_n$ is a pairwise intersecting subfamily of (F) and there is a common element. Thus $\mathcal{F}'$ is Helly.

Note that, if F_1 and F_2 are the same, adding their intersection to $\mathcal{F}$ amounts to duplicating a set already in $\mathcal{F}$. Hence, the mere act of replicating an elament in an ERS family generates a new ERS family.

4 Metric results

In a connected graph G, the *distance* $d(x, y)$ between two vertices is the minimum number of edges in a path from x to y. Our main result in this section follows.

Theorem 12. *Let G be a graph and x, y two vertices of G with $d(x, y) > 2$. Then $G \in K(\mathcal{G})$ if and only if $G + xy \in K(\mathcal{G})$.*

Proof. Since $G \in K(\mathcal{G})$ there is a graph H such that $K(H) = G$. Let C_x, C_y be cliques of H which represent x and y respectively. Observe that

1. $C_x \cap C_y = \emptyset$
2. If $r \in C_x$ and $s \in C_y$ then $rs \notin E(H)$.

The first property follows that x and y are not adjacent. By prove the second we suppose that $rs \in E(H)$, then there is a clique C in H which contains this edge. Because (1), $C \neq C_x$ and $C \neq C_y$. Therefrom $d_G(x, y) = 2$ a contradiction.

Let H' be the graph obtained by adding a new vertex a to C_x and C_y, let $C'_x = C_x \cup \{a\}$ and $C'_y = C_y \cup \{a\}$. It is clear that C'_x and C'_y are cliques of H'. On the other hand, if C is a clique of H' does not meet C_x and C_y, then C is a clique of H. By property (2) C can meet C_x or C_y but not both, then $a \notin C$. Hence C is a clique of H. Therefrom, family of cliques of H' can be obtained that of H by substituting C_x and C_y to C'_x and C'_y respectively. Since these new cliques meet, we obtain that $K(H') = K(H) + xy = G + xy \in K(\mathcal{G})$.

Conversely, let H be graph such that $K(H) = G+xy$ and let C_x, C_y be cliques of H which represent x and y respectively. Since $xy \in E(G + xy)$, $C_x \cap C_y \neq \emptyset$. Let $C_x \cap C_y = \{r_1, ..., r_t\}$. We construct a new graph H' by doubling of these vertices, i. e. $V(H') = V(H) \cup \{r'_1, ..., r'_t\}$ and cliques of H' are the same that of H except C'_y, which is obtained to C_y by changing r_i by r'_i, for all $i = 1, ..., t$.

Since $d(x, y) > 2$, G does not have vertices adjacent to both x and y. Then none clique of H, distict to C_x and C_y, meets $\{r_1, ..., r_t\}$. Therefrom $K(H') = K(H) - xy = G \in K(\mathcal{G})$.

We write $G \triangleleft H$ when there is a pair of vertices x, y in G with $d(x, y) > 2$ and $H \cong G + xy$. Extend this realtion to a symmetric relation by defining $G \sim H$ if and only if $G \triangleleft H$ or $H \triangleleft G$. Now extend this relation to an equivalence relation by defining $G \overset{*}{\sim} H$ if and only if there is a series $G_0, G_1, \ldots, G_k$ of graphs such that

$$G = G_0 \sim G_1 \sim \cdots \sim G_k = H.$$

The following result is immediate from the previous theorem and definitions.

Corollary 13. *Let G and H be two graphs such that $G \overset{*}{\sim} H$. The $G \in K(\mathcal{G})$ if and only if $H \in K(\mathcal{G})$.*

This shows that it is sufficient to deal with graphs of diameter at most 2 when determining whether a graph belongs to $K(\mathcal{G})$.

5 Twin Vertices

We have seen that the graphs in the inverse image $K^{-1}(G)$ are in one-to-one correspondence with the ERS families of G. There is an infinite number of these families. It is easy to see that if a given complete set L of G appears in an ERS family two or more times, this produces twin vertices in the corresponding $H \in K^{-1}(G)$. We would like to simplify the study of $K^{-1}(G)$ by taking only ERS families with no repeated elements, because there is a finite number of such families. For instance, to test whether a given graph G belongs to $K^2(\mathcal{G})$ we could take all reduced (i.e., without twins) graphs in $K^{-1}(G)$ and check each one for pertinence in $K(\mathcal{G})$. To do this, however, we need a result similar to Theorem 12, with the addition of edge xy replaced by the addition of a twin. Unfortunately, only half of such a result is true:

Theorem 14. *Let G be a graph and u, v twin vertices in G. If $G - u \in K(\mathcal{G})$ then $G \in K(\mathcal{G})$.*

Proof. Let $\mathcal{F}$ an RS family of $G - u$. Add u to each member of $\mathcal{F}$ that contains v. The new family is an RS family for G.

The converse of Theorem 14 does not hold. The graph in Figure 5 is obtained from F_2 by adding a twin to one of the central vertices. However, $F_2 \notin K(\mathcal{G})$ while the bigger graph does belong to $K(\mathcal{G})$, because the complete sets $C_1, C_2, \ldots, C_6$ indicated in the figure form an RS family of the graph in question.

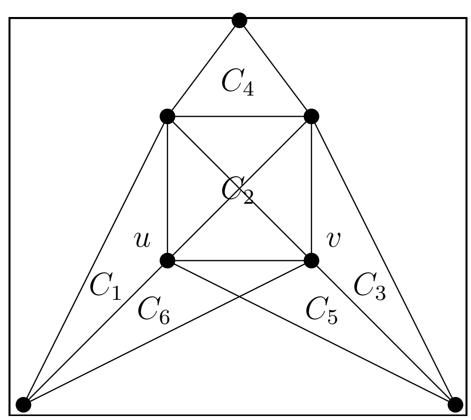

Fig. 5. A graph in $K(\mathcal{G})$ obtained by addition of twin to F_2, which does not belong to $K(\mathcal{G})$.

A weaker converse holds, though.

Theorem 15. *Let u and v be twin vertices of a graph G. If there is an RS family $\mathcal{F}$ of G with the property that every member of $\mathcal{F}$ that has u also has v, then $G - u \in K(\mathcal{G})$.*

Proof. Remove u from all sets in $\mathcal{F}$ that contain it.

A related question has to do with simplicial vertices whose removal does not destroy cliques. A vertex v is *simplicial* when v belongs to only one clique of G. A simplicial vertex v is *superfluous* when the set of its neighbors is a clique in $G - v$. Notice that $K(G) = K(G - v)$ in this case.

Here again we could benefit from the equivalence $G \in K(\mathcal{G}) \iff G - v \in K(\mathcal{G})$ when v is a superfluous simplicial. However, in this case we are not even sure whether any of the implications holds. The only result we were able to prove is for one of the implications in some special circumstances.

Let G be a graph and C a clique of G. We say that an RS family $\mathcal{K}$ of G is C-RS when there is an intersecting subfamily of $\mathcal{K}$ whose union is C. (An intersecting subfamily is one where each two members share at least one common element.)

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Theorem 16. *Let C be a clique of a graph G and $v \in C$ a superfluous simplicial vertex. If there is a $C - v$-RS family of $G - v$, then $G \in K(\mathcal{G})$.*

Proof. Let $(\mathcal{F})$ be a $F_1, ..., F_n$ be an intersecting subfamily of $\mathcal{F}$ such that $F_1 \cup ... \cup F_n = C - v$.

Let $\mathcal{F}'$ be the family obtained from $\mathcal{F}$ added v to each F_i $i = 1, ..., n$.

We will prove that $\mathcal{F}'$ is an RS family of G.

$\mathcal{F}'$ covers all the edges of G because $\mathcal{F}$ covers those of the $G - v$ and those incidente to v are cover because the union of F_i is C.

To show that $\mathcal{F}'$ is Helly, we take $H_1, ..., H_m$ an intersecting subfamily of $\mathcal{F}'$. If for all $i = 1, ..., m$ $H_i = F_i + v$ then v is a common element.

If neither of them has this form then it is an intersecting subfamily of $\mathcal{F}$) and there is a common element.

In other case we have a mixture: $F_1 + v, ..., F_s + v, H_{s+1}, ..., H_m$, since v is not in H_j, $j = s + 1, ..., m$ and $F_1, ..., F_s$ is an intersecting subfamily then we that $F_1, ..., F_s, H_{s+1}, ..., H_m$ is an intersecting subfamily of $\mathcal{F}$ and there is a common element.

6 Conclusions

Our goal in this paper was to shed more light on the clique operator K. In particular, we were interested in whether $K(\mathcal{G}) = K^2(\mathcal{G})$.

The main contributions of this paper can be summarized as follows.

- We prove that nonHelly graphs in $K(\mathcal{G})$ must contain subgraphs isomorphic to either A or B (see Figure 2).
- We prove that recognition of graphs in $K(\mathcal{G})$ can be reduced to graphs with diameter at most 2.
- We show that the addition of twins does not preserve pertinence in $K(\mathcal{G})$.

There is still a lot to be done. We do not know whether $B \in K^2(\mathcal{G})$. We do not know whether superfluous simplicial vertices influence pertinence in $K(\mathcal{G})$.

Depending on the answer to the question "is $K(\mathcal{G}) = K^2(\mathcal{G})$?", the images of other powers of K will be of interest.

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