Network Science

Class 6: Evolving Networks

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Questions

1. Bianconi-Barabasi Model
2. Bose-Einstein Condensation
3. Initial attractiveness
4. Role of internal links.
5. Node deletion.
6. Accelerated growth.

Introduction

EVOLVING NETWORK MODELS

The BA model is only a minimal model.

Makes the simplest assumptions:

- linear growth $\langle k \rangle = 2m$
- linear preferential attachment $\Pi(k_i) \propto k_i$

Does not capture

variations in the shape of the degree distribution
variations in the degree exponent
the size-independent clustering coefficient

Hypothesis:
The BA model can be adapted to describe most features of real networks.

We need to incorporate mechanisms that are known to take place in real networks: addition of links without new nodes, link rewiring, link removal; node removal, constraints or optimization

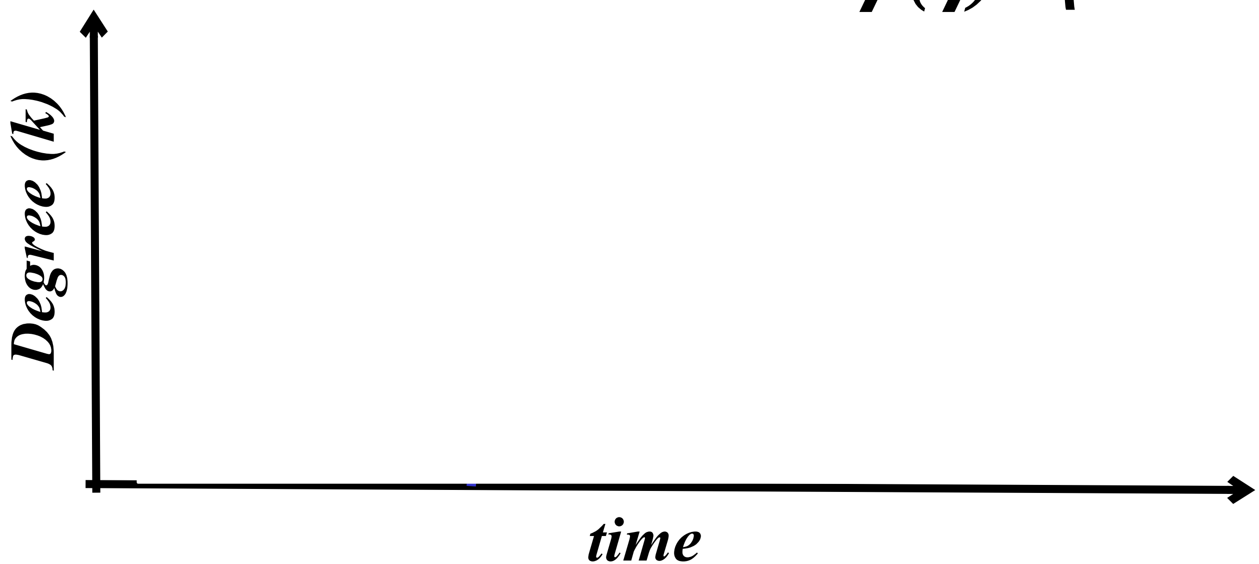
Bianconi-Barabasi model

Can Latecomers Make It?

SF model: $k(t) \sim t^{1/2}$ **(first mover advantage)**

Fitness model: **fitness** (η) $\Pi(k_i) \cong \frac{\eta_i k_i}{\sum_j \eta_j k_j}$ $k(\eta,t) \sim t^{\beta(\eta)}$

$\beta(\eta) = \eta / C$



Bianconi & Barabási, *Physical Review Letters* 2001; *Europhys. Lett.* 2001.

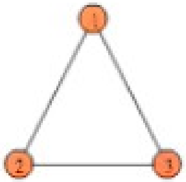
- **Growth**

  In each timestep a new node j with m links and fitness η_j is added to the network, where η_j is a random number chosen from a *fitness distribution* $\rho(\eta)$. Once assigned, a node's fitness does not change.

- **Preferential Attachment**

  The probability that a link of a new node connects to node i is proportional to the product of node i's degree k_i and its fitness η_i,

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}. \tag{6.1}$$


1
2
3

$$\frac{\partial k_i}{\partial t} = m\frac{\eta_i k_i}{\sum_k \eta_j k_j}$$

$$k_{\eta_i}(t,t_i) = m\left(\frac{t}{t_i}\right)^{\beta(\eta_i)}. \quad \text{(6.3)}$$

$$\left\langle \sum_j \eta_j k_j \right\rangle$$

over all possible realizations of the quenched fitnesses η. Since each node is born at a different time t_0, we can write the sum over j as an integral over t_0

$$\left\langle \sum_j \eta_j k_j \right\rangle = \int d\eta\rho(\eta)\eta\int_1^t dt_0 k_\eta(t,t_0). \quad \text{(6.34)}$$

By replacing $k_\eta(t, t_0)$ with (6.3) and performing the integral over t_0, we obtain

$$\left\langle \sum_j \eta_j k_j \right\rangle = \int d\eta\rho(\eta)\eta m\frac{t-t^{\beta(\eta)}}{1-\beta(\eta)}. \quad \text{(6.35)}$$

The dynamic exponent $\beta(\eta)$ is bounded, i.e. $0<\beta(\eta)<1$, because a node can only increase its degree with time ($\beta(\eta)>0$) and $k_i(t)$ cannot increase faster than t ($\beta(\eta)<1$). Therefore in the limit $t\to\infty$ in (6.35) the term $t^{\beta(\eta)}$ can be neglected compared to t, obtaining

$$\left\langle \sum_j \eta_j k_j \right\rangle \overset{t\to\infty}{=} Cmt(1-O(t^{-\varepsilon})), \quad \text{(6.36)}$$

where $\varepsilon = (1 - \max_\eta \beta(\eta)) > 0$ and

$$C = \int d\eta\rho(\eta)\frac{\eta}{1-\beta(\eta)}. \quad \text{(6.37)}$$

$$\frac{\partial k_i}{\partial t} = m \frac{\eta_i k_i}{\sum_k \eta_j k_j}$$

$$C = \int d\eta \rho(\eta) \frac{\eta}{1-\beta(\eta)}$$

$$\frac{\partial k_\eta}{\partial t} = \frac{\eta k_\eta}{Ct},$$

$$k_{\eta_i}(t,t_i) = m\left(\frac{t}{t_i}\right)^{\beta(\eta_i)}.$$

$$\beta(\eta) = \frac{\eta}{C},$$

To complete the calculation we need to determine C from (6.37). After substituting $\beta(n)$ with η/C, we obtain

$$1 = \int_0^{\eta_{\max}} d\eta \rho(\eta) \frac{1}{\frac{C}{\eta} - 1}, \tag{6.40}$$

where η_{max} is the maximum possible fitness in the system. The integral (6.40) is singular. However, since $\beta(\eta)=\eta/C< 1$ for any η, we have $C > \eta_{max}$, thus the integration limit never reaches the singularity. Note also that since

$$Cmt = \sum_j \eta_j k_j \le \eta_{\max} \sum_j k_j = 2mt\eta_{\max} \tag{6.41}$$

we have $C \leqslant 2\eta_{max}$

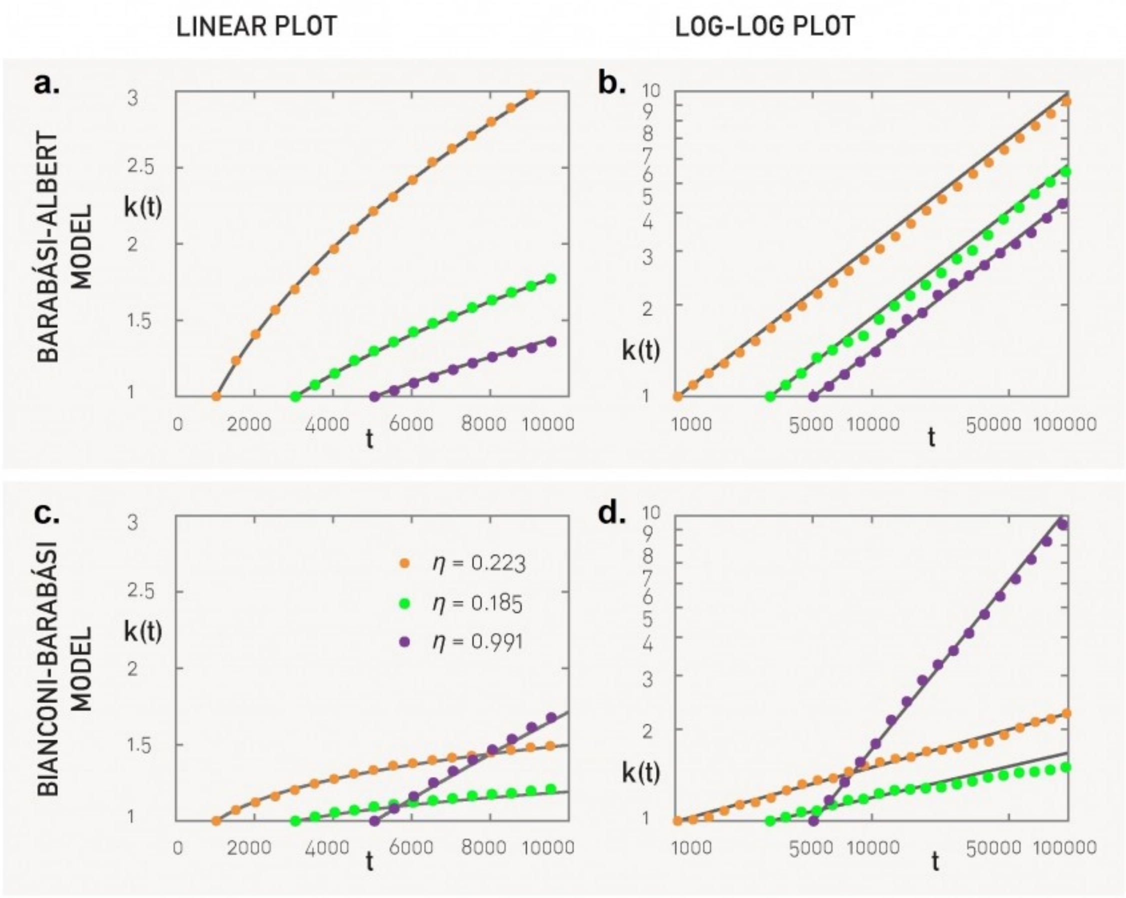
BA model: $k(t) \sim t^{1/2}$

(first mover advantage)

BB model: $k(\eta,t) \sim t^{\beta(\eta)}$

(fit-gets-richer)

$\beta(\eta) = \eta/C$



Fitness Model-Degree distribution

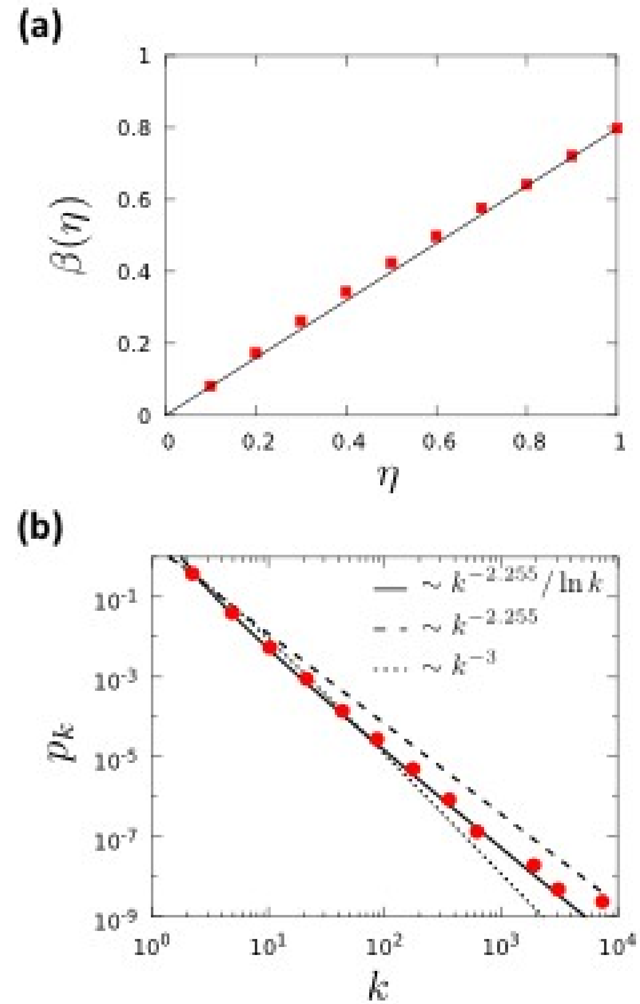
$$p_k \sim C \int d\eta \, \frac{\rho(\eta)}{\eta} \left(\frac{m}{k}\right)^{\frac{C}{\eta}+1}$$

Uniform fitness distribution:

fitness uniformly distributed in the [0,1] interval.

$$p_k \sim \int_1^0 d\eta \frac{C^*}{\eta} \frac{1}{k^{1+C^*/\eta}} \sim \frac{k^{-(1+C^*)}}{\ln k},$$

$C^* = 1.255$ $\quad \gamma = 2.255$



$$k_{\eta_i}(t,t_i) = m\left(\frac{t}{t_i}\right)^{\beta(\eta_i)}.$$

$$\beta(\eta) = \frac{\eta}{C},$$

$$p_k \sim C\int d\eta \frac{\rho(\eta)}{\eta}\left(\frac{m}{k}\right)^{\frac{C}{\eta}+1}$$

If there is a single dynamic exponent β, the degree distribution follows the power law $p_k \sim k^{-\gamma}$ with degree exponent $\gamma=1/\beta+1$. In the Bianconi-Barabási model we have a spectrum of dynamic exponents $\beta(\eta)$, thus p_k is a weighted sum over different power-laws. To calculate p_k we need to determine the cumulative probability that a randomly chosen node's degree satisfies $k_\eta(t)>k$. This cumulative probability is

$$P(k_\eta(t) > k) = P\left(t_0 < t\left(\frac{m}{k}\right)^{C/\eta}\right) = t\left(\frac{m}{k}\right)^{C/\eta}. \qquad \textbf{(6.42)}$$

Thus, the degree distribution is given by the integral

$$p_k = \int_{\eta_{max}}^{0} d\eta \frac{\partial P(k_\eta(t) > k)}{\partial t} \propto \int d\eta \rho(\eta)\frac{C}{\eta}\left(\frac{m}{k}\right)^{\frac{C}{\eta}+1}, \qquad \textbf{(6.43)}$$

$$C = \int d\eta \rho(\eta) \frac{\eta}{1 - \beta(\eta)}$$

- **Equal Fitnesses**
  When all fitnesses are equal, the Bianconi-Barabási model reduces to the Barabási-Albert model. Indeed, let us use $\rho(\eta) = \delta(\eta - 1)$, capturing the fact that each node has the same fitness $\eta = 1$. In this case (6.5) yields $C = 2$. Using (6.4) we obtain $\beta = 1/2$ and (6.6) predicts $p_k \sim k^{-3}$, the known scaling of the degree distribution in the Barabási-Albert model.

$$\beta(\eta) = \frac{\eta}{C} ,$$

$$p_k \sim C \int d\eta \frac{\rho(\eta)}{\eta} \left(\frac{m}{k}\right)^{\frac{C}{\eta}+1}$$

- **Uniform Fitness Distribution**

The model's behavior is more interesting when nodes have different fitnesses. Let us choose η to be uniformly distributed in the [0,1] interval. In this case C is the solution of the transcendental equation (6.5)

$$C = \int d\eta \rho(\eta) \frac{\eta}{1-\beta(\eta)}$$

$$\exp(-2/C) = 1 - 1/C\ , \qquad (6.7)$$

whose numerical solution is $C^* = 1.255$. Consequently, (6.4) predicts that each node i has a different dynamic exponent, $\beta(\eta_i) = \eta_i / C^*$.

$$\beta(\eta) = \frac{\eta}{C}\ ,$$

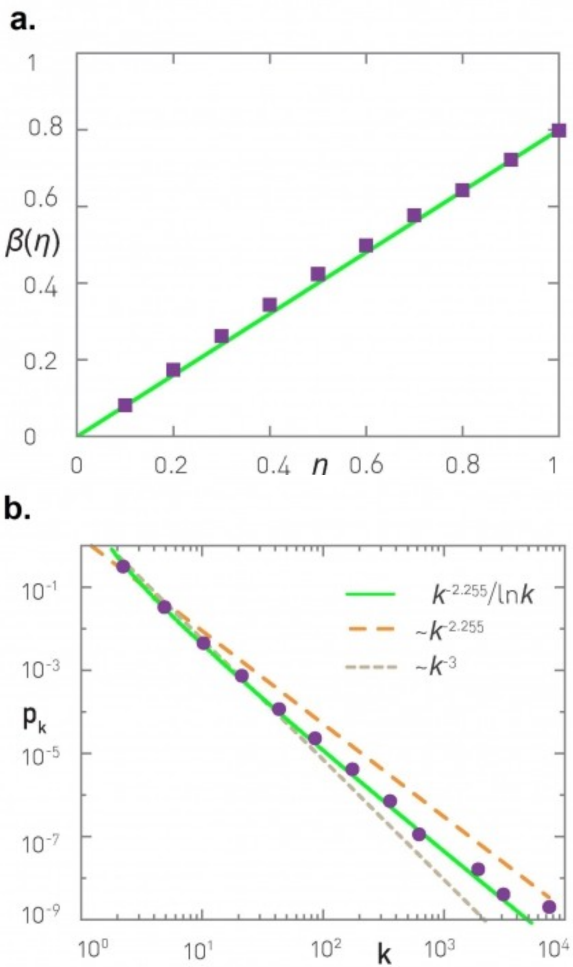
Using (6.6) we obtain

$$p_k \sim C \int d\eta \frac{\rho(\eta)}{\eta} \left(\frac{m}{k}\right)^{\frac{C}{\eta}+1}$$

$$p_k \sim \int_0^1 d\eta \frac{C^*}{\eta} \frac{1}{k^{1+C^*/\eta}} \sim \frac{k^{-(1+C^*)}}{\ln k}, \qquad (6.8)$$

predicting that the degree distribution follows a power law with degree exponent $\gamma = 2.255$. Yet, we do not expect a perfect power law, but the scaling is affected by an inverse logarithmic correction $1/\ln k$.

Uniform Fitnesses



$$\beta(\eta_i) = \eta_i / C^*.$$

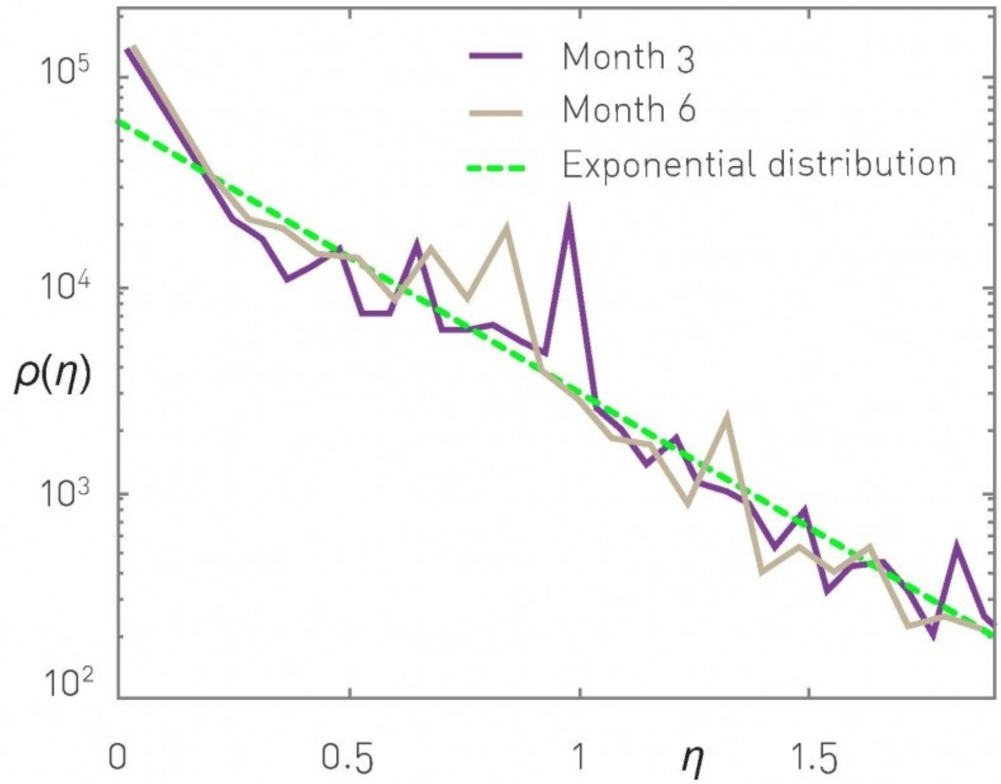
$$C^* = 1.255.$$

$$p_k \sim \int_0^1 d\eta \frac{C^*}{\eta} \frac{1}{k^{1+C^*/\eta}} \sim \frac{k^{-(1+C^*)}}{\ln k},$$

Measuring Fitness

$$k_{\eta_i}(t,t_i) = m\left(\frac{t}{t_i}\right)^{\beta(\eta_i)}.$$

$$\log k_{\eta_i}(t,t_i) = \beta(\eta_i)\log t + B_i,$$

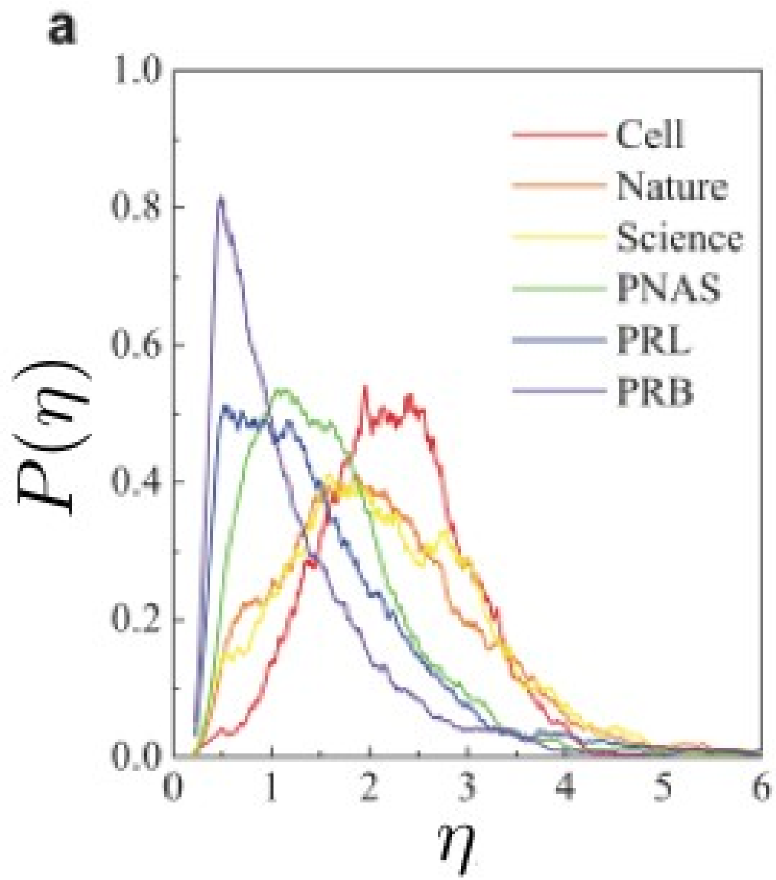


$$\Pi_i \sim \eta_i k_i P_i(t)$$

$$P_i(t) = \frac{1}{\sqrt{2\pi\sigma_i t}} e^{-\frac{(\ln t - \mu_i)^2}{2\sigma_i^2}}.$$

$$k_i(t) = m\left(e^{\frac{\beta \eta_i}{A}\Phi\left(\frac{\ln(t) - \mu_i}{\sigma_i}\right)} - 1\right)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$



The Fitness of a scientific publication

$$k_i(t) = m\left(e^{\frac{\beta \eta_i}{A}\Phi\left(\frac{\ln(t)-\mu_i}{\sigma_i}\right)} - 1\right)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

Ultimate Impact: t → ∞

$$k_i(\infty) = m\left(e^{\frac{\beta \eta_i}{A}} - 1\right)$$