Network Science

Class 6: Evolving Networks

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With

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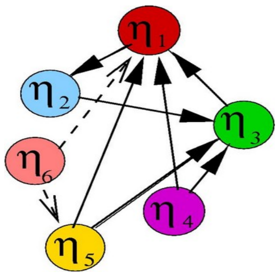
Bose-Einstein condensation

MAPPING TO A QUANTUM GAS

Network



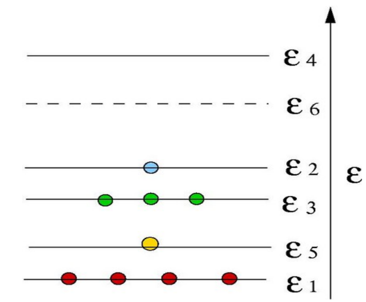
$$\Pi_i = \frac{\eta_i k_i}{\Sigma_j \eta_j k_j}$$

$$\eta \longrightarrow e^{-\beta\varepsilon}$$

$$k_{in}(\eta) \longrightarrow n(\varepsilon)$$

$$\rho(\eta) \longrightarrow g(\varepsilon)$$

Bose gas



Fitness η → Energy level ε

New node with fitness η → New energy level ε

Link pointing to node η → Particle at level ε

Network → quantum gas

G. Bianconi and A.-L. Barabási, Physical Review Letters 2001; cond-mat/0011029

BOSE-EINSTEIN CONDENSATION

$$\frac{\partial k_i(t, t_i, \varepsilon_i)}{\partial t} = m \frac{e^{-\beta\varepsilon_i} k_i(t, t_i, \varepsilon_i)}{\sum_j e^{-\beta\varepsilon_j} k_j(t, t_j, \varepsilon_j)}.$$

$$k(t, t_i, \varepsilon_i) = m\left(\frac{t}{t_i}\right)^{f(\varepsilon_i)} \quad f(\varepsilon) = e^{-\beta(\varepsilon-\mu)}.$$

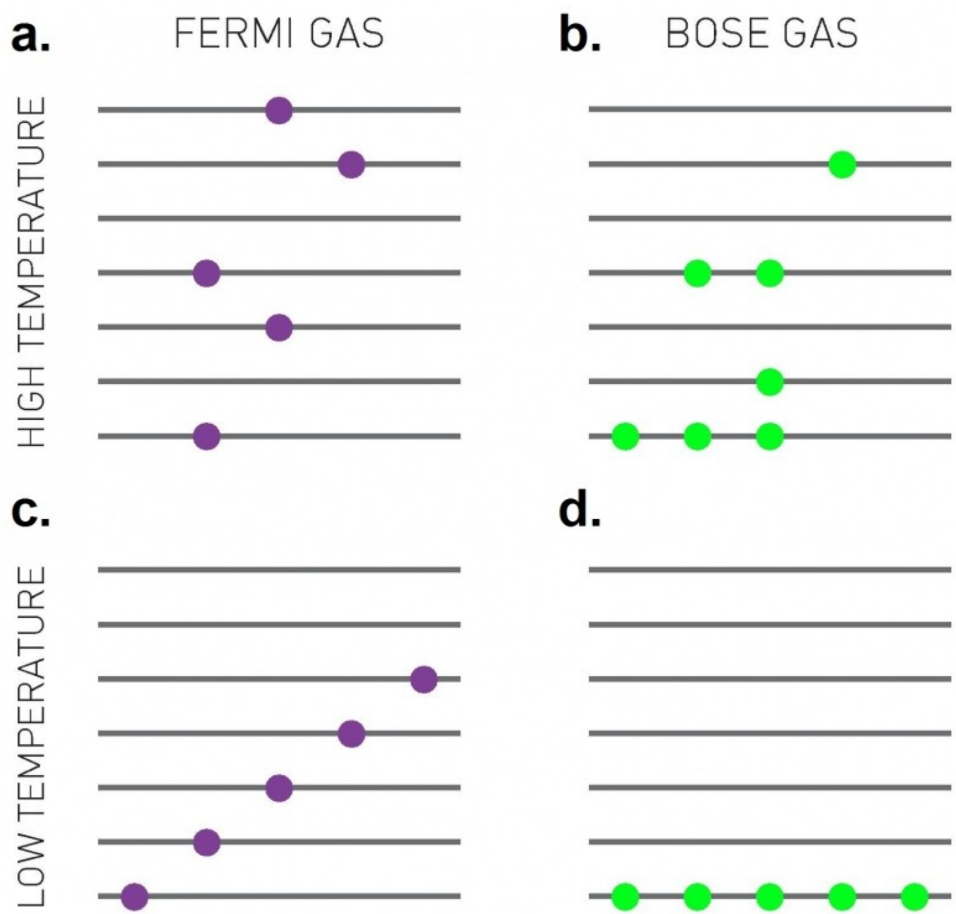
The dynamic exponent *f(e)* depends on **m,** determined by the self-consistent equation:

$$I(\beta, \mu) = \int d\varepsilon\, p(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)} - 1} = 1.$$

$$\int d\varepsilon g(\varepsilon) n(\varepsilon) = 1$$

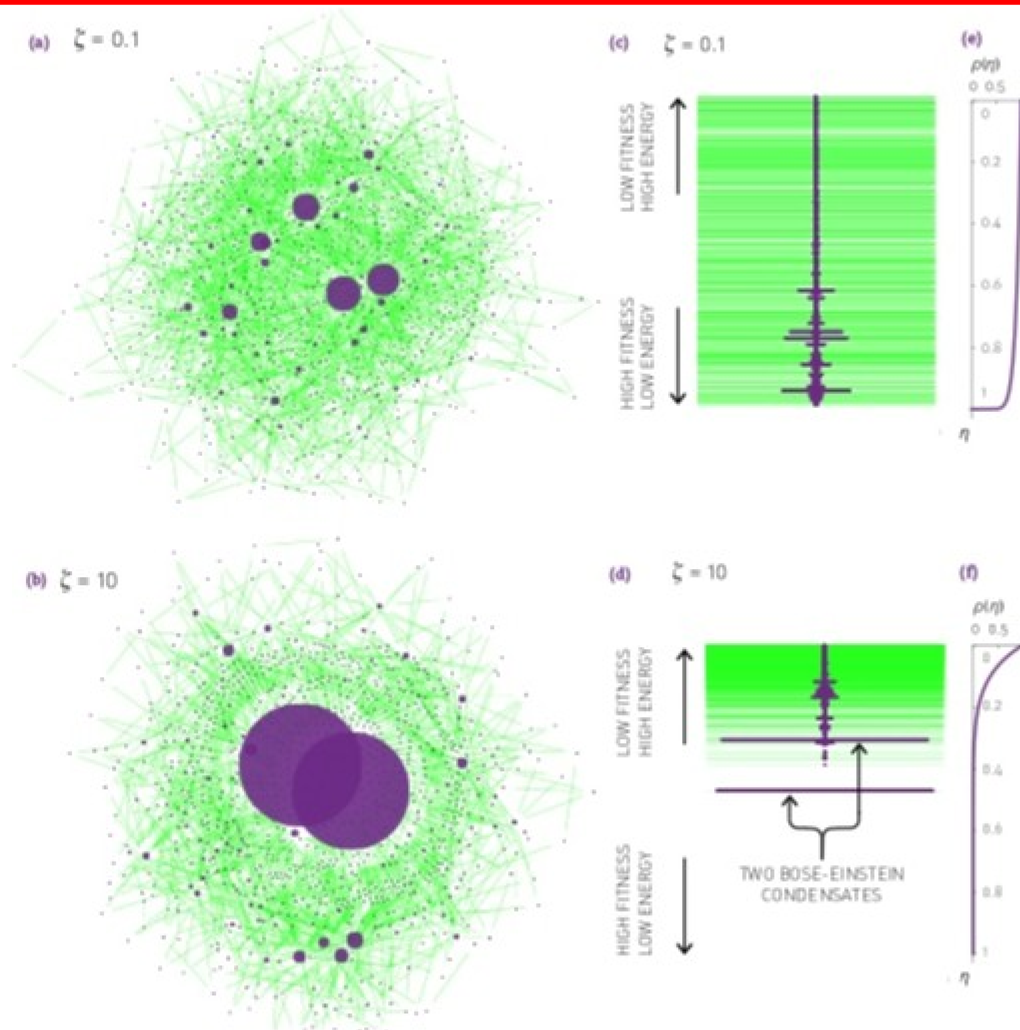
$$n(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1}$$

Section 4 | Bose-Einstein Condensation

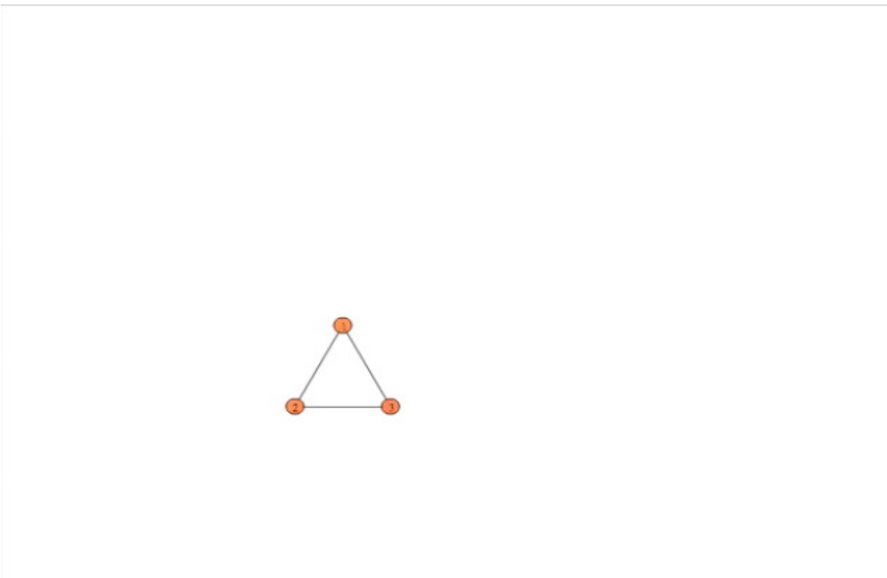


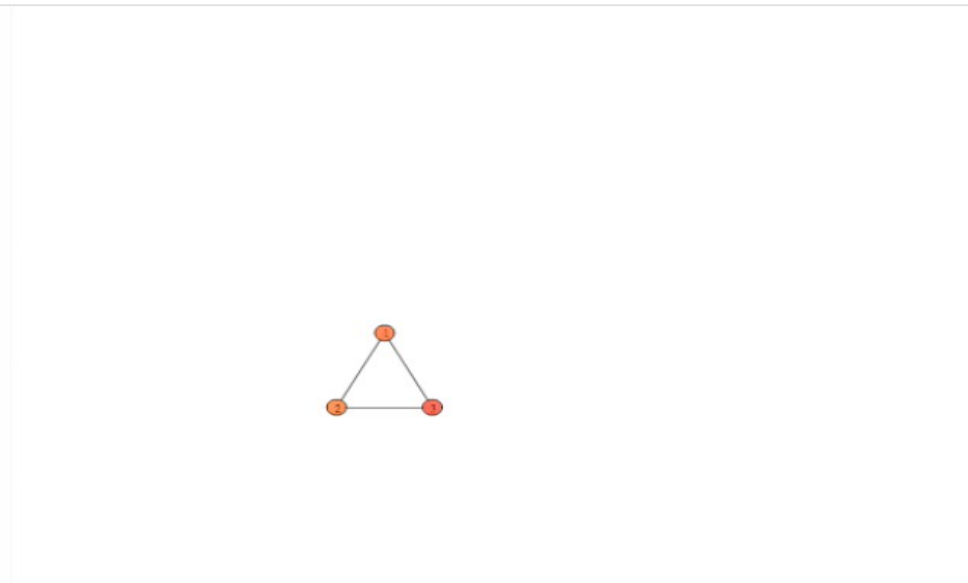
Section 4 Bose-Einstein Condensation

$$\rho(\eta) = (1-\eta)^{\zeta}$$



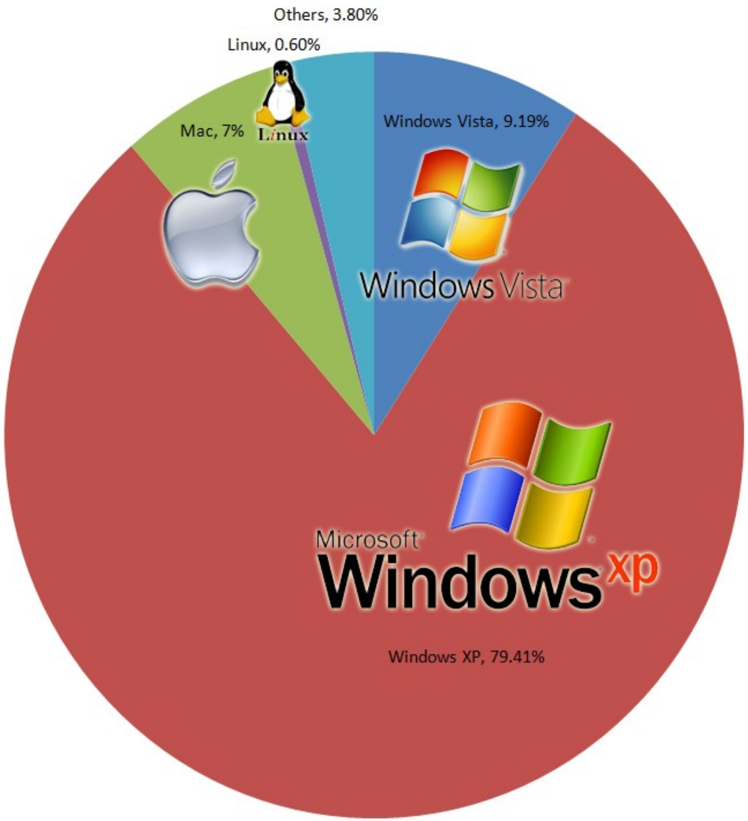
Bose-Einstein Condensation

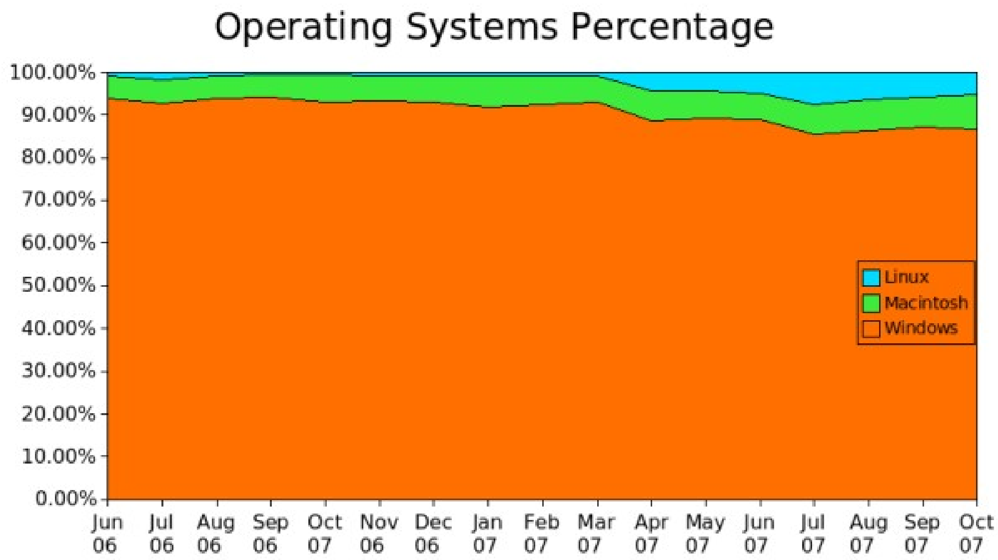




Bianconi & Barabási, *Physical Review Letters* 2001; *Europhys. Lett.* 2001.

FITNESS MODEL: Bose-Einstein Condensation





Bianconi & Barabási, *Physical Review Letters* 2001; *Europhys. Lett.* 2001.

Evolving Networks

(i) The model predicts $\gamma = 3$, while the experimentally observed degree exponents vary between 2 and 5 (**Table 4.1**).

(ii) The model predicts a power-law degree distribution, while in real systems we observe systematic deviations from a pure power-law function, like small-degree saturation or high-degree cutoff (**BOX 4.8**).

(iii) The model ignores a number of elementary processes that are obviously present in many real networks, like the addition of internal links and node or link removal.

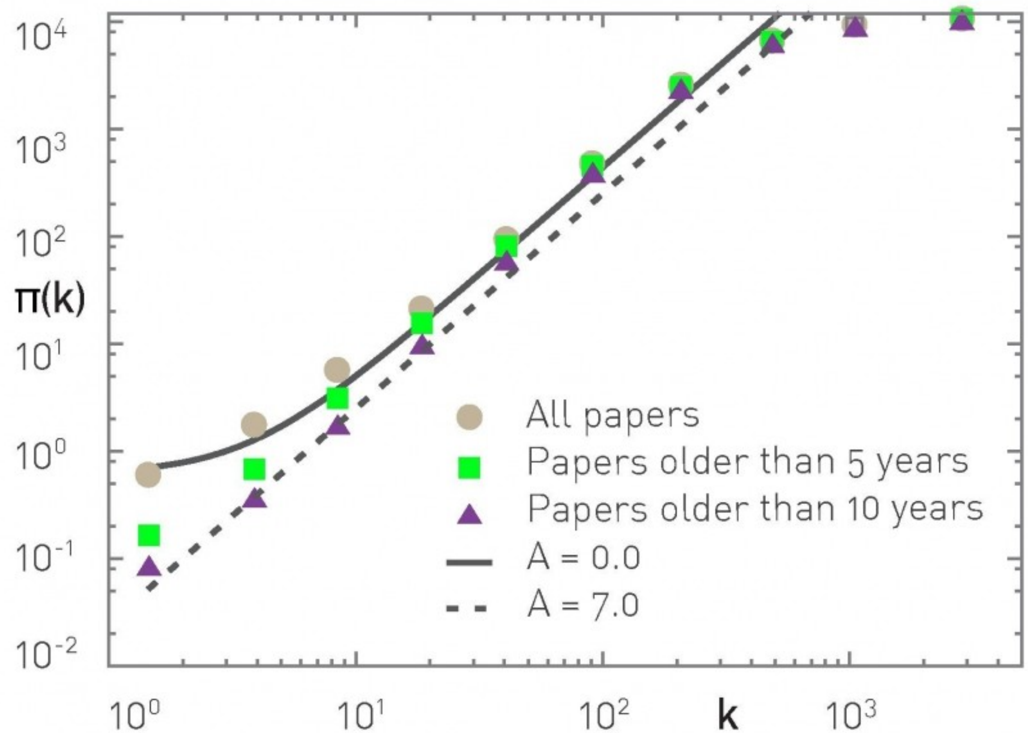
$$\Pi(k) \sim A + k$$

Increases the degree exponent.

$$\gamma = 3 + \frac{A}{m}$$

Generates a small-degree cutoff.

$$p_k = C(k + A)^{-\gamma}$$



$$\Pi(k, k') \sim (A + Bk)(A + Bk')$$

Double preferential attachment (A=0).

$$\gamma = 2 + \frac{m}{m + 2n}$$

Random attachment (B=0).

$$\gamma = 3 + \frac{2n}{m}$$

NODE DELETION

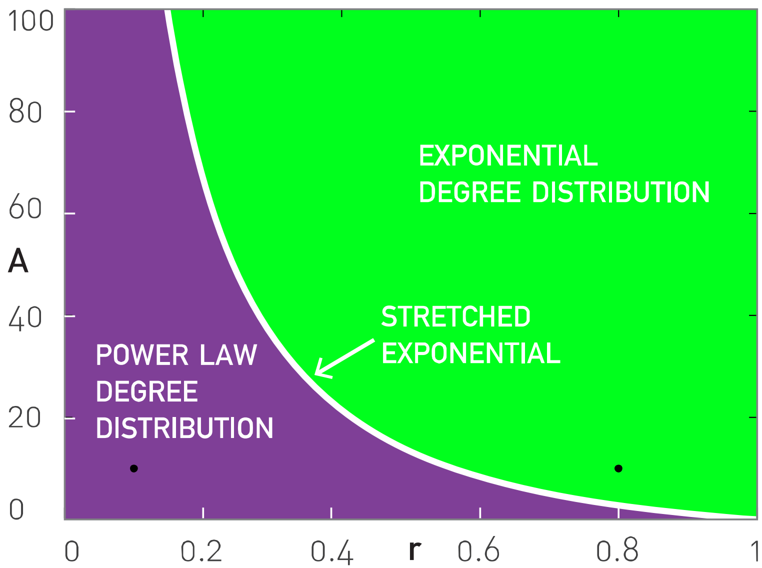
- Start with the Barabási-Albert model.
- In each time step:
  - add a new node with *m links*
  - remove *r* nodes (in average).

***r* < 1**: *Scale-free phase*

$$\gamma = 3 + \frac{2r}{1-r}$$

***r* = 1**: *Exponential phase*

***r* > 1:** *Declining network*



- Start with the Initial Attractiveness model:

  $\Pi(k) \sim A + k$
- In each time step:
  - add a new node with *m links*
  - remove *r* nodes (in average).

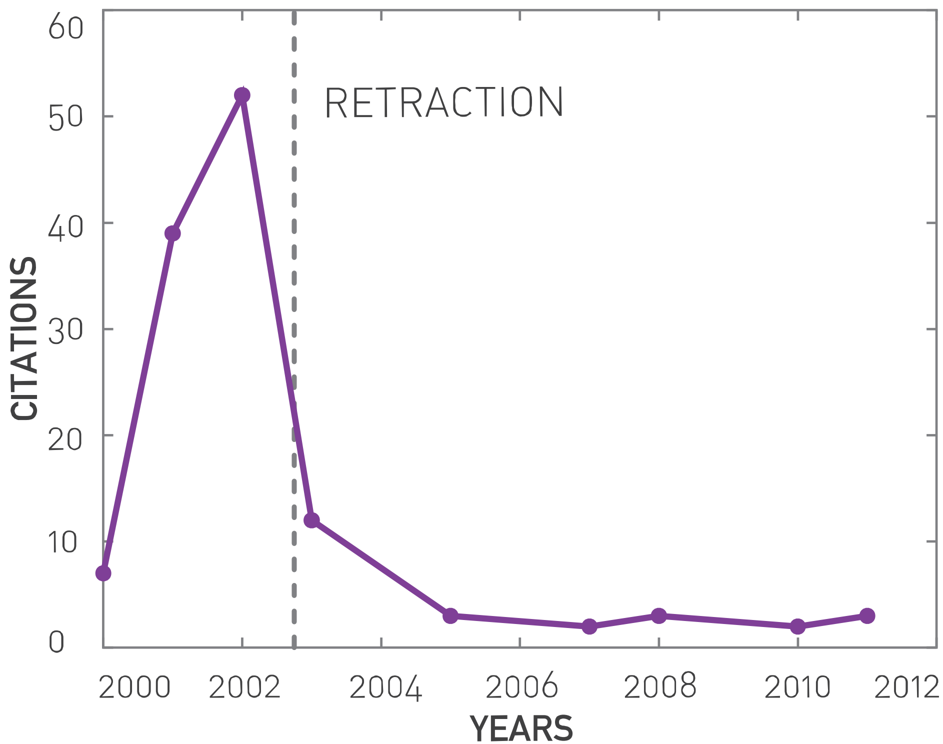
The coexistence of node removal with other elementary processes can lead to interesting topological phase transitions. This is illustrated by a simple model in which the network's growth is governed by **(6.23)**, and we also remove nodes with rate *r* [30]. The network displays three distinct phases, captured by the phase diagram shown above, whose axes are the node removal rate *r* and initial attractiveness *A*:

Subcritical Node Removal: $r < r^*(A)$
If the rate of node removal is under a critical value $r^*(A)$, shown as the white line on the figure, the network will be scale-free.

Critical Node Removal: $r = r^*(A)$
Once *r* reaches a critical value $r^*(A)$, the degree distribution turns into a stretched exponential (**SECTION 4.A**).

Exponential Networks: $r > r^*(A)$
The network looses its scale-free nature, developing an exponential degree distribution.

Section 5 The Impossibility of Node deletion



[23] J.H. Schön, Ch. Kloc, R.C. Haddon, and B. Batlogg. A superconducting field-effect switch. Science, 288: 656–8. 2000.

Jan Hendrik Schön

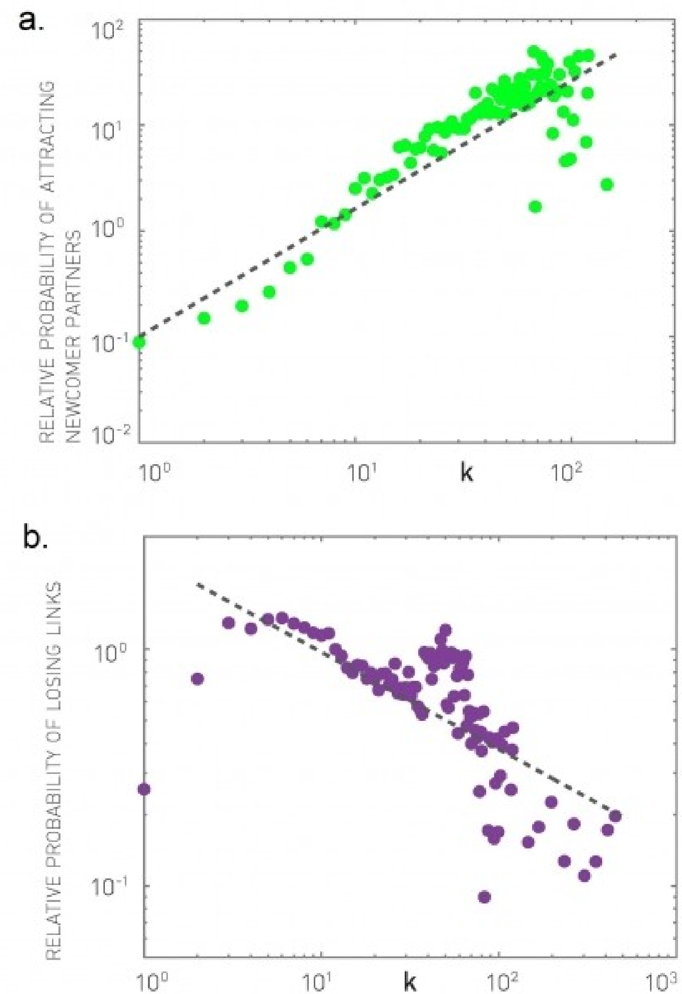
Section 5 Declining Fashion: New York

- **Preferential Attachment**

  While overall the network was shrinking, new nodes continued to arrive. The measurements indicate that the attachment probability of these new nodes follows $\Pi(k) \sim k^{\alpha}$ with α=1.20 ± 0.06 (Figure 6.13a), offering evidence of superlinear preferential attachment (SECTION 5.7).

- **Link Deletion**

  The probability that a firm lost a link follows $k(t)^{-\eta}$ with η = 0.41 ± 0.04, i.e. it decreased with the firms' degree (Figure 6.13b). This documents a *weak-gets-weaker* phenomenon, when the less connected firms are more likely to loose links.



we assumed that $L = \langle k \rangle N$, *where* $\langle k \rangle$ *is independent of time or N.*

- the average degree of the Internet increased from 3.42 (Nov. 1997) to 3.96 (Dec. 1998);
- the WWW increased its average degree from 7.22 to 7.86 during five months;
- in metabolic networks the average degree of the metabolites grows approximately linearly with the number of metabolites [33].

$$m(t) = m_0 t^{\theta}$$

$$\gamma = 3 + \frac{2\theta}{1-\theta}$$

Section 5 Aging

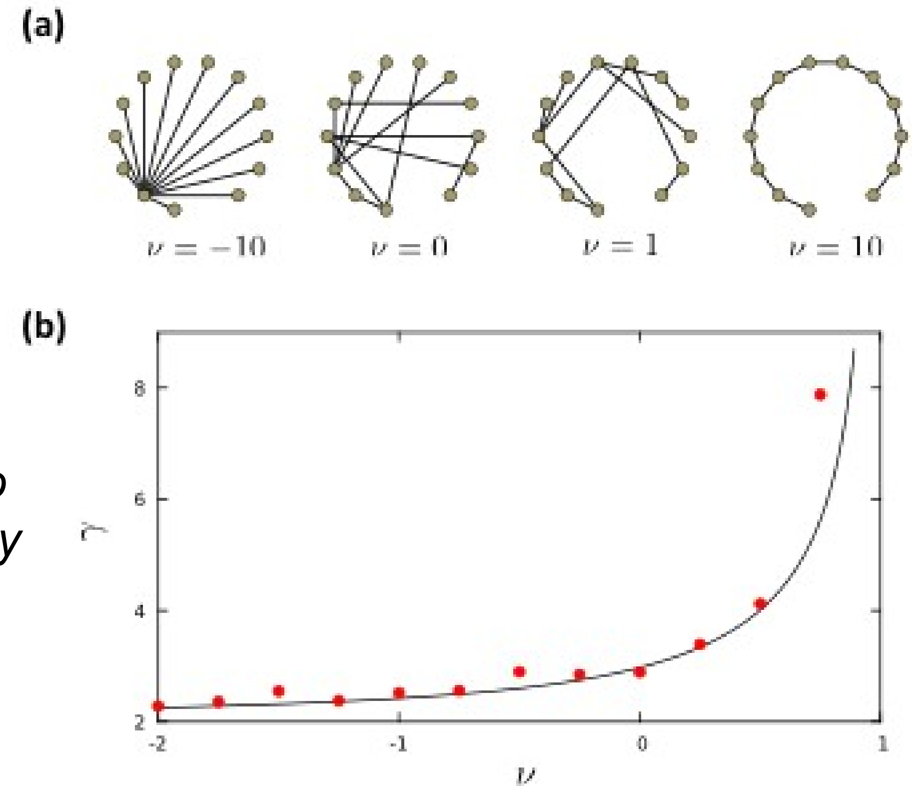
$$\Pi(k, t - t_i) \sim k(t - t_i)^{-\nu}$$

v<0: *new nodes attach to older nodes → enhances the role of preferential attachment.*

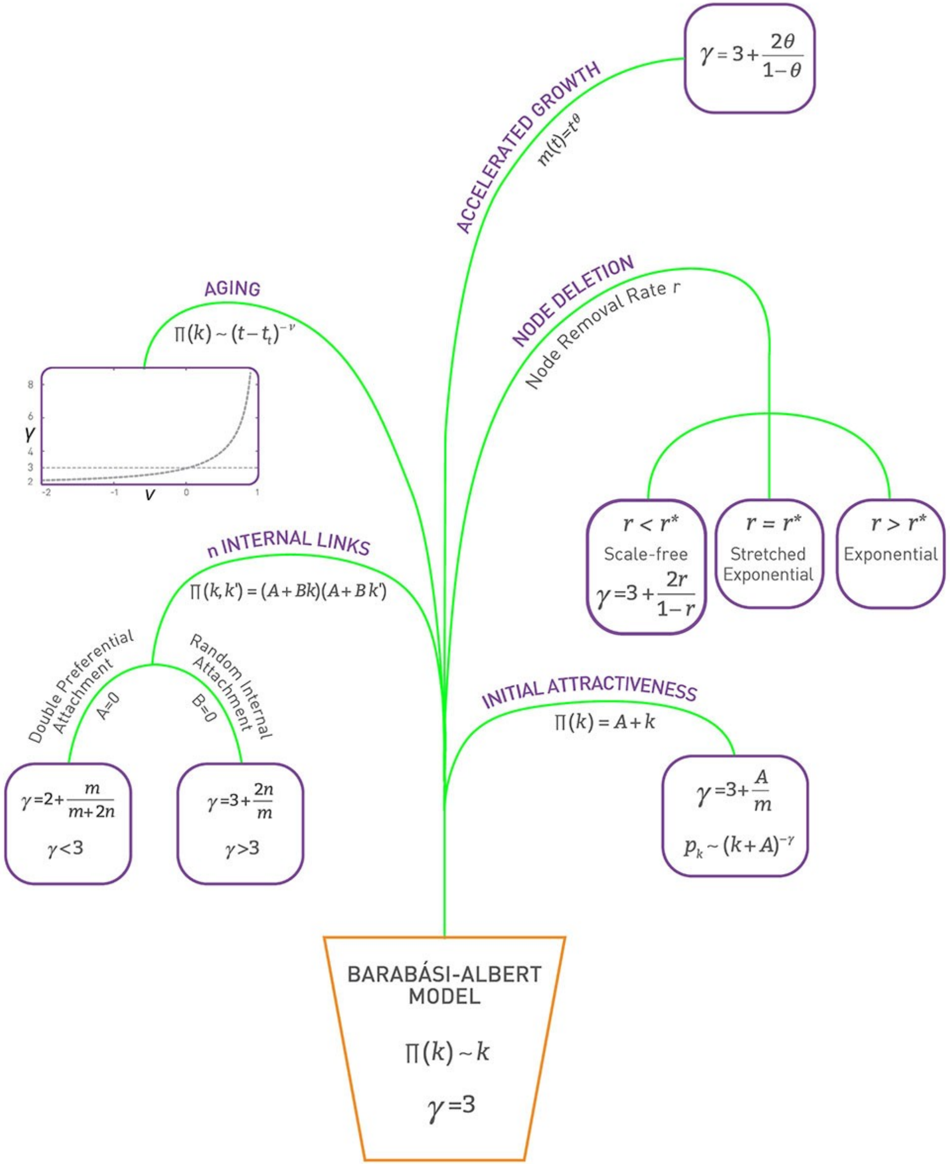
v→ −∞ each new node will only connect to the oldest node → hub-and-spoke topology (Fig 6.10a).

v>0: *new nodes attach to younger nodes*

v→ +∞: each node will connect to its immediate predecessor (Fig. 6.10a).



Section 5



ACCELERATED GROWTH
m(t)=t^θ
γ = 3 + 2θ/(1−θ)
AGING
Π(k) ~ (t−t_i)^−ν
γ
ν
NODE DELETION
Node Removal Rate r
r < r*
Scale-free
γ = 3 + 2r/(1−r)
r = r*
Stretched Exponential
r > r*
Exponential
n INTERNAL LINKS
Π(k,k') = (A+Bk)(A+Bk')
Double Preferential Attachment
A=0
Random Internal Attachment
B=0
γ = 2 + m/(m+2n)
γ<3
γ = 3 + 2n/m
γ>3
INITIAL ATTRACTIVENESS
Π(k) = A+k
γ = 3 + A/m
p_k ~ (k+A)^−γ
BARABÁSI-ALBERT MODEL
Π(k) ~ k
γ = 3

Summary

Section 6 summary : Topological Diversity

- **Power-Law**

  A pure power-law emerges if a growing network is governed by linear preferential attachment only, as predicted by the Barabási-Albert model. It is rare to observe such a pure power law in real systems. This idealized model represents the starting point for understanding the degree distribution of real networks.

- **Stretched Exponential**

  If preferential attachment is sublinear, the degree distribution follows a stretched exponential (**SECTION 4.11**). A similar degree-distribution can also appear under node removal at the critical point (**Figure 6.12**).

Section 6 summary : Topological Diversity

- **Fitness-induced Corrections**
  In the presence of fitness the precise form of p_k depends on the fitness distribution $\rho(\eta)$, which determines p_k via (6.6). For example, a uniform fitness distribution induces a logarithmic correction in p_k as predicted by (6.8). Other forms of $\rho(\eta)$ can lead to rather exotic forms for p_k.

- **Small-degree Saturation**
  Initial attractiveness adds a random component to preferential attachment. Consequently, the degree distribution develops a small-degree saturation, as seen in (6.24).

- **High-degree Cutoffs**
  Node and link removal, present in many real systems, can induce exponential high-degree cutoffs in the degree distribution. Furthermore, random node-removal can deplete the small-degree nodes, inducing a peak in p_k.

Section 6 summary : Topological Diversity

In most real networks several of the elementary processes discussed in this chapter appear together. For example, in the scientific collaboration network we have sublinear preferential attachment with initial attractiveness and the links can be both external and internal. As researchers have different creativity, fitness also plays a role, hence an accurate model requires us to know the appropriate fitness distribution. Therefore, the degree distribution is expected to display small degree saturation (thanks to initial attractiveness), stretched exponential cutoff at high degrees (thanks to sublinear preferential attachment), and some unknown corrections due to the particular form of the fitness distribution $\rho(\eta)$.

In general if wish to obtain an accurate fit to the degree distribution, we first need to build a generative model that analytically predicts the functional form of p_k. Yet, in many systems developing an accurate theory for p_k may be an overkill. It is often sufficient, instead, to establish if we are dealing with a bounded or an unbounded degree distribution (SECTION 4.9), as the system's properties will be primarily driven by this distinction.

MODEL CLASS	EXAMPLES	CHARACTERISTICS
Static Models	Erdős-Rényi Watts-Strogatz	• N fixed • p_k bounded • Static, time independent topologies
Generative Models	Configuration Model Hidden Parameter Model	• Arbitrary pre-defined p_k • Static, time independent topologies
Evolving Network Models	Barabási-Albert Model Bianconi-Barabási Model Initial Attractiveness Model Internal Links Model Node Deletion Model Accelerated Growth Model Aging Model	• p_k is determined by the processes that contribute to the network's evolution. • Time-varying network topologies

LESSONS LEARNED: evolving network models

1. There is no universal exponent characterizing all networks.
2. Growth and preferential attachment are responsible for the emergence of the scale-free property.
3. The origins of the preferential attachment are system-dependent.
4. Modeling real networks:
    - identify the low-level processes in the system
    - measure their frequency from real data
    - develop dynamical models to capture these processes.
5. If the model is correct, it should correctly predict not only the degree exponent, but both small and large k-cutoffs.

The end