Network Science

Class 4: Scale-free property

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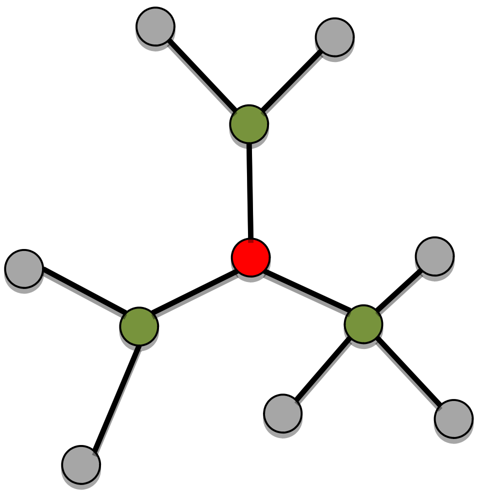
Ultra-small property

DISTANCES IN RANDOM GRAPHS

Random graphs tend to have a tree-like topology with almost constant node degrees.



- nr. of first neighbors: $N_1 \cong \langle k \rangle$
- nr. of second neighbors: $N_2 \cong \langle k \rangle^2$
- nr. of neighbours at distance d: $N_d \cong \langle k \rangle^d$
- estimate maximum distance:

$$1 + \sum_{l=1}^{l_{max}} \langle k \rangle^i = N \Rightarrow l_{max} = \frac{\log N}{\log \langle k \rangle}$$

SMALL WORLD BEHAVIOR IN SCALE-FREE NETWORKS

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

Ultra Small World

$$\langle l \rangle \sim \begin{cases} const. & \gamma = 2 \\ \dfrac{\ln\ln N}{\ln(\gamma-1)} & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln\ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

Size of the biggest hub is of order O(N). Most nodes can be connected within two layers of it, thus the average path length will be independent of the system size.

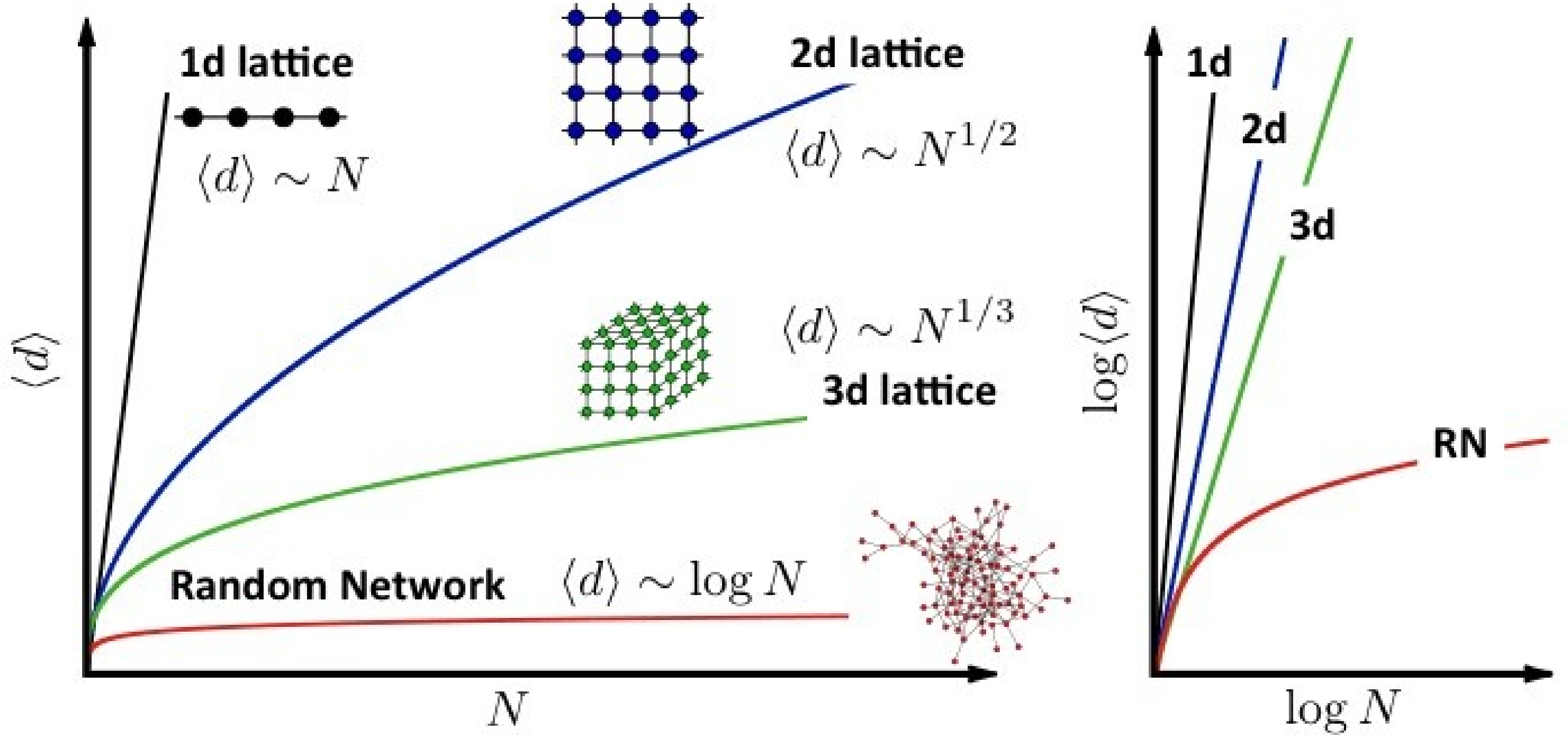
The average path length increases slower than logarithmically. In a random network all nodes have comparable degree, thus most paths will have comparable length. In a scale-free network the vast majority of the path go through the few high degree hubs, reducing the distances between nodes.

Some key models produce γ=3, so the result is of particular importance for them. This was first derived by Bollobas and collaborators for the network diameter in the context of a dynamical model, but it holds for the average path length as well.

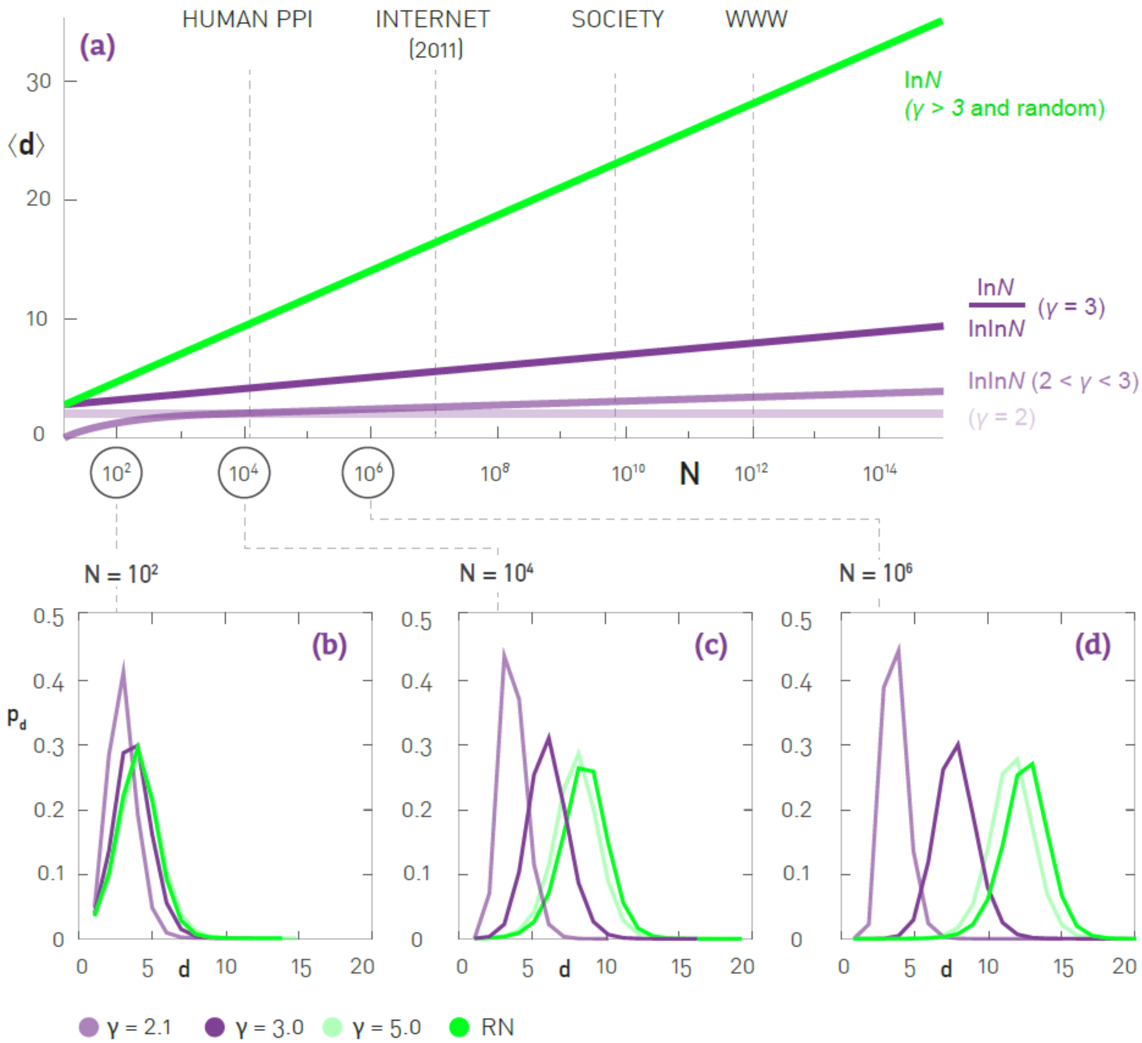
The second moment of the distribution is finite, thus in many ways the network behaves as a random network. Hence the average path length follows the result that we derived for the random network model earlier.

Cohen, Havlin Phys. Rev. Lett. 90, 58701(2003); Cohen, Havlin and ben-Avraham, in Handbook of Graphs and Networks, Eds. Bornholdt and Shuster (Willy-VCH, NY, 2002) Chap. 4; Confirmed also by: Dorogovtsev et al (2002), Chung and Lu (2002); (Bollobas, Riordan, 2002; Bollobas, 1985; Newman, 2001

Why are small worlds surprising? Suprising compared to what?



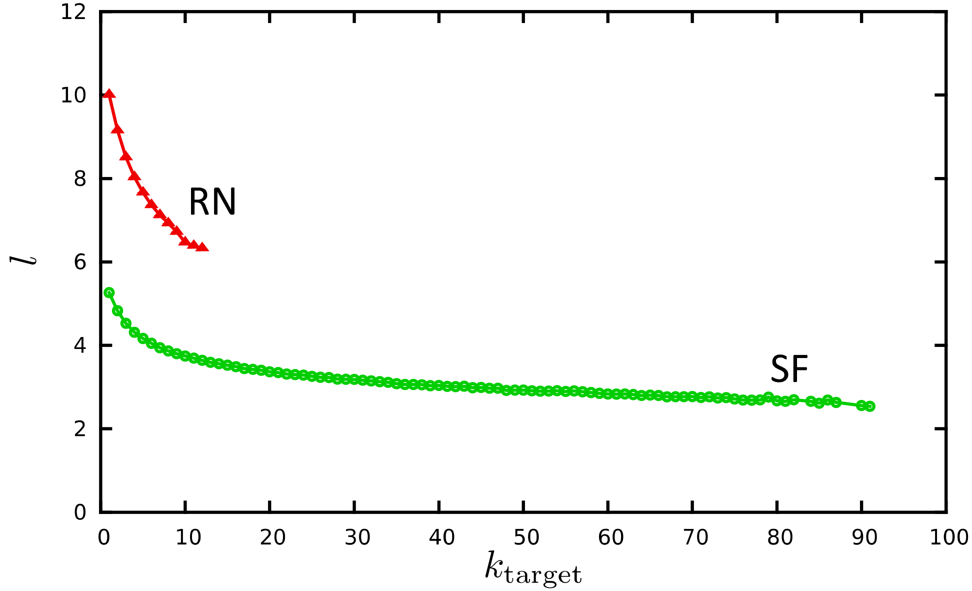
SMALL WORLD BEHAVIOR IN SCALE-FREE NETWORKS



$$\langle d \rangle \sim \begin{cases} \text{const.} & \gamma = 2, \\ \dfrac{\ln \ln N}{\ln(\gamma - 1)} & 2 < \gamma < 3, \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3, \\ \ln N & \gamma > 3. \end{cases}$$

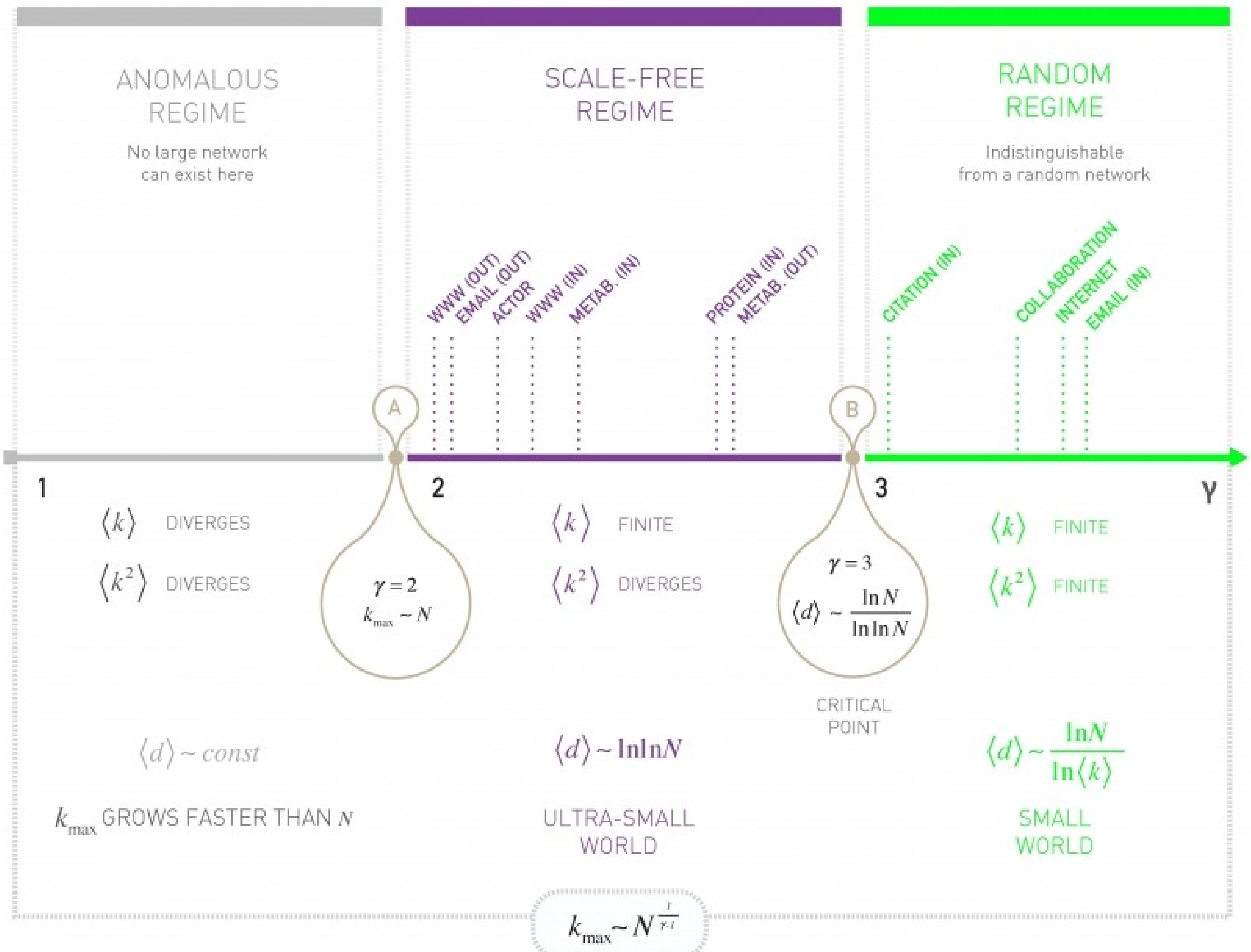
We are always close to the hubs

" it's always easier to find someone who knows a famous or popular figure than some run-the-mill, insignificant person."
(Frigyes Karinthy, 1929)

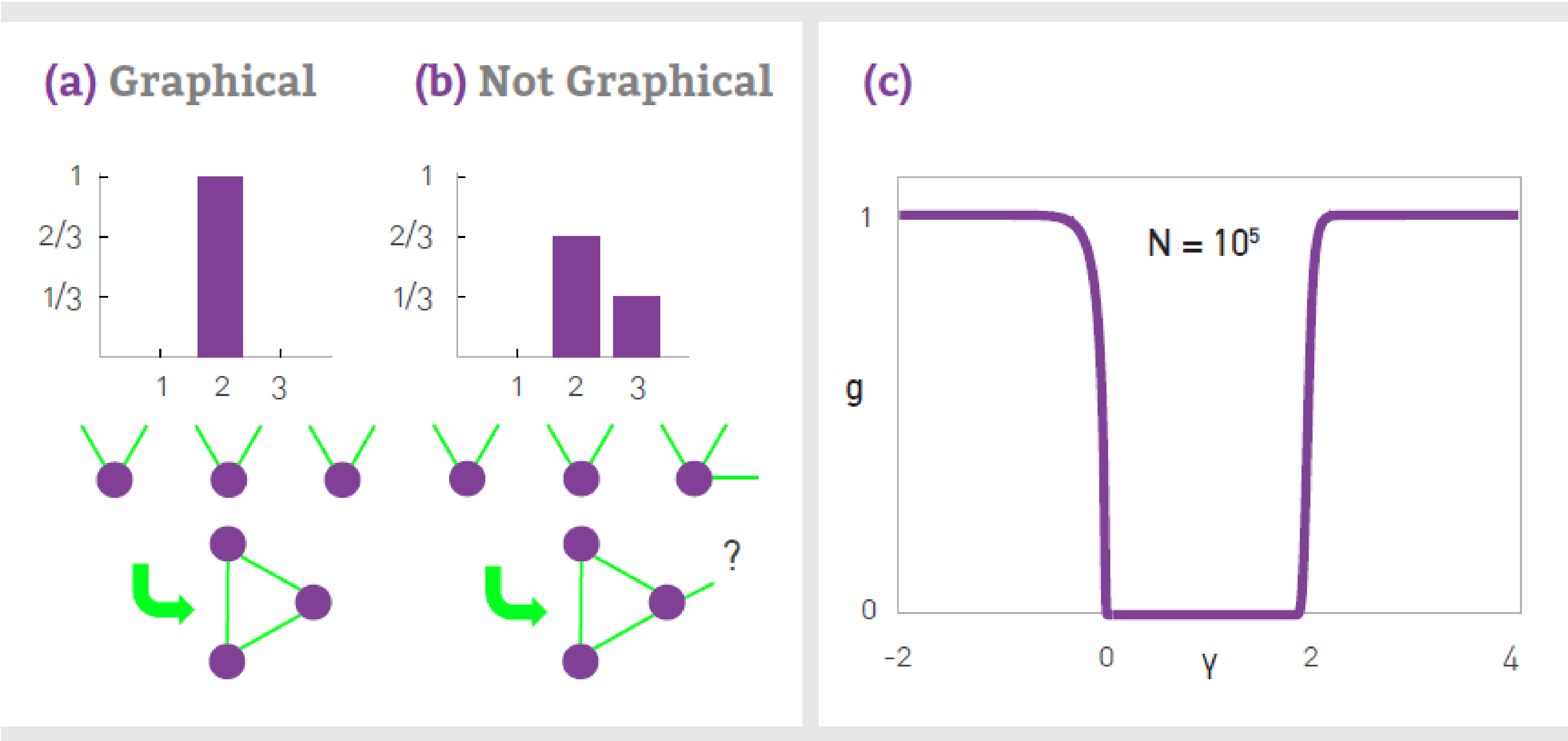


The role of the degree exponent

SUMMARY OF THE BEHAVIOR OF SCALE-FREE NETWORKS



Graphicality: No large networks for γ<2



In scale-free networks: $k_{max} = k_{min} N^{\frac{1}{\gamma - 1}}$ For γ<2: 1/(γ-1)>1

Why don't we see networks with exponents in the range of γ=4,5,6, etc?

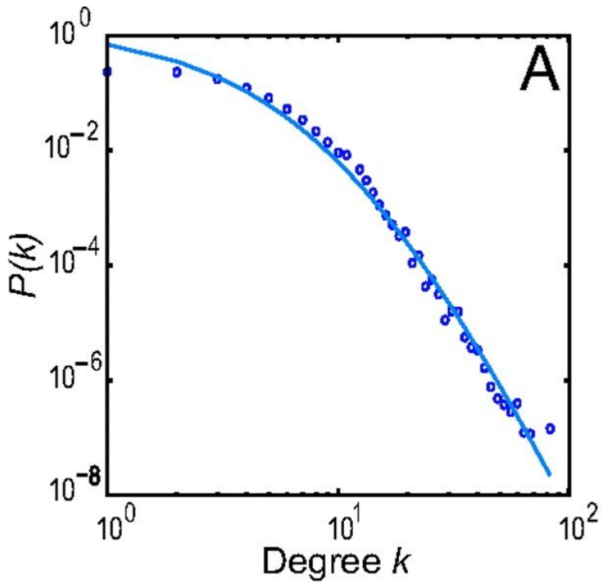
In order to document scale-free networks, we need 2-3 orders of magnitude scaling.
That is, $K_{max} \sim 10^2 K_{min}$ *to* $10^3 K_{min}$

However, that constrains on the system size we require to document it.
For example, to measure an exponent *γ=5*,we need to maximum degree a system size of the order of

$$K_{\max} = K_{\min} N^{\frac{1}{\gamma-1}}$$

$$N = \left(\frac{K_{\max}}{K_{\min}}\right)^{\gamma-1} \approx 10^8$$

Mobile Call Network



N=4.6x10[6]
γ=8.4

Fig. 1.

Characterizing the large-scale structure and the tie strengths of the mobile call graph. (A and B) Vertex degree (A) and tie strength distribution (B). Each distribution was fitted with *P(k) = a(k + k_0)*$^{-\gamma}$ exp(−*k/k_c*).

Onella et al. PNAS 2007