A Tight Linear Time (1/2)-Approximation for Unconstrained Submodular Maximization

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In this presentation we will focus on the Unconstrained Submodular Maximization problem (USM).

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Submodular Functions

A function is submodular if, for every \( A \subseteq B \subseteq N \) and \( u \in N \), we have:

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f(A \cup \{u\}) - f(A) \geq f(B \cup \{u\}) - f(B).
\]

An equivalent definition is, for any subsets \( A \) and \( B \):

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f(A) + f(B) \geq f(A \cup B) + f(A \cap B).
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As an example, consider the cardinality of a cut in a graph.
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They design a linear time deterministic (1/3)-approximation algorithm for USM, using a greedy based approach.

Then, modifying the deterministic algorithm using randomness, they design a (1/2)-approximation algorithm for USM.

This result is tight, because there is an upper bound of $(1/2 + \epsilon)$ to the approximation ratio of any algorithm for USM [2].
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Techniques

Let's show two straightforward greedy approaches.

First, define $\bar{f}(S) = f(N \setminus S)$.

Once $f(S)$ is submodular so it is $\bar{f}(S)$.

$$
\bar{f}(A) + \bar{f}(B) = f(N \setminus A) + f(N \setminus B) \\
\geq f((N \setminus A) \cup (N \setminus B)) + f((N \setminus A) \cap (N \setminus B)) \\
= f(N \setminus (A \cap B)) + f(N \setminus (A \cup B)) \\
= \bar{f}(A \cap B) + \bar{f}(A \cup B).
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\]
Now, let's define a greedy algorithm that starts from an empty solution and iteratively adds elements to it.

This algorithm decides to add an element by checking if the submodular function increases when it is added.

It works both for $f$ and $\bar{f}$, and for the later it corresponds to start with $\mathcal{N}$ and to iteratively remove elements from it.

Although they seem reasonable, neither gives a constant approximation ratio.
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Algorithm 1: DeterministicUSM.

**Data:** $f$, $N$

$X_0 \leftarrow \emptyset; \quad Y_0 \leftarrow N$

**for** $i = 1$ to $|N|$ **do**

  $a_i \leftarrow f(X_{i-1} \cup \{u_i\}) - f(X_{i-1})$

  $b_i \leftarrow f(Y_{i-1} \setminus \{u_i\}) - f(Y_{i-1})$

  **if** $a_i \geq b_i$ **then**

    $X_i \leftarrow X_{i-1} \cup \{u_i\}; \quad Y_i \leftarrow Y_{i-1}$

  **else** $a_i < b_i$

    $X_i \leftarrow X_{i-1}; \quad Y_i \leftarrow Y_{i-1} \setminus \{u_i\}$

**end**

**end**

**return** $X_n$ (or equivalently $Y_n$).
Lemma (1)

For every $1 \leq i \leq |N|$ we have that $a_i + b_i \geq 0$.

Demonstração.

By submodularity, we have:

$$f(X_{i-1} \cup \{u_i\}) - f(X_{i-1}) \geq f(Y_{i-1}) - f(Y_{i-1} \setminus \{u_i\}).$$

So:

$$a_i + b_i = f(X_{i-1} \cup \{u_i\}) - f(X_{i-1}) + f(Y_{i-1} \setminus \{u_i\}) - f(Y_{i-1})$$
$$= (f(X_{i-1} \cup \{u_i\}) - f(X_{i-1})) - (f(Y_{i-1}) - f(Y_{i-1} \setminus \{u_i\}))$$
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Analysis of the Deterministic USM Algorithm

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$$\geq 0.$$
Let's define $OPT_i = (OPT \cup X_i) \cap Y_i$.

Realize that $OPT_0 = OPT$ and $OPT_{|N|} = X_{|N|} = Y_{|N|}$.

**Lemma (2)**

For every $1 \leq i \leq |N|$ we have:

$$f(OPT_{i-1}) - f(OPT_i) \leq f(X_i) - f(X_{i-1}) + f(Y_i) - f(Y_{i-1}).$$
Analysis of the DeterministicUSM (cont.)

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Using lemma 2 we have:

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\sum_{i=1}^{N} (f(OPT_{i-1}) - f(OPT_i)) \leq \sum_{i=1}^{N} (f(X_i) - f(X_{i-1})) + \sum_{i=1}^{N} (f(Y_i) - f(Y_{i-1})).
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Proving theorem (cont).

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Once the previous sums are telescopic we have:

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\begin{align*}
  f(OPT_0) - f(OPT_{|N|}) & \leq f(X_{|N|}) - f(X_0) + f(Y_{|N|}) - f(Y_0) \\
  & \leq f(X_{|N|}) + f(Y_{|N|}).
\end{align*}
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So,

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Assume that $a_i \geq b_i$ (the other case is similar).

In this case, $OPT_i = (OPT \cup X_i) \cap Y_i = OPT_{i-1} \cup \{u_i\}$ and $Y_i = Y_{i-1}$.

So, we have to prove that:

$$f(OPT_{i-1}) - f(OPT_{i-1} \cup \{u_i\}) \leq f(X_i) - f(X_{i-1}) = a_i.$$
Analysis of the Deterministic USM (cont.)

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Analysis of the Deterministic USM (cont.)

Proving lemma 2 (cont).

Demonstração.

Now we consider two cases.

If $u_i \in OPT$ then $f(OPT_{i-1}) - f(OPT_{i-1}) = 0$ and $a_i \geq 0$.

If $u_i \notin OPT$ then $u_i \notin OPT_{i-1}$ and

$$f(OPT_{i-1}) - f(OPT_{i-1} \cup \{u_i\}) \leq f(Y_{i-1} \setminus \{u_i\}) - f(Y_{i-1}) = b_i \leq a_i.$$
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U. Feige, V.S. Mirrokni, and J. Vondrák.
Maximizing non-monotone submodular functions.
Thank you!

Questions?
Acknowledgements

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