The Online Prize-Collecting Facility Location Problem

LAGOS 2015

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Combinatorial Optimization Problems
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Problems in which an objective function needs to be minimized or maximized.
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Minimization problems in which we are interested:
- Facility Location problem,
- Prize-Collecting Facility Location problem.
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- Prize-Collecting Facility Location problem.

These problems are NP-hard and constant factor approximation algorithms are known for them.
Facility Location Problem

\[
\text{f}=2
\]

\[
\begin{align*}
\text{Total cost } &= 2 + 3 \\
&= 5.
\end{align*}
\]
The Online Prize-Collecting Facility Location Problem

Facility Location Problem

\[
\text{Total cost} = 2 + 3 = 5.
\]
The Online Prize-Collecting Facility Location Problem

Facility Location Problem

\[
\min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, F^a)
\]

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Facility Location Problem

\[ \min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, F^a) \]

Total cost = 2
Facility Location Problem

\[ \min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, F^a) \]

Total cost = 2 + 3
The Online Prize-Collecting Facility Location Problem

Facility Location Problem

$$\min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, F^a)$$

Total cost = 2 + 3 = 5.
Online Computation
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Parts of the input are revealed one at a time.
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Parts of the input are revealed one at a time.

Each part must be served before the next one arrives.
Online Computation

Parts of the input are revealed one at a time.

Each part must be served before the next one arrives.

No decision can be changed in the future.
Competitive Analysis
Competitive Analysis

Worst case technique used to analyze online algorithms.
Competitive Analysis

Worst case technique used to analyze online algorithms.

An online algorithm $\text{ALG}$ is $c$-competitive if:

$$\text{ALG}(I) \leq c \cdot \text{OPT}(I) + \kappa,$$

for every input $I$ and some constant $\kappa$. 
Online Problems

Minimization problems in which we are interested:
Online Problems

Minimization problems in which we are interested:

- Online Facility Location (OFL),
Online Problems

Minimization problems in which we are interested:
- Online Facility Location (OFL),
- Online Prize-Collecting Facility Location (OPFL).
Online Facility Location Problem

\[ \text{Total cost} = 2 + 2 + 2 = 6. \]

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Online Facility Location Problem

\[ \min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, a(j)) \]

Total cost = 2 + 2 + 2 = 6.
Online Facility Location Problem

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Total cost = 2 + 2 + 2 = 6.
Online Facility Location Problem

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Total cost = 2
Online Facility Location Problem

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\min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, a(j))
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Total cost = 2
Online Facility Location Problem

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\min \sum_{i \in F^a} f(i) + \sum_{j \in D} d(j, a(j))
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Total cost = 2 + 2
Online Facility Location Problem

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Online Facility Location Problem

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Total cost = 2 + 2 + 2 = 6.
Online Facility Location Results

The Online Prize-Collecting Facility Location Problem

Online Facility Location Results

The OFL has competitive ratio $\Theta\left(\log n \log \log n\right)$ [Fotakis 2008].

There are randomized and deterministic $O(\log n)$-competitive algorithms known for it [Meyerson 2001, Fotakis 2007].

[Nagarajan and Williamson 2013] give a dual-fitting analysis for the algorithm by [Fotakis 2007].
Online Facility Location Results

The OFL has competitive ratio $\Theta\left(\frac{\log n}{\log \log n}\right)$ [Fotakis 2008].
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Online Facility Location LP Formulation
Online Facility Location LP Formulation

Linear programming relaxation

\[
\begin{align*}
\min & \quad \sum_{i \in F} f(i)y_i + \sum_{j \in D} \sum_{i \in F} d(j, i)x_{ji} \\
\text{s.t.} & \quad x_{ji} \leq y_i \quad \text{for } j \in D \text{ and } i \in F, \\
& \quad \sum_{i \in F} x_{ji} \geq 1 \quad \text{for } j \in D, \\
& \quad y_i \geq 0, x_{ji} \geq 0 \quad \text{for } j \in D \text{ and } i \in F,
\end{align*}
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Online Facility Location LP Formulation

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& \quad \sum_{i \in F} x_{ji} \geq 1 \quad \text{for } j \in D, \\
& \quad y_i \geq 0, x_{ji} \geq 0 \quad \text{for } j \in D \text{ and } i \in F,
\end{align*}
\]

and its dual

\[
\begin{align*}
\text{max} & \quad \sum_{j \in D} \alpha_j \\
\text{s.t.} & \quad \sum_{j \in D} (\alpha_j - d(j, i))^+ \leq f(i) \quad \text{for } i \in F, \\
& \quad \alpha_j \geq 0 \quad \text{for } j \in D.
\end{align*}
\]
Algorithm 1: OFL Algorithm.

**Input:** $(G, d, f, F)$

$F^a \leftarrow \emptyset$; $D \leftarrow \emptyset$;

**while a new client $j'$ arrives do**

increase $\alpha_{j'}$ until one of the following happens:

(a) $\alpha_{j'} = d(j', i)$ for some $i \in F^a$; /* connect only */

(b) $f(i) = (\alpha_{j'} - d(j', i)) + \sum_{j \in D} (d(j, F^a) - d(j, i))^+$ for some $i \in F \setminus F^a$; /* open and connect */

$F^a \leftarrow F^a \cup \{i\}$; $D \leftarrow D \cup \{j'\}$; $a(j') \leftarrow i$;

**end**

**return** $(F^a, a)$;
Online Prize-Collecting Facility Location

The total cost is $2 + 2 + 2 = 6$. 

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Online Prize-Collecting Facility Location

\[ \min \sum_{i \in F^a} f(i) + \sum_{j \in D^c} d(j, a(j)) + \sum_{j \in D^p} p(j) \]
Online Prize-Collecting Facility Location

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Total cost = 2 + 2 + 2 = 6.
Online Prize-Collecting Facility Location

\[
\min \sum_{i \in F^a} f(i) + \sum_{j \in D^c} d(j, a(j)) + \sum_{j \in D^p} p(j)
\]

Total cost = 2
Online Prize-Collecting Facility Location

\[
\min \sum_{i \in F^a} f(i) + \sum_{j \in D^c} d(j, a(j)) + \sum_{j \in D^p} p(j)
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Total cost = 2
Online Prize-Collecting Facility Location

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\min \sum_{i \in F^a} f(i) + \sum_{j \in D^c} d(j, a(j)) + \sum_{j \in D^p} p(j)
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The Online Prize-Collecting Facility Location Problem

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Online Prize-Collecting Facility Location

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The Online Prize-Collecting Facility Location Problem

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\text{Total cost} & = 2 + 2 + 2 = 6.
\end{align*}
\]
OPFL Results
Our contribution: we proposed the problem and showed a primal-dual $(6 \log n)$-competitive algorithm for it, by extending the algorithm from [Fotakis 2007, Nagarajan and Williamson 2013].
OPFL Results

*Our contribution:* we proposed the problem and showed a primal-dual $(6 \log n)$-competitive algorithm for it, by extending the algorithm from [Fotakis 2007, Nagarajan and Williamson 2013].

Since the OPFL is a generalization of the OFL, the lower bound of $\Omega \left( \frac{\log n}{\log \log n} \right)$ applies to it.
The Online Prize-Collecting Facility Location Problem

OPFL LP Formulation
OPFL LP Formulation

Linear programming relaxation

$$\min \sum_{i \in F} f(i)y_i + \sum_{j \in D} \sum_{i \in F} d(j, i)x_{ji} + \sum_{j \in D} p(j)z_j$$

s.t.  \( x_{ji} \leq y_i \) for \( j \in D \) and \( i \in F \),

\( \sum_{i \in F} x_{ji} + z_j \geq 1 \) for \( j \in D \),

\( y_i \geq 0, x_{ji} \geq 0, z_j \geq 0 \) for \( j \in D \) and \( i \in F \),
The Online Prize-Collecting Facility Location Problem

OPFL LP Formulation

Linear programming relaxation

\[
\begin{align*}
\min \quad & \sum_{i \in F} f(i) y_i + \sum_{j \in D} \sum_{i \in F} d(j, i) x_{ji} + \sum_{j \in D} p(j) z_j \\
\text{s.t.} \quad & x_{ji} \leq y_i \quad \text{for } j \in D \text{ and } i \in F, \\
& \sum_{i \in F} x_{ji} + z_j \geq 1 \quad \text{for } j \in D, \\
& y_i \geq 0, \ x_{ji} \geq 0, \ z_j \geq 0 \quad \text{for } j \in D \text{ and } i \in F,
\end{align*}
\]

and its dual

\[
\begin{align*}
\max \quad & \sum_{j \in D} \alpha_j \\
\text{s.t.} \quad & \sum_{j \in D} (\alpha_j - d(j, i))^+ \leq f(i) \quad \text{for } i \in F, \\
& \alpha_j \leq p(j) \quad \text{for } j \in D, \\
& \alpha_j \geq 0 \quad \text{for } j \in D.
\end{align*}
\]
OPFL Algorithm

Algorithm 2: OPFL Algorithm.

Input: \((G, d, f, p, F)\)

\[ D \leftarrow \emptyset; \quad F^a \leftarrow \emptyset; \]

while a new client \(j'\) arrives do

increase \(\alpha_{j'}\) until one of the following happens:
(a) \(\alpha_{j'} = d(j', i)\) for some \(i \in F^a\); /* connect only */
(b) \(f(i) = (\alpha_{j'} - d(j', i)) + \sum_{j \in D}(\min\{d(j, F^a), p(j)\} - d(j, i))^+\) for some \(i \in F \setminus F^a\); /* open and connect */
(c) \(\alpha_{j'} = p(j')\); /* pay the penalty */

(in this case \(i\) is choose to be null, i.e., \(\{i\} = \emptyset\))

\[ F^a \leftarrow F^a \cup \{i\}; \quad D \leftarrow D \cup \{j'\}; \quad a(j') \leftarrow i; \]

end

return \((F^a, a)\);
Analysis

Towards the $O(\log n)$ competitive ratio

1. Resulting assignment is feasible.
Analysis

Towards the $O(\log n)$ competitive ratio

1. Resulting assignment is feasible.
2. Total cost of the assignment is bounded by $2 \cdot \sum_{j} \alpha_j$. 

Analysis

Towards the \( O(\log n) \) competitive ratio

1. Resulting assignment is feasible.
2. Total cost of the assignment is bounded by \( 2 \cdot \sum_j \alpha_j \).
3. \( \left\{ \frac{\alpha_j}{3H_n} \right\}_j \) is feasible to the dual problem, so \( \sum_j \frac{\alpha_j}{3H_n} \leq \text{OPT} \).
Analysis

Towards the $O(\log n)$ competitive ratio

1. Resulting assignment is feasible. ✓
2. Total cost of the assignment is bounded by $2 \cdot \sum_j \alpha_j$.
3. \( \left\{ \frac{\alpha_j}{3H_n} \right\}_{j} \) is feasible to the dual problem, so \( \sum_j \frac{\alpha_j}{3H_n} \leq \text{OPT} \).
Analysis

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- total cost for opening facilities $\leq \sum_j \alpha_j$.
- total cost for connections and penalties $\leq \sum_j \alpha_j$. 
Analysis

2. Total cost of the assignment is bounded by $2 \cdot \sum_j \alpha_j$. ✓
   - total cost for opening facilities $\leq \sum_j \alpha_j$.
   - total cost for connections and penalties $\leq \sum_j \alpha_j$. 
Analysis

Towards $O(\log n)$ competitive ratio

1. Resulting assignment is feasible. ✓
2. Total cost of the assignment is bounded by $2 \cdot \sum_j \alpha_j$. ✓
3. $\left\{ \frac{\alpha_j}{3H_n} \right\}_j$ is feasible to the dual problem, so $\sum_j \frac{\alpha_j}{3H_n} \leq \text{OPT}$. 
Analysis

\[
\left\{ \frac{\alpha_j}{3H_n} \right\}_j \text{ is feasible to the dual problem, so } \sum_j \frac{\alpha_j}{3H_n} \leq \text{OPT}.
\]
Analysis

\[ \{ \frac{\alpha_j}{3H_n} \}_{j} \] is feasible to the dual problem, so \( \sum_j \frac{\alpha_j}{3H_n} \leq \text{OPT} \).

\( D^c := \{ \text{connected clients} \} \quad D^p := \{ \text{penalized clients} \} \)
The Online Prize-Collecting Facility Location Problem

Analysis

\[ \left\{ \frac{\alpha_j}{3H_n} \right\}_j \text{ is feasible to the dual problem, so } \sum_j \frac{\alpha_j}{3H_n} \leq \text{OPT}. \]

\[ D^c := \{ \text{connected clients} \} \quad D^p := \{ \text{penalized clients} \} \]

(i) \[ \sum_{j \in D^c} \left( \frac{\alpha_j}{3H_n} - d(j, i) \right)^+ \leq \frac{f_i}{2} \text{ for each } i \in F. \]
Analysis

\[ \left\{ \frac{\alpha_j}{3H_n} \right\}_j \]

is feasible to the dual problem, so \( \sum_j \frac{\alpha_j}{3H_n} \leq \text{OPT} \).

\( D^c := \{ \text{connected clients} \} \quad D^p := \{ \text{penalized clients} \} \)

(i) \( \sum_{j \in D^c} \left( \frac{\alpha_j}{3H_n} - d(j, i) \right)^+ \leq \frac{f_i}{2} \) for each \( i \in F \).

(ii) \( \sum_{j \in D^p} \left( \frac{\alpha_j}{3H_n} - d(j, i) \right)^+ \leq \frac{f_i}{2} \) for each \( i \in F \).
The Online Prize-Collecting Facility Location Problem

Analysis:  
(i) \( \sum_{j \in D^C} \left( \frac{\alpha_j}{3H_n} - d(j, i) \right)^+ \leq \frac{f_i}{2} \) for each \( i \in F \)

Techniques in [NW, 2013]

For each \( i \in F \),

\[
\begin{align*}
    f(i) & \geq (\alpha[k] - d(j[k], i)) + \sum_{j \in D^C_{[k-1]}} \left( \min\{d(j, F^a[k]), p(j)\} - d(j, i) \right)^+ \quad (j \text{ connected } \Rightarrow d(j, F^a[k]) \text{ is smaller}) \\
    & = (\alpha[k] - d(j[k], i)) + \sum_{j \in D^C_{[k-1]}} \left( d(j, F^a[k]) - d(j, i) \right)^+ \quad (\text{triangle inequality}) \\
    & \geq (\alpha[k] - d(j[k], i)) + \sum_{j \in D^C_{[k-1]}} \left( \alpha[k] - d(j[k], i) - 2d(j, i) \right)^+ \\
    \Rightarrow f(i) & \geq (1 + (k - 1)) \cdot (\alpha[k] - d(j[k], i)) - 2 \sum_{j \in D^C_{[k-1]}} d(j, i) \\
\end{align*}
\]
Analysis: (i) $\sum_{j \in D^c} \left( \frac{\alpha_j}{3H_n} - d(j, i) \right)^+ \leq \frac{f_i}{2}$ for each $i \in F$

Techniques in [NW, 2013]

For each $i \in F$,

$$f(i) \geq k \cdot (\alpha[k] - d(j[k], i)) - 2 \sum_{j \in D_{[k-1]}^c} d(j, i)$$

$$\Rightarrow \frac{f(i)}{k} \geq (\alpha[k] - d(j[k], i)) - 2 \sum_{j \in D_{[k-1]}^c} \frac{d(j, i)}{k}$$

$$\Rightarrow H_{|D^c|} \cdot f(i) \geq \sum_{k=1}^{|D^c|} (\alpha[k] - d(j[k], i)) - 2 \sum_{k=1}^{|D^c|} \frac{1}{k} \cdot \sum_{j \in D_{[k-1]}^c} d(j, i)$$

$$\Rightarrow H_{|D^c|} \cdot f(i) \geq \sum_{k=1}^{|D^c|} (\alpha[k] - d(j[k], i)) - (2H_{|D^c|} - 1) \cdot \sum_{j \in D^c} d(j, i)$$

$$\Rightarrow H_{|D^c|} \cdot f(i) \geq \sum_{j \in D^c} (\alpha[k] - 2H_{|D^c|} d(j[k], i))$$
Analysis:  
\[ (i) \sum_{j \in D^c} \left( \frac{\alpha_j}{3H_n} - d(j, i) \right)^+ \leq \frac{f_i}{2} \]  
for each \( i \in F \)

Techniques in [NW, 2013]

For each \( i \in F \),

\[ f(i) \geq k \cdot (\alpha[k] - d(j[k], i)) - 2 \sum_{j \in D^c_{[k-1]}} d(j, i) \]

\[ \Rightarrow \frac{f(i)}{k} \geq (\alpha[k] - d(j[k], i)) - 2 \frac{\sum_{j \in D^c_{[k-1]}} d(j, i)}{k} \]

\[ \Rightarrow H_{|D^c|} \cdot f(i) \geq \sum_{k=1}^{n} (\alpha[k] - d(j[k], i)) - 2 \sum_{k=1}^{n} \frac{1}{k} \cdot \sum_{j \in D^c_{[k-1]}} d(j, i) \]

\[ \Rightarrow H_{|D^c|} \cdot f(i) \geq \sum_{k=1}^{n} (\alpha[k] - d(j[k], i)) - (2H_{|D^c|} - 1) \cdot \sum_{j \in D^c} d(j, i) \]

\[ \Rightarrow H_{|D^c|} \cdot f(i) \geq \sum_{j \in D^c} (\alpha[k] - 2H_{|D^c|} \cdot d(j[k], i)) \]

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Analysis: (ii) \( \sum_{j \in D^p} \left( \frac{\alpha_j}{3H_n} - d(j, i) \right)^+ \leq \frac{f_i}{2} \) for each \( i \in F \)

Same trick for \( D^p \)?

For each \( i \in F \),

\[
f(i) \geq \left( \alpha_{[k]} - d(j_{[k]}, i) \right) + \left( \sum_{j \in D^p_{[k-1]}} \min\{d(j, F^a_{[k]}), p(j)\} - d(j, i) \right)^+ = \alpha_j
\]

\( (j \text{ not connected } \not\Rightarrow d(j, F^a_{[k]}) \text{ is smaller}) \)

\[
\bar{D}^p_{[k-1]} := \{ j \in D^p_{[k-1]} : \alpha_j \geq d(j, F^a_{[k]}) \}
\]

\[
\geq \left( \alpha_{[k]} - d(j_{[k]}, i) \right) + \sum_{j \in \bar{D}^p_{[k-1]}} \left( d(j, F^a_{[k]}) - d(j, i) \right)^+
\]

(triangle inequality)

\[
\geq \left( \alpha_{[k]} - d(j_{[k]}, i) \right) + \sum_{j \in \bar{D}^p_{[k-1]}} \left( \alpha_{[k]} - d(j_{[k]}, i) - 2d(j, i) \right)
\]

\( \Rightarrow f(i) \geq (1 + |\bar{D}^p_{[k-1]}|) \cdot (\alpha_{[k]} - d(j_{[k]}, i)) - 2 \sum_{j \in D^c_{[k-1]}} d(j, i)
\]

\( \text{could } < (k-1) \)
Analysis: (ii) \( \sum_{j \in D^p} \left( \frac{\alpha_j}{3H_n} - d(j, i) \right)^+ \leq \frac{f_i}{2} \) for each \( i \in F \)

How to fix?

Goal: harmonic series as coefficients, e.g. \( f(i) \geq k \cdot (\alpha_k - d(j_k, i)) \) (\( \ast \)) for some ordering \( \{j_k\} \) over \( D^p \).

Observe: For larger \( \alpha_k - d(j_k, i) \) value, smaller coefficient \( k \) should be assigned.

New argument: For each \( i \in F \), order \( \alpha_k \in D^p \) in nonincreasing order of \( \alpha_k - d(j_k, i) \) and show that inequality (\( \ast \)) holds.

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Analysis: (ii) $\sum_{j \in D^p} \left( \frac{\alpha_j}{3H_n} - d(j, i) \right)^+ \leq \frac{f_i}{2}$ for each $i \in F$

How to fix?

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$$f(i) \geq k \cdot (\alpha_k - d(j_k, i))$$

for some ordering $\{j_k\}$ over $D^p$. 
Analysis: (ii) \( \sum_{j \in D^p} \left( \frac{\alpha_j}{3H_n} - d(j, i) \right)^+ \leq \frac{f_i}{2} \) for each \( i \in F \)

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f(i) \geq k \cdot (\alpha_k - d(j_k, i)) \quad (\star)
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for some ordering \( \{j_k\} \) over \( D^p \). Observe: For larger \( \alpha_k - d(j_k, i) \) value, smaller coefficient \( k \) should be assigned.
The Online Prize-Collecting Facility Location Problem

Analysis: (ii) \( \sum_{j \in D^p} \left( \frac{\alpha_j}{3H_n} - d(j, i) \right)^+ \leq \frac{f_i}{2} \) for each \( i \in F \)

How to fix?

Goal: harmonic series as coefficients, e.g.

\[ f(i) \geq k \cdot (\alpha_k - d(j_k, i)) \quad (\ast) \]

for some ordering \( \{j_k\} \) over \( D^p \). Observe: For larger \( \alpha_k - d(j_k, i) \) value, smaller coefficient \( k \) should be assigned.

New argument: For each \( i \in F \), order \( \alpha_k \in D^p \) in nonincreasing order of \( \alpha_k - d(j_k, i) \) and show that inequality \((\ast)\) holds.
Analysis: (ii) \( \sum_{j \in D^p} \left( \frac{\alpha_j}{3H_n} - d(j, i) \right)^+ \leq \frac{f_i}{2} \) for each \( i \in F \)

How to fix?

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for some ordering \( \{j_k\} \) over \( D^p \).  

Observe: For larger \( \alpha_k - d(j_k, i) \) value, smaller coefficient \( k \) should be assigned.

New argument: For each \( i \in F \), order \( \alpha_k \in D^p \) in nonincreasing order of \( \alpha_k - d(j_k, i) \) and show that inequality (\( * \)) holds.
Analysis

Towards $O(\log n)$ competitive ratio

1. Resulting assignment is feasible. ✓
2. Total cost of the assignment is bounded by $2 \cdot \sum_j \alpha_j$. ✓
3. $\left\{ \frac{\alpha_j}{3H_n} \right\}_j$ is feasible to the dual problem, so $\sum_j \frac{\alpha_j}{3H_n} \leq \text{OPT}$. ✓
The Online Prize-Collecting Facility Location Problem

Analysis

Towards $O(\log n)$ competitive ratio

1. Resulting assignment is feasible. ✓
2. Total cost of the assignment is bounded by $2 \cdot \sum_j \alpha_j$. ✓
3. \( \left\{ \frac{\alpha_j}{3H_n} \right\}_j \) is feasible to the dual problem, so \( \sum_j \frac{\alpha_j}{3H_n} \leq \text{OPT} \). ✓

Conclusion

Our algorithm has $(6 \log n)$-competitive ratio.
Future Research:
Future Research:

Other variants of Facility Location, like:

- Online Robust Facility Location,
- Online Multicommodity Facility Location,
- Online Prize-Collecting Facility Leasing.
Acknowledgements

Thank you!

Questions?
References

A. Meyerson.
*Online Facility Location.*

D. Fotakis.
*On the Competitive Ratio for Online Facility Location.*
References (cont.)

D. Fotakis.  
*A Primal-Dual Algorithm for Online Non-Uniform Facility Location.*  

C. Nagarajan and D.P. Williamson.  
*Offline and Online Facility Leasing.*  