The cover time of the preferential attachment graph

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Abstract

The preferential attachment graph $G_m(n)$ is a random graph formed by adding a new vertex at each time step, with m edges which point to vertices selected at random with probability proportional to their degree. Thus at time n there are n vertices and mn edges. This process yields a graph which has been proposed as a simple model of the world wide web [2]. In this paper we show that **whp** the cover time of a simple random walk on $G_m(n)$ is asymptotic to $\frac{2m}{m-1} n \ln n$.

1 Introduction

Let $G = (V, E)$ be a connected graph, let $|V| = n$, and $|E| = m$. A *random walk* $\mathcal{W}_u$, $u \in V$ on the undirected graph $G = (V, E)$ is a Markov chain $X_0 = u, X_1, \ldots, X_t, \ldots \in V$ associated to a particle that moves from vertex to vertex according to the following rule: the probability of a transition from vertex i, of degree $d(i)$, to vertex j is $1/d(i)$ if $\{i, j\} \in E$, and 0 otherwise. For $u \in V$ let C_u be the expected time taken for $\mathcal{W}_u$ to visit every vertex of G. The *cover time* C_G of G is defined as $C_G = \max_{u \in V} C_u$. The cover time of connected graphs has been extensively studied. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [1] that $C_G \leq 2m(n-1)$. It was shown by Feige [7], [8], that for any connected graph G

$$(1 - o(1)) n \log n \leq C_G \leq (1 + o(1)) \frac{4}{27} n^3.$$

The lower bound is achieved by (for example) the complete graph K_n, whose cover time is determined by the Coupon Collector problem.

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In a previous paper [5] we studied the cover time of random graphs $G_{n,p}$ when $np = c \log n$ where $c = O(1)$ and $(c-1)\log n \to \infty$. This extended a result of Jonasson, who proved in [11] that when the expected average degree $(n-1)p$ grows faster than $\log n$, **whp** a random graph has the same cover time (asymptoticaly) as the complete graph K_n, whereas, when $np = \Omega(\log n)$ this is not the case.

Theorem 1. *[5] Suppose that $np = c\log n = \log n + \omega$ where $\omega = (c-1)\log n \to \infty$ and $c \geq 1$. If $G \in G_{n,p}$, then* **whp**[1]

$$C_G \sim c \log \left(\frac{c}{c-1}\right) n \log n.$$

The notation $A_n \sim B_n$ means that $\lim_{n\to\infty} A_n/B_n = 1$.

In another paper [6] we used a different technique to study the cover time of random regular graphs. We proved the following:

Theorem 2. *Let $r \geq 3$ be constant. Let $\mathcal{G}_r$ denote the set of r-regular graphs with vertex set $V = \{1, 2, \ldots, n\}$. If G is chosen randomly from $\mathcal{G}_r$, then* **whp**

$$C_G \sim \frac{r-1}{r-2}\, n \log n.$$

In this paper we turn our attention to the preferential attachment graph $G_m(n)$ introduced by Barabási and Albert [2] as a simplified model of the WWW. The preferential attachment graph $G_m(n)$ is a random graph formed by adding a new vertex at each time step, with m edges which point to vertices selected at random with probability proportional to their degree. Thus at time n there are n vertices and mn edges. We use the generative model of [3] (see also [4]) and build a graph sequentially as follows:

- At each time step t, we add a vertex v_t, and we add an edge from v_t to some other vertex u, where u is chosen at random according to the distribution:

$$\mathbf{Pr}(u = v_i) = \begin{cases} \frac{d_t(v_i)}{2t-1}, & \text{if } v_i \neq v_t; \\ \frac{1}{2t-1}, & \text{if } v_i = v_t; \end{cases}$$

  where $d_t(v)$ denotes the degree of vertex v at time t.

- For some constant m, every m steps we contract the most recently added m vertices to form a single vertex.

Let $G_m(n)$ denote the random graph at time step mn after n contractions of size m. Thus $G_m(n)$ has n vertices and mn edges and may be a multi-graph.

[1] A sequence of events $\mathcal{E}_n$ occurs *with high probability* **whp** if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.

This is a very nice clean model, but we warn the reader that it allows loops and multiple edges, although **whp** there will be very few of them.

We write $d(j)$ in place of $d_n(j)$.

We prove

Theorem 3. *If $m \geq 3$ then* **whp** *the preferential attachment graph $G = G_m(n)$ satisfies*

$$C_G \sim \frac{2m}{m-1}\, n \log n.$$

Conjecture: The theorem holds for $m = 2$ as well.

2 The first visit time lemma.

2.1 Convergence of the random walk

In this section G denotes a fixed connected graph with n vertices. u is some arbitrary vertex from which a walk $\mathcal{W}_u$ is started. Let $\mathcal{W}_u(t)$ be the vertex reached at step t, let P be the matrix of transition probabilities of the walk and let $P_u^{(t)}(v) = \mathbf{Pr}(\mathcal{W}_u(t) = v)$. Let π be the steady state distribution of the random walk $\mathcal{W}_u$. Let $\pi_v = \pi(v)$ denote the stationary distribution of the vertex v. Let $\lambda_{\max}$ be the second largest eigenvalue of P. Then,

$$|P_u^{(t)}(x) - \pi_x| \leq (\pi_x/\pi_u)^{1/2} \lambda_{\max}^t. \tag{1}$$

See for example, Jerrum and Sinclair [10].

2.2 Generating function formulation

Fix two vertices u, v. Let h_t be the probability $\mathbf{Pr}(\mathcal{W}_u(t) = v) = P_u^{(t)}(v)$, that the walk $\mathcal{W}_u$ visits v at step t. Let $H(s)$ generate h_t.

Similarly, considering the walk $\mathcal{W}_v$, starting at v, let r_t be the probability that this walk returns to v at step $t = 0, 1, \ldots$. Let $R(s)$ generate r_t. We note that $r_0 = 1$.

Let $f_t(u{\to}v)$ be the probability that the first visit of the walk $\mathcal{W}_u$ to v occurs at step t. If $u \neq v$ then $f_0(u{\to}v) = 0$. Let $F(s)$ generate $f_t(u{\to}v)$. Thus

$$H(s) = F(s)R(s). \tag{2}$$

Let

$$T = 4 \log_{\lambda_{\max}^{-1}} n \tag{3}$$

and note that

$$\max_{x\in V}|P_u^{(t)}(x)-\pi_x| \le n^{-3} \qquad \text{for } t\ge T. \tag{4}$$

For $R(s)$ let

$$R_T(s)=\sum_{j=0}^{T-1} r_j s^j. \tag{5}$$

Thus $R_T(s)$ generates the probability of a return to v during steps $0,...,T-1$ of a walk starting at v. Similarly for $H(s)$, let

$$H_T(s)=\sum_{j=0}^{T-1} h_j s^j. \tag{6}$$

2.3 First visit time: Single vertex v

The following lemma was proved in [6]. Here ϵ_1,ϵ_2 are constants in $(0,1)$.

Lemma 4. *Let T be as defined in (3). Suppose that*

(a) $H_T(1)<(1-\epsilon_1)R_T(1)$.

(b) $\max_{|s|=1}\frac{|R_T(s)-R_T(1)|}{R_T(1)}\le 1-\epsilon_2$.

(c) $T\pi_v=o(1)$, $T\pi_v=\Omega(n^{-2})$.

Let

$$\lambda=\frac{1}{K_0T}. \tag{7}$$

for some sufficently large constant K_0.

Let

$$p_v = \frac{\pi_v}{R_T(1)(1+O(T\pi_v))}, \tag{8}$$

$$c_{u,v} = 1-\frac{H_T(1)}{R_T(1)(1+O(T\pi_v))}. \tag{9}$$

Then

$$f_t(u\to v)=c_{u,v}\frac{p_v}{(1+p_v)^{t+1}}+O(e^{-\lambda t/2}) \qquad \textit{for all } t\ge T. \tag{10}$$

Corollary 5. *Let $\boldsymbol{A}_t(v)$ be the event that $\mathcal{W}_u$ has not visited v by step t. Then for $t\ge T$,*

$$\mathbf{Pr}(\boldsymbol{A}_t(v))=\frac{c_{u,v}}{(1+p_v)^t}+O(\lambda e^{-\lambda t/2}).$$

As we leave this section we introduce the notation R_v,H_v to replace $R_T(1),H_T(1)$ (which are not attached to v).

3 The random graph $G_m(n)$

In this section we prove some properties of $G_m(n)$. Let

$$\omega = \ln \ln \ln n.$$

We first derive crude bounds on degrees.

Lemma 6.
(a)

$$\mathbf{Pr}(\exists i \leq n^{1/10} : \; d(i) \leq n^{1/4}) = o(1).$$

(b)

$$\mathbf{Pr}(\exists s, t : \; d_t(s) \geq (t/s)^{1/2} (\ln n)^3) = O(n^{-3}).$$

Proof For this we use a model devised in [3]. Let $Y = X^{1/2}$ where X is uniform $[0,1]$. Let $y_1 < y_2 < \cdots < y_{mn}$ be the order statistics of mn independent copies of Y. Let $y_0 = 0$, $y_{mn+1} = 1$. $I_j = [y_{(j-1)m}, y_{jm})$ for $j = 1, 2, \ldots, n$. Then for $k = 1, 2, \ldots, mn$ choose x_k uniformly from $[0, y_k)$. If $x_k \in I_j$ then add the edge $(j, \lceil k/m \rceil)$ to G. The resulting graph G at the end of this process has precisely the same distribution as $G_m(n)$.

The random variables y_i can be generated via

$$y_i^2 = \frac{A_1 + A_2 + \cdots + A_i}{A_1 + A_2 + \cdots + A_{mn+1}}$$

where $A_1, A_2, \ldots, A_{mn+1}$ are independent exponentially distributed with mean 1.

It is not difficult to prove that $S = A_1 + A_2 + \cdots + A_{mn+1} \in J = [mn - n^{2/3}, mn + n^{2/3}]$ **qs**. Next let $\mathcal{E}$ be the event that $S \notin J$ or that there exists $i \leq mn^{1/10}$ such that $A_i \leq n^{-1/9}$ or $A_i \geq \log n$. Then

$$\mathbf{Pr}(\mathcal{E}) \leq o(1) + mn^{1/10}((1 - e^{-n^{-1/9}}) + n^{-1}) = o(1).$$

If $\mathcal{E}$ does not occur then $i \leq mn^{1/10}$ implies that

$$\begin{aligned} y_i - y_{i-1} &= S^{-1/2}(A_1 + \cdots + A_i)^{1/2} \left(1 - \left(1 - \frac{A_i}{A_1 + \cdots + A_i}\right)^{1/2}\right) \\ &\geq S^{-1/2} \frac{A_i}{2(A_1 + \cdots + A_i)^{1/2}} \\ &\geq \frac{1}{n^{11/18} i^{1/2} \log n}. \end{aligned} \tag{11}$$

Now put $W_i = y_{im} - y_{(i-1)m}$ and condition of the values of W_i for $i = 1, 2, \ldots, n$. Then for $j \leq n^{1/10}$ we find that $d(j)$ dominates $Bin(mn/2, W_j)$ in distribution. This is becaue for

$k > mn/2$, the probability that $x_k \in I_j$ is at least W_j. Part (a) now follows from (11) and the Chernoff bounds.

(b) Fix $s \le t$ and let $X_\tau = d(s,\tau)$ for $\tau = s, s+1, \ldots, t$ and let $\lambda = \frac{(s/t)^{1/2}}{10 \ln n}$. Now conditional on $X_\tau = x$, we have

$$X_{\tau+1} = X_\tau + B\left(m, \frac{x}{2m\tau}\right). \tag{12}$$

Thus

$$\begin{aligned} \mathbf{E}\left(e^{\lambda X_{\tau+1}} \mid X_\tau = x\right) &= e^{\lambda x}\left(1 - \frac{x}{2m\tau} + \frac{x}{2m\tau}e^{\lambda}\right)^m \\ &\le \exp\left\{\lambda x - \frac{x}{2\tau} + \frac{x}{2\tau}(1+\lambda+\lambda^2)\right\} \\ &= \exp\left\{\lambda x\left(1 + \frac{1+\lambda}{2\tau}\right)\right\}. \end{aligned}$$

Thus

$$\mathbf{E}\left(e^{\lambda X_{\tau+1}}\right) \le \mathbf{E}\left(\exp\left\{X_\tau \lambda\left(1 + \frac{1+\lambda}{2\tau}\right)\right\}\right).$$

If we put $\lambda_t = \lambda$ and $\lambda_{\tau-1} = \lambda_\tau\left(1 + \frac{1+\lambda_\tau}{2\tau}\right)$ then provided $\lambda_s \le 1$ we will have

$$\mathbf{E}\left(e^{\lambda X_t}\right) \le e^{m\lambda_s}.$$

Now provided $\lambda_\tau \le \Lambda = \frac{1}{\ln n}$ we can write

$$\lambda_{\tau-1} \le \lambda_\tau\left(1 + \frac{1+\Lambda}{2\tau}\right)$$

and then

$$\begin{aligned} \lambda_s &\le \lambda \prod_{\tau=s}^{t}\left(1 + \frac{1+\Lambda}{2\tau}\right) \\ &\le 10\lambda(t/s)^{1/2} \end{aligned}$$

which is $\le \Lambda$ by the definition of λ.

Putting $u = (t/s)^{1/2}(\ln n)^3$ we get

$$\begin{aligned} \mathbf{Pr}(X_t \ge u) &\le e^{m\lambda_s - \lambda u} \\ &\le \exp\{\lambda(10m(t/s)^{1/2} - u)\} \\ &= O(n^{-5}) \end{aligned}$$

and part (b) follows. □

Let a cycle C be small if $|C| \le 2\omega + 1$. Let a vertex v be *locally tree-like* if the sub-graph H_v induced by the vertices at distance 2ω or less is a tree. Thus a locally tree-like vertex is at distance at least 2ω from any small cycle. Suppose that v is locally tree-like. We say that v is *locally regular* if H_v is a tree of depth 2ω, rooted at v, in which every non-leaf has branching factor m.

For $j \in [n]$ we let $X(j)$ denote the set of neighbours of j in $[j-1]$ i.e. the vertices "chosen" by j (although not including j, loops are allowed in this construction).

Lemma 7. Whp, *$G_m(n)$ contains at least $n^{1-o(1)}$ locally regular vertices.*

Proof Let $I_k = \left[n\left(1-\frac{1}{2^k}\right), n\left(1-\frac{1}{2^{k+1}}\right)\right)$ for $1 \le k \le \omega$. Let

$$J_2 = \{j \in I_2 : \; X(j) \subseteq I_1, \, |X(j)| = m, \, X(j') \cap X(j) = \emptyset, \text{ for } j \neq j'\}.$$

($|X(j) = m|$ so that there are no parallel edges emanating from j.)

Then for $2 < k \le \omega$ we let

$$J_k = \{j \in I_k : \; X(j) \subseteq J_{k-1}, \, |X(j)| = m, \, X(j') \cap X(j) = \emptyset, \text{ for } j \neq j'\}.$$

By construction, J_ω consists of locally regular vertices and so it remains to bound $|J_\omega|$.

For $j \in I_2$, $i_{m+1} = j-1$,

$$\begin{aligned}
&\mathbf{Pr}(j \in J_2) = \\
&\sum_{\{i_1<\cdots<i_m\}\subseteq I_1} \prod_{k=1}^{m} \prod_{\tau=mi_k+1}^{mi_{k+1}} \left(1-\frac{km}{2\tau-1}\right) \cdot m! \prod_{i=1}^{m} \frac{m}{2mj+2i-1} \cdot \prod_{\tau=mj+1}^{mn} \left(1-\frac{m^2}{2\tau-1}\right) \\
&\sim \sum_{\{i_1<\cdots<i_m\}\subseteq I_1} \left(\prod_{k=1}^{m} \frac{i_k}{j}\right)^{m/2} \cdot \frac{m!}{(2j)^m} \cdot \frac{j^{m^2}}{n^{m^2}} \\
&\sim \left(\frac{j^{m-3/2}}{2n^m}\right)^m m! \sum_{\{i_1<\cdots<i_m\}\subseteq I_1} \prod_{k=1}^{m} i_k^{1/2} \\
&\sim \left(\frac{j^{m-3/2}}{2n^m}\right)^m \left(\sum_{i\in I_1} i^{1/2}\right)^m \\
&\sim \frac{j^{m^2}}{n^{m^2}} \left(\frac{\sqrt{3}-\sqrt{2}}{3}\right)^m .
\end{aligned}$$

So we can write, for some A_m, B_m which depend only on m,

$$\mathbf{E}\left(|J_2|\right) \sim \frac{A_m}{n^{m^2}} \sum_{j=3n/4+1}^{7n/8} j^{m^2} \sim B_m n.$$

We argue next that $|J_2|$ is concentrated around its mean. Let $Y_1, Y_2, \ldots, Y_{mn}$ denote the sequence of single choices of edges added. Here when vertex i is choosing one of its m neighbours, we consider each edge $\{u, v\}$ of $G_m(i-1)$ as being 2 directed arcs (u, v) and (v, u).

When choosing a neighbour, i chooses (x, y) randomly from the $2m(i-1)$ arcs available and adds the edge $\{i, y\}$. In this way, each vertex is chosen proportioanl to its degree in $G_m(i-1)$. We let

$$Z_i = \mathbf{E}\left(|J_2| \mid Y_1, Y_2, \ldots, Y_t, \mathcal{A}\right) - \mathbf{E}\left(|J_2| \mid Y_1, Y_2, \ldots, Y_{t-1}, \mathcal{A}\right)$$

and prove that

$$|Z_i| \leq 4. \tag{13}$$

The Azuma-Hoeffding martingale inequality then implies that

$$\mathbf{Pr}(||J_2| - \mathbf{E}\left(|J_2|\right)| \geq u \mid \mathcal{A}) \leq \exp\left\{-\frac{u^2}{8mn}\right\}. \tag{14}$$

It follows that **qs**[2]

$$||J_2| - \mathbf{E}\left(|J_2|\right)| \leq n^{1/2} \ln n. \tag{15}$$

Fix $Y_1, Y_2, \ldots, Y_i$ and let $Y_i = (x, v)$, $\hat{Y}_i = (\hat{x}, \hat{v})$. Of course $\hat{x} = x$ if m does not divide i. Then for each complete outcome $\mathbf{Y} = Y_1, Y_2, \ldots, Y_T$ we define a corresponding outcome $\hat{\mathbf{Y}} = Y_1, Y_2, \ldots, Y_{i-1}, \hat{Y}_i, \ldots, \hat{Y}_T$ where for $j > i$, $\hat{Y}_j$ is obtained from Y_j as follows: If Y_j creates a new edge (w, v) by randomly choosing edge (x, v) arising from Y_i, then in $\hat{Y}_j$, (w, v) is replaced by $(w, \hat{v})$, otherwise $\hat{Y}_j = Y_j$.

The map $\mathbf{Y} \to \hat{\mathbf{Y}}$ is measure preserving and in going from $\mathbf{Y}$ to $\hat{\mathbf{Y}}$ $|J_2|$ changes by at most 4. (13) follows.

Repeating the argument for $\mathbf{Pr}(j \in J_2)$ we see that for $j \in I_3$ and some $B_m < 1$,

$$\mathbf{Pr}(j \in J_3 \mid J_2) \sim \left(\frac{4j^{m-1}}{7n^{m+1}}\right)^m \left(\sum_{i \in J_2} i\right)^m \geq B_m \frac{|J_2|^{2m+1}}{n^{2m}} \sim B_m A_m^{2m+1} n$$

and given J_2 of size $\sim A_m n$, $|J_3|$ will be concentrated around its mean.

Proceeding in this way we find that for $2 \leq k < \omega$ we have **qs**

$$|J_{k+1}| \geq C_k \frac{|J_k|^{2m+1}}{n^{2m}}$$

where we can choose $C_k < 1$, depending only on m.

From this we get

$$|J_\omega| \geq n \prod_{k=0}^{\omega-1} C_{\omega-k}^{(2m+1)^k} \cdot A_m^{(2m+1)^{\omega-1}} = n^{1-o(1)}.$$

□

A small cycle is *light* if it contains no vertex $v \leq n^{1/10}$ (it has no "heavy" vertices).

[2] A sequence of events $\mathcal{E}_n$ occurs *quite surely* (**qs**) if $\mathbf{Pr}(\mathcal{E}_n) = 1 - O(n^{-K})$ for any constant $K > 0$.

Lemma 8. Whp $G_m(n)$ *does not contain a small cycle within* 2ω *of a light cycle.*

Proof First consider the number pairs Z_1 of disjoint cycles C_1, C_2 joined by a short path P. Here C_1 is light and C_2 is small. Then (with $a_{r+1} = a_1$ and $b_{s+1} = b_1$)

$$\mathbf{E}(Z_1) \leq o(1) + \sum_{\substack{3 \leq r,s \leq 2\omega+1 \\ 0 \leq t \leq 2\omega \\ 1 \leq i \leq r, 1 \leq j \leq s}} \sum_{\substack{a_1,\ldots,a_r \geq n^{1/10} \\ b_1\ldots,b_s \\ c_1,\ldots,c_t}} \frac{(\log n)^3}{(a_i b_j)^{1/2}} \prod_{k=1}^{r} \frac{(\log n)^3}{a_k a_{k+1}} \prod_{l=1}^{s} \frac{(\log n)^3}{b_l b_{l+1}} \prod_{\mu=1}^{t-1} \frac{(\log n)^3}{c_\mu c_{\mu+1}} \tag{16}$$

Explanation: $a_1, \ldots, a_r$, $b_1, \ldots, b_s$ are the vertices of C_1, C_2 respectively and $c_1, \ldots, c_t$ are the internal vertices of the path P which joins a_i to b_j. Next suppose that $1 \leq \alpha < \beta \leq n$. Then $\mathbf{Pr}(G_m(n)$ contains edge $(\alpha, \beta))$ is at most $\frac{(\log n)^3}{(\alpha\beta)^{1/2}}$. This is because when β chooses its neighbours, the probability it chooses α is at most $\frac{m(\log n)^3(\beta/\alpha)^{1/2}}{2m(\beta-1)}$. Here the numerator is a bound on the degree of α in $G_m(\beta-1)$. We are using Lemma 6 here and the $o(1)$ term accounts for the failure of this bound. Furthermore, this remains an upper bound if we condition on the existence of some of the other edges of C_1, C_2, P.

Thus, if H_k is the kth harmonic number $1 + \frac{1}{2} + \cdots + \frac{1}{k}$,

$$\begin{aligned} \mathbf{E}(Z_1) &\leq o(1) + \sum_{\substack{3 \leq r,s \leq 2\omega+1 \\ 0 \leq t \leq 2\omega \\ 1 \leq i \leq r, 1 \leq j \leq s}} \frac{(\log n)^{3(1+r+s+t)}}{n^{1/20}} H_n^{r+s+t} \\ &= O((\log n)^{16\omega+3} \omega^2 n^{-1/20}) \\ &= o(1). \end{aligned}$$

A similar argument deals with the case of a light cycle with a short path joining two of its vertices. □

We need to deal with the possibility that $G_m(n)$ contains many cycles.

Lemma 9. Whp $G_m(n)$ *contains at most* $(\log n)^{5\omega}$ *small cycles.*

Proof Let Z be the number of small cycles in $G_m(n)$ (including parallel edges). Then

$$\mathbf{E}(Z) \leq \sum_{k=2}^{2\omega+1} \sum_{a_1,\ldots,a_k} \prod_{i=1}^{k} \frac{(\log n)^3}{a_i a_{i+1}} = O((\log n)^{4\omega})$$

and the result follows from the Markov inequality. □

We estimate the number of non tree-like vertices.

Lemma 10. Whp *there are at most* $O(n^{1/2+o(1)})$ *non tree-like vertices.*

Proof A non tree-like vertex is within ω of a small cycle. So the expectation of the number Z of such vertices satisfies

$$\begin{aligned} \mathbf{E}(Z) &\leq o(1) + \sum_{\substack{0\leq r\leq \omega \\ 3\leq s\leq 2\omega+1 \\ 1\leq i\leq s}} \sum_{\substack{a_0,\dots,a_r \\ b_1\dots,b_s}} \frac{(\log n)^3}{(a_0 b_i)^{1/2}} \prod_{k=1}^{r-1} \frac{(\log n)^3}{a_k a_{k+1}} \prod_{l=1}^{s} \frac{(\log n)^3}{b_l b_{l+1}} \\ &= O(n^{1/2+o(1)}). \end{aligned}$$

The result follows from the Markov inequality. □

Finally, we state the following consequence of a result in Mihail, Papadimitriou and Saberi [12]:

Lemma 11. *For $m \geq 2$,* **whp***,*

$$\lambda_{\max} \leq 1 - \zeta$$

where ζ is a positive constant.

4 Cover time of $G_m(n)$

4.1 Parameters

Asume now that $G_m(n)$ (i) has $n^{1-o(1)}$ locally regular vertices, (ii) $d(s) \geq n^{1/4}$ for $s \leq n^{1/10}$, (iii) no small cycle close to a light cycle, (iv) $O((\log n)^{5\omega})$ small cycles and (v) $O(n^{1/2+o(1)})$ non tree-like vertices.

Cosider first a locally regular vertex v. It was shown in [6] (Lemma 6) that $R_v = \frac{r-1}{r-2} + o(\omega^{-1})$ for a locally tree-like vertex w of an r-regular graph. We obtain the same result for v by puting $m = r+1$. Note that the degree of v is irrelevant here. It is the branching factor of the rest of the tree H_v that matters.

Lemma 12. *Suppose that $m \geq 2$. Suppose that v is locally tree-like.*

(a) *If v is locally regular then $R_v = \frac{m}{m-1} + o(\omega^{-1})$.*

(b) *In general, $R_v \leq \frac{m^2}{m^2-m-1} + o(\omega^{-1})$.*

(c) *If $d(v) = m$ and v is not locally regular then $R_v < \frac{m}{m-1} + o(\omega^{-1})$.*

Proof (a) We project the first ω steps of $\mathcal{W}_v$ onto a random walk $\mathcal{X}$ on $\{0, 1, 2, \dots, \}$. Here 0 replaces v. The probability of going right at a point $\ell \neq 0$ is $p = \frac{m}{m+1}$. Let E_i be the expected number of visits to 0 for such a walk starting at i and continuing indefinitely. Then

$$E_0 = 1 + E_1 = 1 + E_0(1-p)/p.$$

This is because E_1 is E_0 times the expected number of visits to 0 between right moves from 1. Solving gives

$$\sum_{i=0}^{\infty} \rho_i = E_0 = \frac{m}{m-1}. \tag{17}$$

Note that $\rho_{2i+1} = 0, \rho_{2i} \leq \binom{2i}{i}(p(1-p))^i$ and so

$$\sum_{i=\omega+1}^{\infty} \rho_i \leq \sum_{j=\omega/2} \binom{2j}{j} \left(\frac{m}{(m+1)^2}\right)^j = o(\omega^{-1}). \tag{18}$$

We compare this with R_v. First observe that $r_i = \rho_i$ for $i \leq \omega$. Then from (1) we see that

$$\sum_{i=\omega+1}^{T} r_i \leq \sum_{i=\omega+1}^{T} (\pi_v + \lambda_{\max}^i) = o(\omega^{-1}).$$

Part (a) now follows from (17) and (18).

(b) If every non-leaf vertex other than v of H_v has degree at least $m+1$ then the projection $\mathcal{X}$ on $\{0, 1, 2, \ldots, \}$ is such that probability of going right at a point ℓ depends on the degree of the vertex w where $\mathcal{W}_v$ finds itself but is at least $\frac{m}{m+1}$ and so the expected number of returns to 0 by $\mathcal{X}$ will be at most $\frac{m}{m-1}$. The rest of the argument in (a) is unchanged.

Unfortunately, the situation is a little more complicated due to H_v posibly having vertices of degree m, in which case the probability of moving right is only $\frac{m-1}{m}$. However, if this is the probability of moving right at ℓ and $\ell \neq 1$ then after moving to $\ell - 1$ the probability of going right will be at least $\frac{m+1}{m+2}$ and after moving to $\ell + 1$ the probability of going right will be at least $\frac{m}{m+1}$. This is because the move right probability is $\frac{m-1}{m}$ only if the degree of w is exactly m and then $w > w'$ where w' is the ancestor of w in H_v. This forces w' to have degree at least $m+2$ and the successors of w to have degree at least $m+1$. To bound R_v we consider 2 cases. The first will be used when $d(v) \geq m+1$. Then to maximise the number of posible go right probabilities of $\frac{m-1}{m}$ we sandwich them between go right probabilities of $\frac{m}{m+1}$, except for the first. This will prove (b). To prove (c) we sandwich a go right probability of $\frac{m-1}{m}$ between a go right probability of $\frac{m+1}{m+2}$ and a go right probability of $\frac{m}{m+1}$.

We first analyse a *non-uniform* random walk on $\{0, 1, 2, \ldots, \}$ wth reflecting barrier at 0 which we will use when $d(v) \geq m+1$. At $2k+1$ the probability of moving right is $p_{2k+1} = \frac{m-1}{m}$ and at $2k$ the probability of moving right is $p_{2k} = \frac{m}{m+1}$. Suppose now that E_i denotes the expected number of visits to 0 for such a random walk, started at i. Then

$$\begin{aligned} E_0 &= 1 + E_1 \\ E_1 &= \frac{1}{m}E_0 + \frac{m-1}{m}E_2 \\ E_2 &= E_0 Z \end{aligned}$$

where Z is the expected number of visits to 0 between right moves from 2.

Now consider starting the random walk at i and going left immediately. Define Z_i to the expected number of visits to 0 before the next return to i. Then $Z = Z_2/m$. The factor $1/m = (1-p_2)/p_2$ is the expected number of left moves at 2 between going right. Then $Z_1 = 1$ and $Z_2 = Z_1(1-p_1)/p_1 = 1/(m-1)$. Thus $Z = \frac{1}{m(m-1)}$ giving

$$E_0 = 1 + \frac{1}{m}E_0 + \frac{1}{m^2}E_0$$

implying that

$$E_0 = \frac{m^2}{m^2 - m - 1}.$$

We now analyse a *non-uniform* random walk on $\{0, 1, 2, \dots,\}$ wth reflecting barrier at 0 which we will use when $d(v) = m$. At $3k+1$ the probability of moving right is $p_{3k+1} = \frac{m+1}{m+2}$, at $3k+2$ the probability of moving right is $p_{3k+2} = \frac{m-1}{m}$ and at $3k$ the probability of moving right is $p_{3k} = \frac{m}{m+1}$. Suppose now that E_i denotes the expected number of visits to 0 for such a random walk, started at i. Then

$$\begin{aligned}
E_0 &= 1 + E_1 \\
E_1 &= \frac{1}{m+2}E_0 + \frac{m+1}{m+2}E_2 \\
E_2 &= \frac{1}{m}E_1 + \frac{m-1}{m}E_3 \\
E_3 &= E_0 Z_3(1-p_3)/p_3
\end{aligned}$$

Here $Z_1 = 1$, $Z_2 = (1-p_1)/p_1 = 1/(m+1)$ and $Z_3 = Z_2(1-p_2)/p_2$ which implies

$$E_0 = \frac{m(m^2+m-1)}{m^3-m-1} < \frac{m}{m-1}.$$

□

We deal with non-tree like vertices in a somewhat piece-meal fashion.

Lemma 13. *Suppose that $m \geq 2$.*

(a) *Suppose that G_v contains a unique light cycle C_v. Assume that $v \notin C_v$ and that the shortest path $P = (w_0 = v, w_1, \dots, w_k)$ from v to C_v is such that* $\max\{d(w_1), \dots, d(w_k)\}$ $\geq \omega^3$*. Then*

$$(i)\ R_v \leq \frac{m}{m-1} + o(\omega^{-1}) \text{ or } (ii)\ d(v) \geq m+1 \text{ and } R_v \leq \frac{m^2}{m^2-m-1} + o(\omega^{-1}). \quad (19)$$

(b) *Suppose that H_v contains only heavy cycles. Then* (19) *holds in this case too.*

Proof
(a) Let H_v' be obtained from H_v by deleting those vertices, other than w, whose only path to v in H_v goes through w. Let R_v' be the expected number of returns to v in a random walk of length ω on H_v' where w is an absorbing state. We claim that

$$R_v \leq R_v' + o(\omega^{-1}). \tag{20}$$

Once we verify this, the proof of (a) follows from (the proof of) Lemma 12. To verify (20) we couple random walks on H_v, H_v' until w is visited. In the latter the process stops. In the former, we find that when at w, the probability we get closer to v in the next step is at most ω^{-3} and so the expected number of returns from now on is at most $\omega \times \omega^{-3}$ and (20) follows.

(b) Now consider the case where H_v contains only heavy cycles. We argue first that a random walk of length ω that starts at v might as well terminate if it reaches a vertex $w \leq n^{1/10}$, $w \neq v$. We can assume $d(w) \geq n^{1/4}$. Now we can assume from Lemma 9 at least $n_0 = n^{1/4} - (\log n)^{5\omega}$ of the edges incident with w are not in cycles contained in H_v. But then a walk that arrives at w has a more than $\frac{n_0}{n^{1/4}}$ chance of entering a sub-tree T_w of H_v rooted at w for which every vertex is separated from v by w. But then the probability of leaving T_w in ω steps is $O(\omega(\log n)^{5\omega}/n^{1/4})$ and so once a walk has reached w, the expected number of further returns to v is $o(\omega^{-1})$. We can therefore remove T_w from H_v and then replace an edge (x, w) by an edge (x, w_x) and make all the vertices w_x absorbing. Repeating this argument, we are left with a tree to which we can apply the argument of Lemma 12. □

Note that if $v \in V_B$ then no bound on R_v has been established:

$$V_B = \{v:\ G_v \text{ contains a unique light cycle } C_v \text{ and the path from } v \text{ to } C_v \text{ contains no vertex of degree at least } \omega^3\}$$

We now establish Part (b) of Lemma 4.

Lemma 14. *If $m \geq 3$ and $v \notin V_B$ then*

$$\max_{|s|=1} \frac{|R_v(s) - R_v|}{R_v} \leq 8/9 + o(1).$$

Proof For any s,

$$|R_v(s) - R_v| \leq \sum_{j=1}^{T} r_j |s^j - 1|.$$

As $|s| = 1$ we have that

$$\sum_{j=1}^{T} r_j |s^j - 1| \leq 2 \sum_{j=1}^{T} r_j.$$

But, Lemmas 12 and 13 imply

$$2R_v^{-1}\sum_{j=1}^{T} r_j = 2 - 2R_v^{-1} \leq 2 - \frac{2(m^2-m-1)}{m^2} + o(1) \leq \frac{8}{9} + o(1).$$

□

We now establish Part (a) of Lemma 4.

Lemma 15. *If $m \geq 3$ and $v \notin V_B$ then*

$$H_v < C_m R_v + o(1)$$

where $C_m < 1$.

Proof Let f_t' be the probability that $\mathcal{W}_u$ has a first visit to v at time t. As $H(s) = F(s)R(s)$ we have

$$\begin{aligned} H_v &\leq \mathbf{Pr}(\mathcal{W}_u \text{ visits } v \text{ by time } T-1)R_v \\ &= R_v \sum_{t=1}^{T-1} f_t'. \end{aligned}$$

Equation (1) implies that

$$\sum_{t=\omega}^{T-1} f_t' = o(1). \tag{21}$$

We now estimate $\sum_{t=0}^{\omega} f_t'$, the probability that $\mathcal{W}_u$ visits v by time ω. Let $v_1, v_2, \ldots, v_k$ be the neighbours of v and let w be the first neighbour of v visited by $\mathcal{W}_u$. Then

$$\begin{aligned} \mathbf{Pr}(\mathcal{W}_u \text{ visits } v \text{ by time } \omega) &= \sum_{i=1}^{k} \mathbf{Pr}(\mathcal{W}_u \text{ visits } v \text{ by time } \omega \mid w = v_i)\mathbf{Pr}(w = v_i) \\ &\leq \sum_{i=1}^{k} \mathbf{Pr}(\mathcal{W}_{v_i} \text{ visits } v \text{ by the time } \omega)\mathbf{Pr}(w = v_i). \end{aligned}$$

So it suffices to prove the lemma when u is a neighbour of v.

Let the neighbours of u be $u_1, u_2, \ldots, u_d$, $d \geq m$ and that $v = u_d$. Suppose first that u is tree-like.

$$\mathbf{Pr}(\mathcal{W}_u \text{ does not visit } v \text{ by time } \omega) \geq \frac{m-1}{m}\left(2 - \frac{m^2}{m^2-m-1}\right) - o(1) > 0. \tag{22}$$

Here we use the fact that $\frac{m^2}{m^2-m-1} - 1 + o(1)$ bounds the probability of return to u_i, $i \leq d-1$, assuming that the first neighbour of u visited by $\mathcal{W}_u$ is u_i.

If H_v contains a unique light cycle then because $v \notin V_B$ we have either (i) $d > \omega^3$ or (ii) $u \notin C_u$. In the former case we have $\mathbf{Pr}(\mathcal{W}_u \text{ visits } v \text{ by time } \omega) = O(\omega^{-2})$. In the latter case either (a) the distance from u to C_u is at least $\omega/2$ and we can replace ω by $\omega/2$ from (21) onwards i.e. act as though u is tree-like to obtain (22) or (b) the path from u to C_u contains a vertex x of degree more than ω^3. In which case we could "truncate" H_v at x as in the proof of Lemma 13(b) and proceed as if H_u is tree-like to obtain (22) .

If there is more than one cycle then we deal with them as we did in Lemma 13(c) and then once again obtain (22) . □

4.2 Upper bound on cover time

From now on, assume that $m \geq 3$.

Let $t_0 = \lceil \frac{2m}{m-1} n \log n \rceil$. We prove that **whp**, for $G_m(n)$, for any vertex $u \in V$,

$$C_u \leq t_0 + o(t_0). \tag{23}$$

- V_1 denotes the set of vertices which are either (i) locally regular or (ii) are locally tree-like but not locally regular and have degree m or are not locally tree-like but (iii) satisfy condition (a) or (b) of Lemma 13 and satisfy (i) of (19).

- V_2 denotes the set of vertices which have degree $\geq m+1$ and are either (i) locally tree-like, not locally regular or are not locally tree-like but (ii) satisfy conditions (a) or (b) of Lemma 13 and satisfy (ii) of (19).

- V_3 denotes the set of not locally tree-like vertices for which there is a path of length $\leq \omega$ to a short cycle which only uses vertices of degree ω^3 or less. (V_3 contains the vertices of small cycles). Note that

$$|V_3| \leq \omega (\log n)^{5\omega} \omega^{3\omega} \leq (\log n)^{6\omega}.$$

Note that V_1, V_2, V_3 partition $V \setminus V_B$ where V_B was defined following Lemma 13.

Let $T_G(u)$ be the time taken to visit every vertex of G by the random walk $\mathcal{W}_u$. Let U_t be the number of vertices of G which have not been visited by $\mathcal{W}_u$ at step t. We note the following:

$$C_u = \mathbf{E}\, T_G(u) = \sum_{t>0} \mathbf{Pr}(T_G(u) \geq t), \tag{24}$$

$$\mathbf{Pr}(T_G(u) > t) = \mathbf{Pr}(U_t > 0) \leq \min\{1, \mathbf{E}\, U_t\}. \tag{25}$$

It follows from (24), (25) that for all t

$$
\begin{aligned}
C_u &\leq t+\sum_{s\geq t}\mathbf{E}\,U_s \\
&= t+\sum_{v\in V}\sum_{s\geq t}\mathbf{Pr}(\boldsymbol{A}_s(v)) \\
&= t+\sum_{v\in V_B}\sum_{s\geq t}\mathbf{Pr}(\boldsymbol{A}_s(v))+\sum_{v\in V\setminus V_B}\sum_{s\geq t}\left(\frac{c_{u,v}}{(1+p_v)^s}+O(\lambda e^{-\lambda s/2})\right) \\
&\leq t+\sum_{v\in V_B}\sum_{s\geq t}\mathbf{Pr}(\boldsymbol{A}_s(v))+(1+O(T\pi_v))\sum_{v\in V\setminus V_B}\left(\frac{R_v}{\pi_v}e^{-(1+O(T\pi_v))t\pi_v/R_v}+O(\lambda e^{-\lambda t/2})\right)
\end{aligned} \tag{26}
$$

Let $t_1=(1+\epsilon)t_0$ where $\epsilon=n^{-1/3}$ can be assumed by Lemma 6 to satisfy $T\pi_v=o(\epsilon)$ for all $v\in V$.

If $v\in V_1$ then by Lemmas 12 and 13,

$$t_1(1+O(T\pi_v))\pi_v/R_v\geq\frac{2m}{m-1}n\log n\cdot\frac{m}{2mn}\cdot\frac{m-1}{m}=\log n. \tag{27}$$

If $v\in V_2$ then by Lemmas 12 and 13,

$$t_1(1+O(T\pi_v))\pi_v/R_v\geq\frac{2m}{m-1}n\log n\cdot\frac{m+1}{2mn}\cdot\frac{m^2-m-1}{m^2}\geq\log n. \tag{28}$$

If $v\in V_3$ then,

$$t_1(1+O(T\pi_v))\pi_v/R_v\geq\frac{2m}{m-1}n\log n\cdot\frac{m}{2mn}\cdot\omega^{-1}\geq\omega^{-1}\log n. \tag{29}$$

Plugging (27), (28), (29) into (26) and using $R_v\leq 10$ and $\pi_v\geq\frac{1}{2n}$ for all $v\in V\setminus V_B$ we get

$$
\begin{aligned}
C_u &\leq t_1+o(1)+\sum_{v\in V_B}\sum_{s\geq t}\mathbf{Pr}(\boldsymbol{A}_s(v))+2((|V_1|+|V_2|)\cdot 20n\cdot n^{-1}+|V_3|\cdot 20n\cdot n^{-1/\omega}) \\
&\leq t_1+o(1)+\sum_{v\in V_B}\sum_{s\geq t}\mathbf{Pr}(\boldsymbol{A}_s(v))+40n(1+(\log n)^{6\omega}n^{-1/\omega}) \\
&= (1+o(1))t_0+\sum_{v\in V_B}\sum_{s\geq t}\mathbf{Pr}(\boldsymbol{A}_s(v)).
\end{aligned} \tag{30}
$$

It remains to deal with V_B We first observe that

$$|V_B|\leq(\log n)^{5\omega}\omega^{3\omega}. \tag{31}$$

Now fix $v\in V_B$. By "walking away from C_v" we see that there exists $w\notin V_B$ such that there is a path P from v to w of length $\sigma\leq\omega$ such that all internal vertices of P have degree at most ω^3. Let $d\leq n^{1/2+o(1)}$ (see Lemma 6(b)) be the degree of w.

Let $\nu = \sigma + 21mn/d$. If our walk visits w, it will with probability at least $d^{-1}\omega^{-3\omega}$ visit v within the next σ steps. Therefore

$$\begin{aligned}
\mathbf{Pr}(\boldsymbol{A}_\nu(v)) &\leq (1 - \mathbf{Pr}(\boldsymbol{A}_{\nu-\sigma}(v)))d^{-1}\omega^{-3\omega} \\
&\leq \left(1 - \exp\left\{-(1+o(1))\frac{d\nu}{20mn}\right\}\right) d^{-1}\omega^{-3\omega} \\
&\leq (1-e^{-1})d^{-1}\omega^{-3\omega}.
\end{aligned}$$

Thus if $\gamma = (1-e^{-1})d^{-1}\omega^{-3\omega}$,

$$\begin{aligned}
\sum_{s \geq t_0} \mathbf{Pr}(\boldsymbol{A}_s(v)) &\leq \sum_{s \geq t_0} (1-\gamma)^{\lfloor s/\nu \rfloor} \\
&\leq \sum_{s \geq t_0} (1-\gamma)^{s/(2\nu)} \\
&= \frac{(1-\gamma)^{t_0/(2\nu)}}{1-(1-\gamma)^{1/(2\nu)}} \\
&\leq 2\nu\gamma^{-1}e^{-t_0\gamma/(2\nu)} \\
&\leq 50m\omega^{3\omega}n^{1-1/(100m\omega^{3\omega})}
\end{aligned}$$

and so, using (31),

$$\sum_{v \in V_B} \sum_{s \geq t_0} \mathbf{Pr}(\boldsymbol{A}_s(v)) \leq 50m(\log n)^{5\omega}\omega^{6\omega}n^{1-1/(100m\omega^{3\omega})} = o(t_0).$$

This completes our proof of the upper bound on cover time for $m \geq 3$.

4.3 Lower bound on cover time

For some vertex u, we can find a set of vertices S such that at time $t_1 = t_0(1-\epsilon)$, $\epsilon \to 0$, the probability the set S is covered by the walk $\mathcal{W}_u$ tends to zero. Hence $T_G(u) > t_1$ **whp** which implies that $C_G \geq t_0 - o(t_0)$.

We construct S as follows. Let S be some maximal set of locally regular vertices all of which are at least distance $2\omega + 1$ apart. Thus $|S| \geq n^{1-o(1)}m^{-(2\omega+1)}$.

Let $S(t)$ denote the subset of S which has not been visited by $\mathcal{W}_u$ after step t. Now, provided $t \geq T$

$$\mathbf{E}\left(|S(t)|\right) \geq (1-o(1)) \sum_{v \in S} \left(\frac{c_{u,v}}{(1+p_v)^t} + o(n^{-2}) \right).$$

Let u be a fixed vertex of S. Let $v \in S$ and let $H_T(1)$ be given by (6), then (1) implies that

$$H_T(1) \leq \sum_{t=\omega}^{T-1} (\pi_v + \lambda_{\max}^t) = o(1). \tag{32}$$

Thus $c_{uv} = 1 - o(1)$. Setting $t = t_1 = (1-\epsilon)t_0$ where $\epsilon = 2\omega^{-1}$, we have

$$\begin{aligned}\mathbf{E}\,(|S(t_1)|) &= (1+o(1))|S|e^{-(1-\epsilon)t_0 p_v} \\ &\geq n^{1/\omega}. \end{aligned} \tag{33}$$

Let $Y_{v,t}$ be the indicator for the event that $\mathcal{W}_u$ has not visited vertex v at time t. Let $Z = \{v, w\} \subset S$. We will show (below) that that for $v, w \in S$

$$\mathbf{E}\,(Y_{v,t_1} Y_{w,t_1}) = \frac{c_{u,Z}}{(1+p_Z)^{t+2}} + o(n^{-2}), \tag{34}$$

where $c_{u,Z} \sim 1$ and $p_Z \sim (m-1)/(mn)$. Thus

$$\mathbf{E}\,(Y_{v,t_1} Y_{w,t_1}) = (1+o(1))\mathbf{E}\,(Y_{v,t_1})\mathbf{E}\,(Y_{w,t_1}). \tag{35}$$

It follows from (33) and (35), that

$$\mathbf{Pr}(S(t_1) \neq 0) \geq \frac{\mathbf{E}\,(|S(t_1)|)^2}{\mathbf{E}\,(|S(t_1)|^2)} = \frac{1}{\frac{\mathbf{E}(|S_{t_1}|(|S_{t_1}|-1))}{\mathbf{E}(|S(t_1)|)^2} + \mathbf{E}\,(|S_{t_1}|)^{-1}} = 1 - o(1).$$

Proof of (34). Let Γ be obtained from G by merging v, w into a single node Z. This node has degree $2m$.

There is a natural measure preserving mapping from the set of walks in G which start at u and do not visit v or w, to the corresponding set of walks in Γ which do not visit Z. Thus the probability that $\mathcal{W}_u$ does not visit v or w in the first t steps is equal to the probability that a random walk $\widehat{\mathcal{W}}_u$ in Γ which also starts at u does not visit Z in the first steps.

We apply Lemma 4 to Γ. That $\pi_Z = \frac{1}{n}$ is clear, and $c_{u,Z} = 1 - o(1)$ is argued as in (32). The derivation of R_Z in Lemma 12 is also valid. The vertex Z is tree-like up to distance ω in Γ. The fact that the root vertex of the corresponding infinite tree has degree $2m$ does not affect the calculation of E_0. □

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