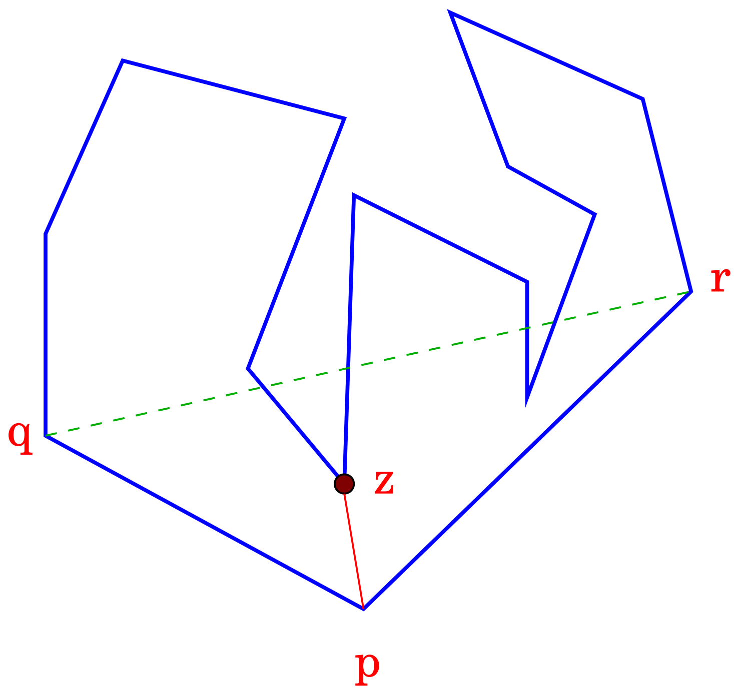
# Triangulation: Theory

**Theorem: Every polygon has a triangulation.**

- **Proof by Induction. Base case $n = 3$.**

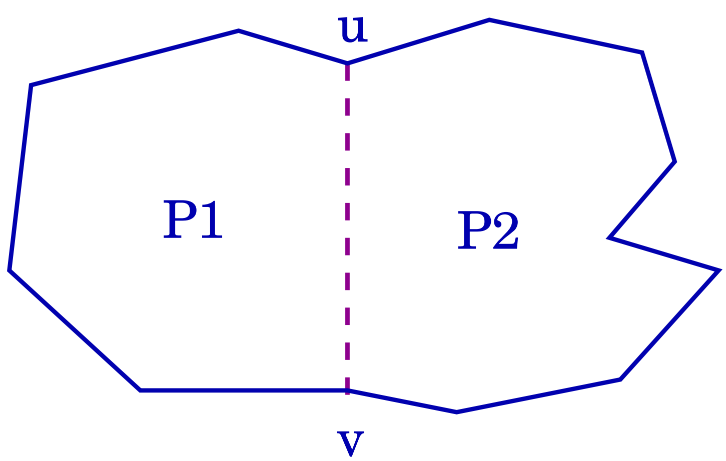



- **Pick a convex corner $p$. Let $q$ and $r$ be pred and succ vertices.**
- **If $qr$ a diagonal, add it. By induction, the smaller polygon has a triangulation.**
- **If $qr$ not a diagonal, let $z$ be the reflex vertex farthest to $qr$ inside $\triangle pqr$.**
- **Add diagonal $pz$; subpolygons on both sides have triangulations.**

# Triangulation: Theory

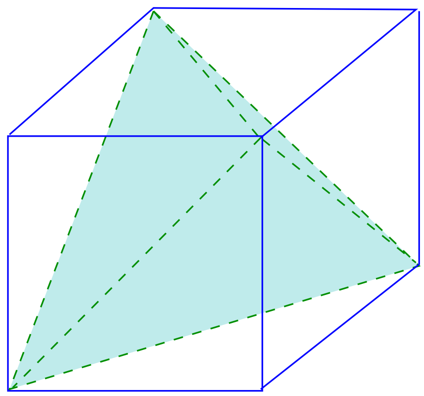
**Theorem: Every triangulation of an $n$-gon has $n-2$ triangles.**

- **Proof by Induction. Base case $n = 3$.**

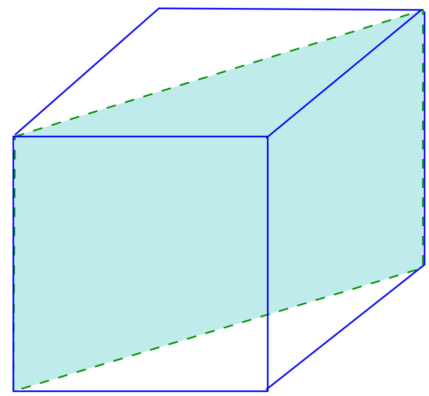



- **Let $t(P)$ denote the number of triangles in any triangulation of $P$.**
- **Pick a diagonal $uv$ in the given triangulation, which divides $P$ into $P_1$, $P_2$.**
- $t(P) = t(P_1) + t(P_2) = n_1 - 2 + n_2 - 2$.
- **Since $n_1 + n_2 = n + 2$, we get $t(P) = n - 2$.**

# Triangulation in 3D



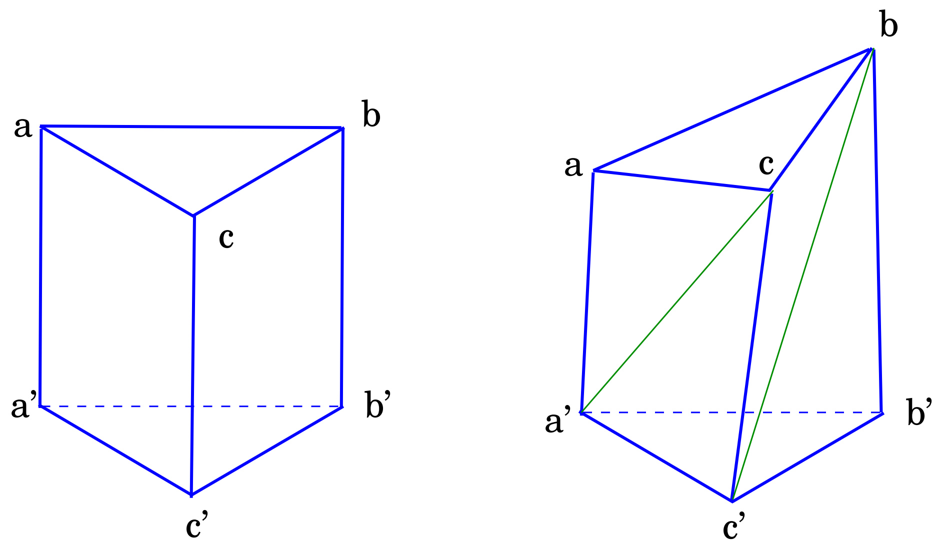
5 Tetrahedra



6 Tetrahedra

- **Different triangulations can have different number of tetrahedra (3D triangles).**

# Untriangulable Polyhedron




- Smallest example of a polyhedron that cannot be triangulated without adding new vertices. (Schoenhardt [1928]).

- It is NP-Complete to determine if a polyhedron requires Steiner vertices for triangulation.

- Every 3D polyhedron with $N$ vertices can be triangulated with $O(N^2)$ tetrahedra.

# Triangulation History

1. **A really naive algorithm is $O(n^4)$: check all $n^2$ choices for a diagonal, each in $O(n)$ time. Repeat this $n-1$ times.**

2. **A better naive algorithm is $O(n^2)$; find an ear in $O(n)$ time; then recurse.**

3. **First non-trivial algorithm: $O(n \log n)$ [GJPT-78]**

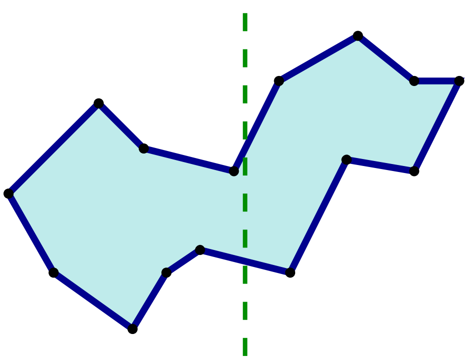
4. **A long series of papers and algorithms in 80s until Chazelle produced an optimal $O(n)$ algorithm in 1991.**

5. **Linear time algorithm insanely complicated; there are randomized, expected linear time that are more accessible.**

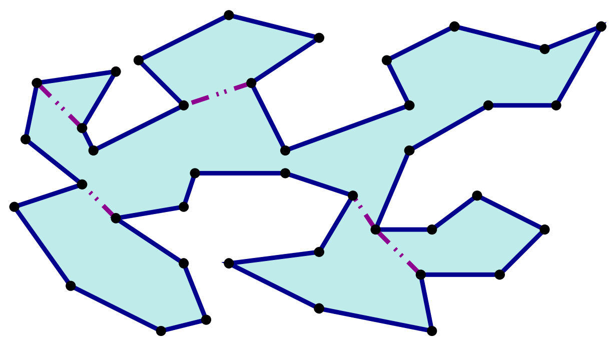
6. **We content ourselves with $O(n \log n)$ algorithm.**

# Algorithm Outline

1. Partition polygon into trapezoids.
2. Convert trapezoids into monotone subdivision.
3. Triangulate each monotone piece.



x−monotone polygon

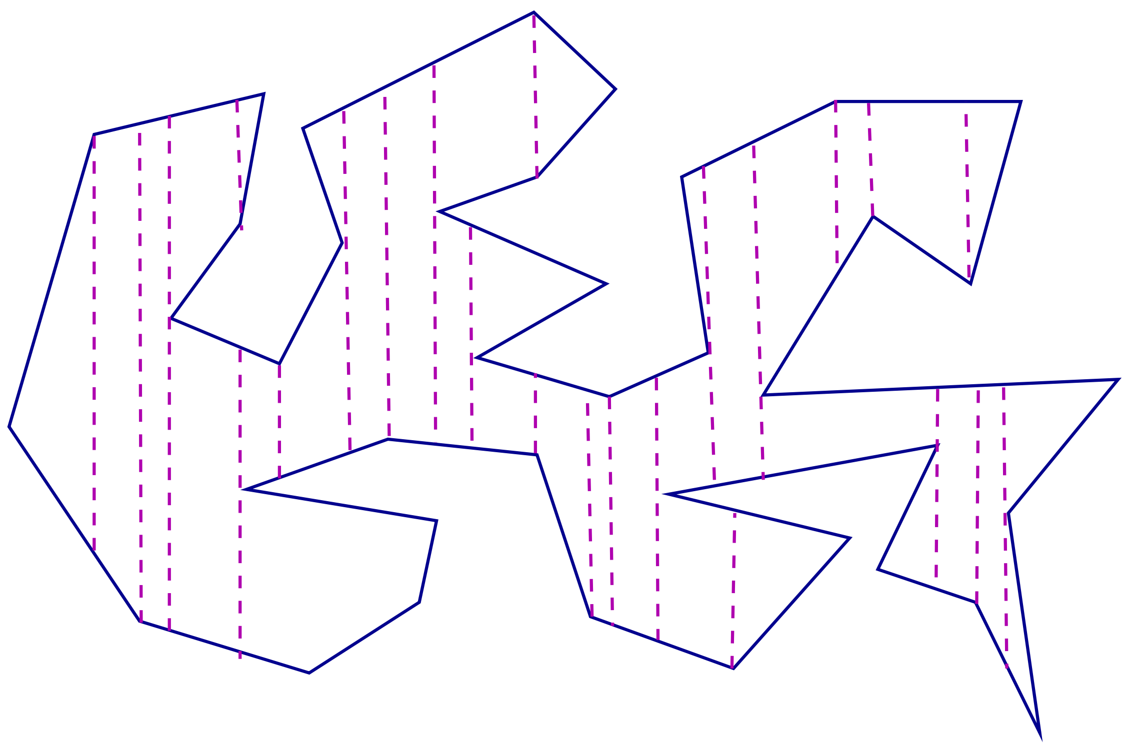


Monotone decomposition

4. A polygonal chain $C$ is monotone w.r.t. line $L$ if any line orthogonal to $L$ intersects $C$ in at most one point.
5. A polygon is monotone w.r.t. $L$ if it can be decomposed into two chains, each monotone w.r.t. $L$.
6. In the Figure, $L$ is $x$-axis.

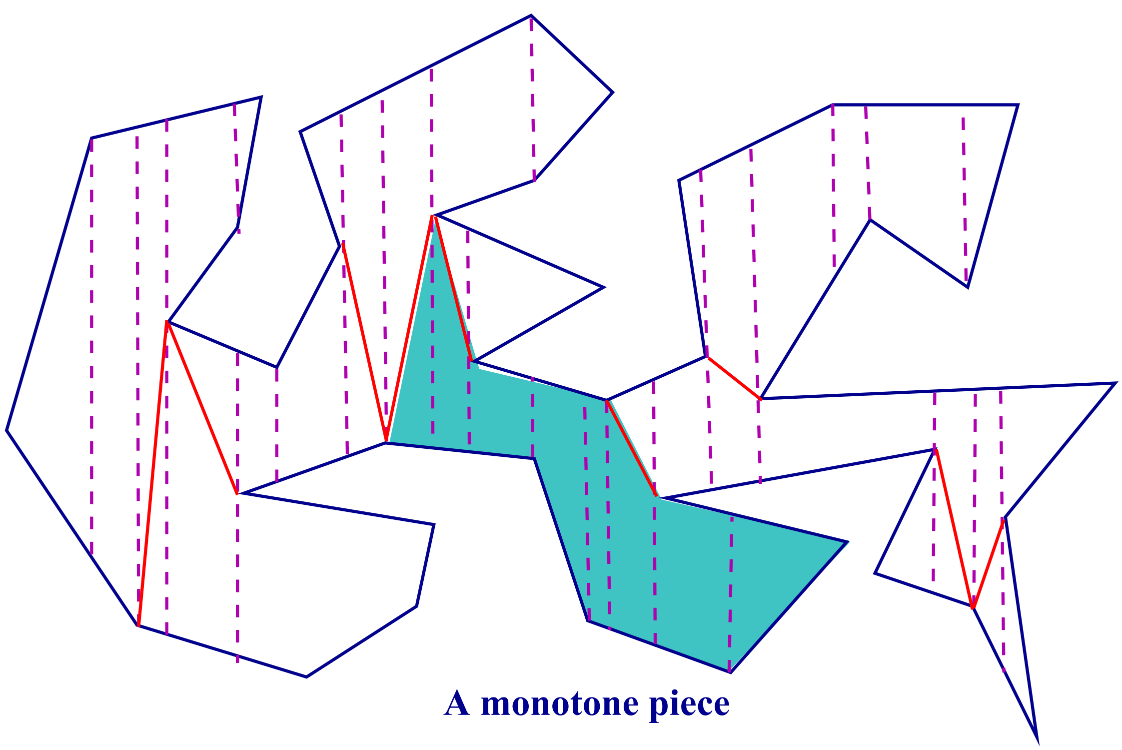
# Trapezoidal Decomposition

- **Use plane sweep algorithm.**
- **At each vertex, extend vertical line until it hits a polygon edge.**
- **Each face of this decomposition is a trapezoid; which may degenerate into a triangle.**
- **Time complexity is $O(n \log n)$.**



# Monotone Subdivision

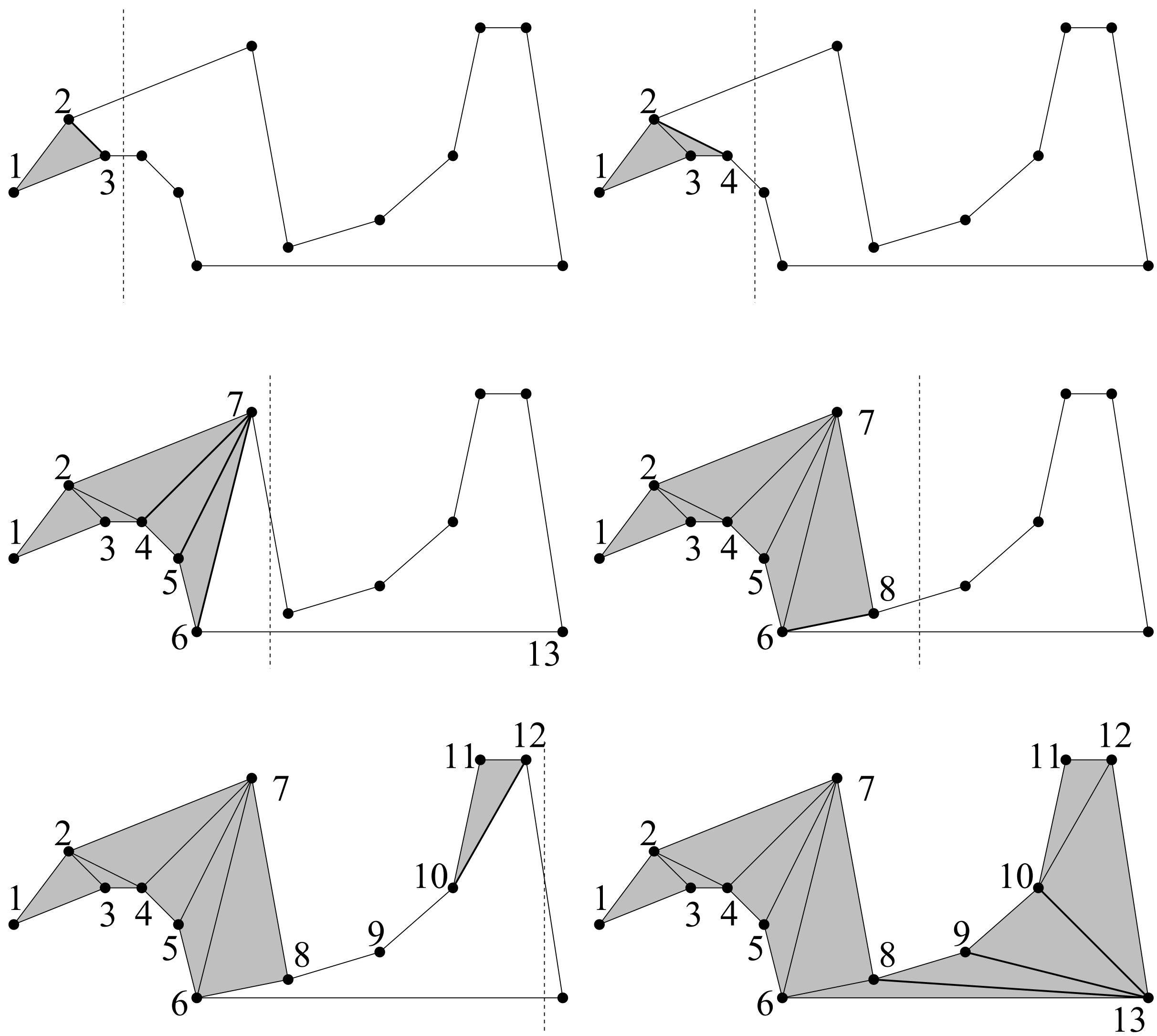
- Call a reflex vertex with both rightward (leftward) edges a split (merge) vertex.
- Non-monotonicity comes from split or merge vertices.
- Add a diagonal to each to remove the non-monotonicity.
- To each split (merge) vertex, add a diagonal joining it to the polygon vertex of its left (right) trapezoid.



# Monotone Subdivision

- **Assume that trap decomposition represented by DCEL.**
- **Then, matching vertex for split and merge vertex can be found in $O(1)$ time.**
- **Remove all trapezoidal edges. The polygon boundary plus new split/merge edges form the monotone subdivision.**
- **The intermediate trap decomposition is only for presentation clarity—in practice, you can do monotone subdivision directly during the plane sweep.**

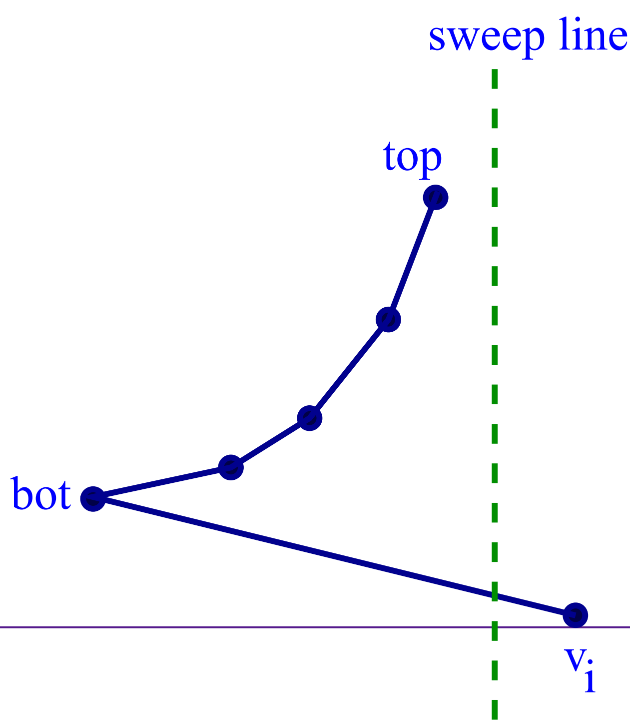
# Triangulation



# Triangulation

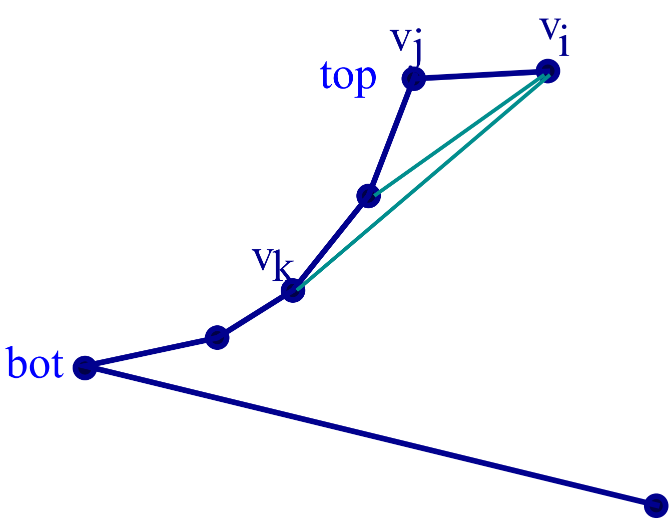
- $\langle v_1, v_2, \ldots, v_n \rangle$ **sorted left to right.**
- **Push** $v_1, v_2$ **onto stack.**
- **for** $i = 3$ **to** $n$ **do**

  **if** $v_i$ **and** $top(stack)$ **on same chain**
  **Add diagonals** $v_i v_j, \ldots, v_i v_k$**, where**
  $v_k$ **is last to admit legal diagonal**
  **Pop** $v_j, \ldots, v_{k-1}$ **and Push** $v_i$
  **else**
  **Add diagonals from** $v_i$ **to all vertices**
  **on the stack and pop them**
  **Save** $v_{top}$**; Push** $v_{top}$ **and** $v_i$



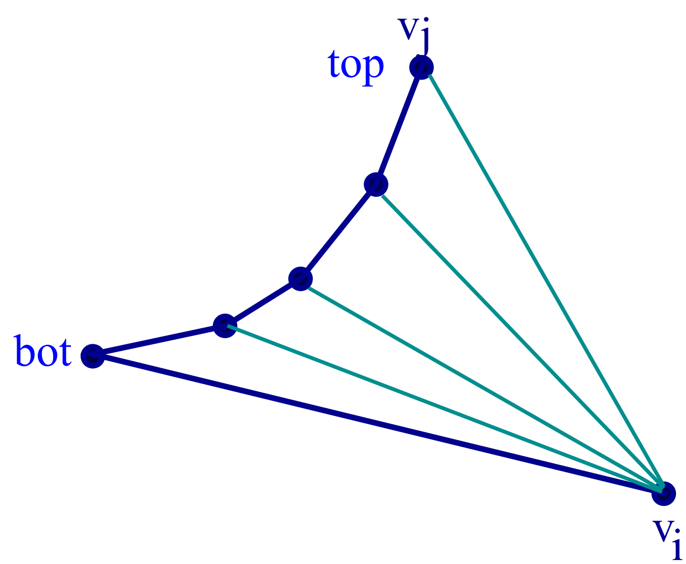
# Correctness

- **Invariant:** **Vertices on current stack form a single reflex chain. The leftmost unscanned vertex in the other chain is to the right of the current scan line.**




New stack: (bot, ..., vk, vi)

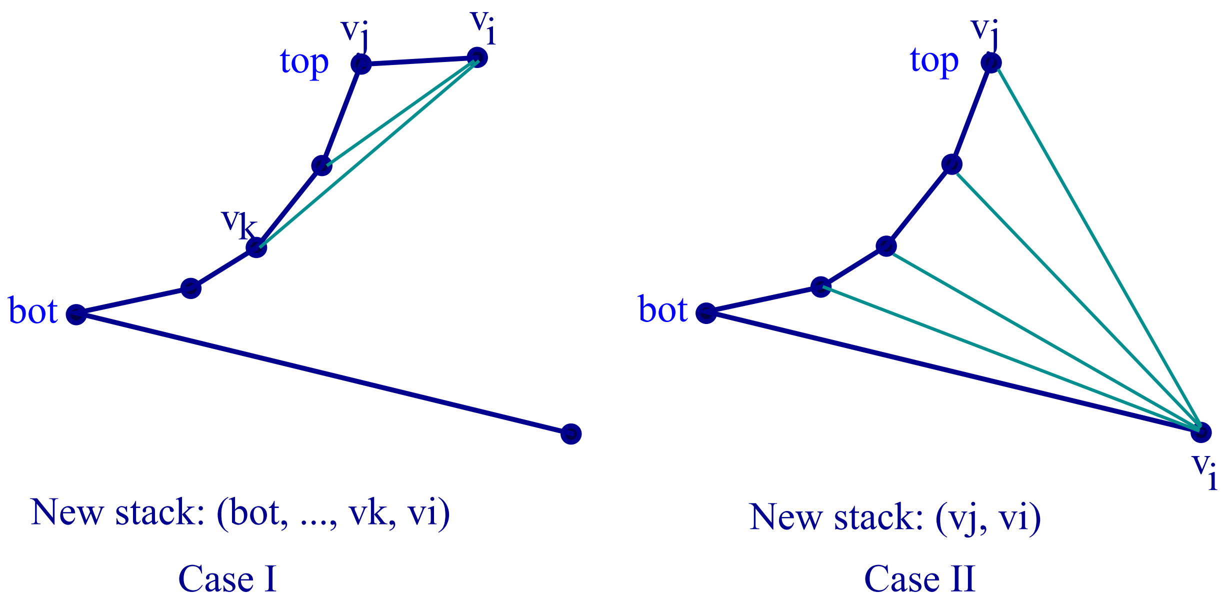
Case I




New stack: (vj, vi)

Case II

# Time Complexity




- **A vertex is added to stack once. Once it's visited during a scan, it's removed from the stack.**
- **In each step, at least one diagonal is added; or the reflex stack chain is extended by one vertex.**
- **Total time is $O(n)$.**
- **Total time for polygon triangulation is therefore $O(n \log n)$.**