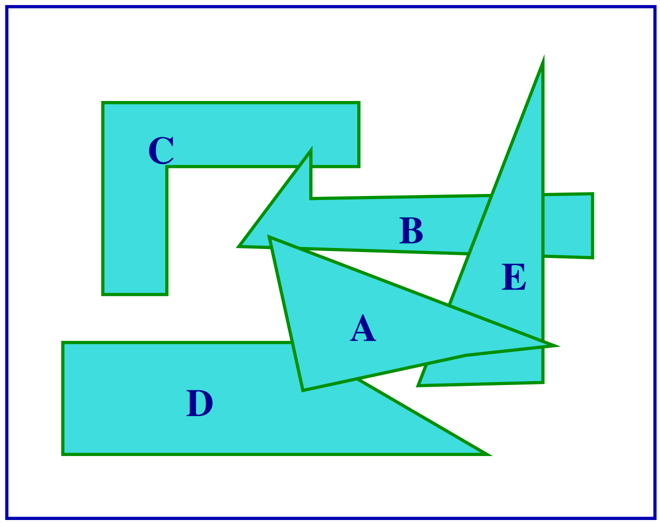
Referências:

① P&S cap. 7.   §7.2 ...

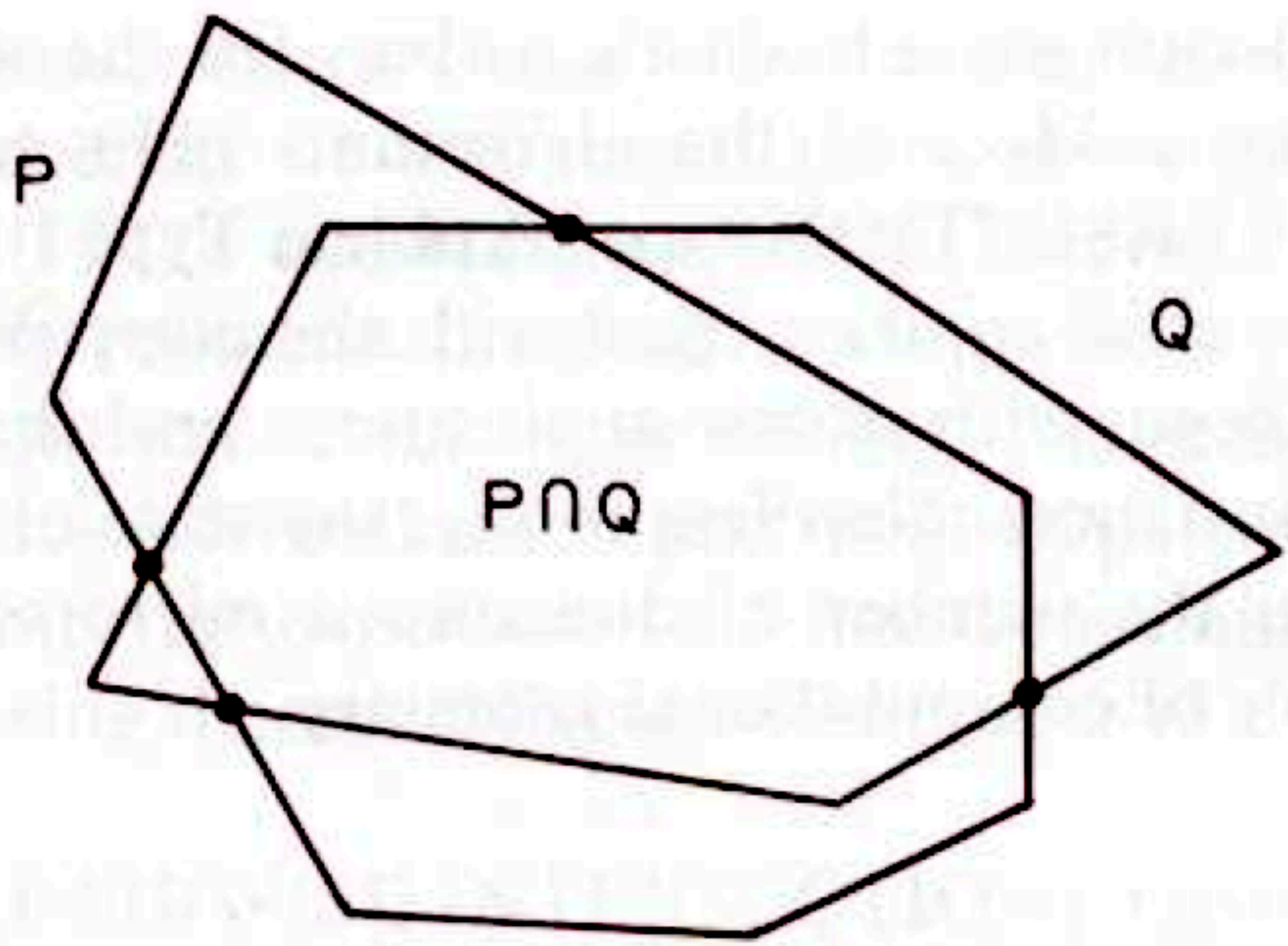
② dB, vK, O, S cap. 2

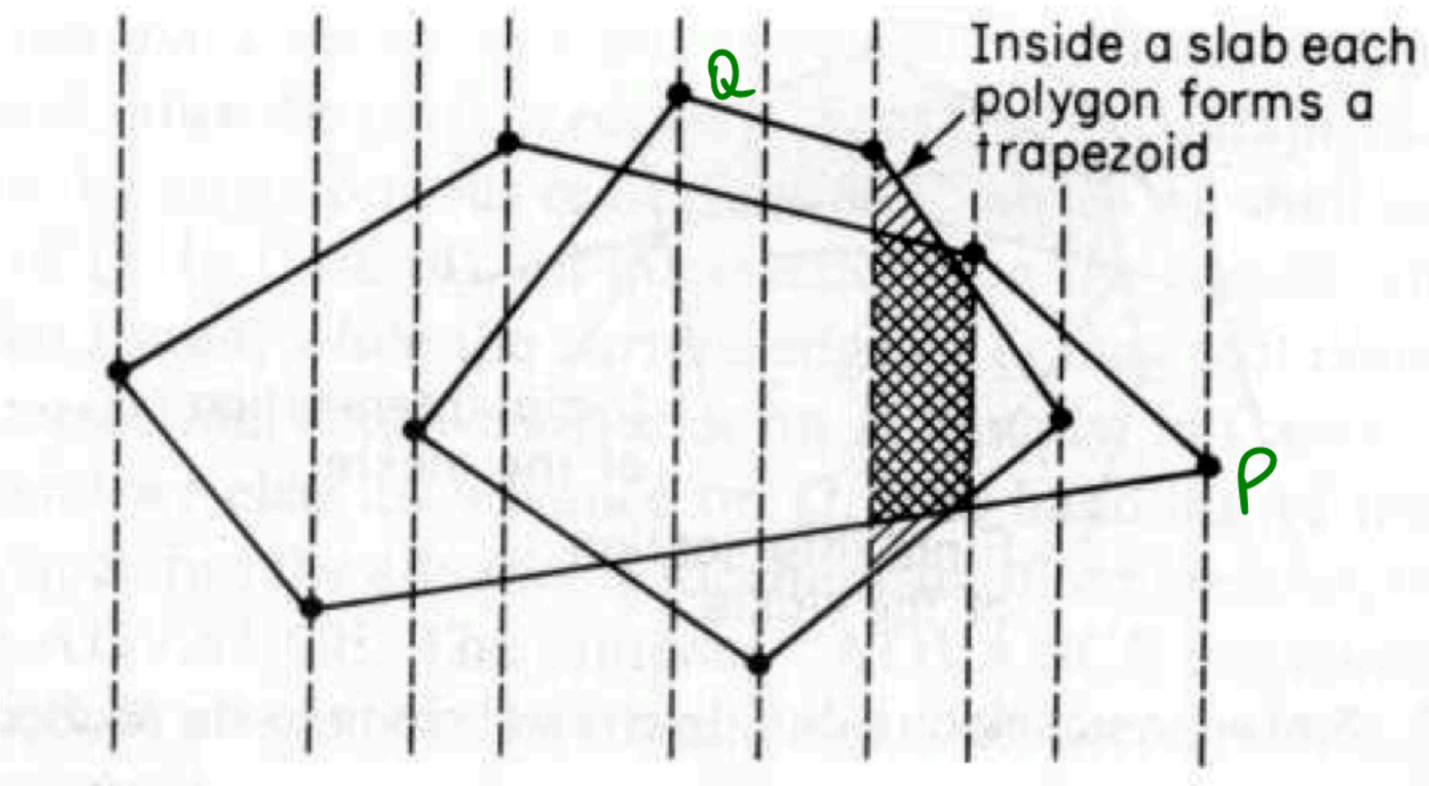
Intersection Problems

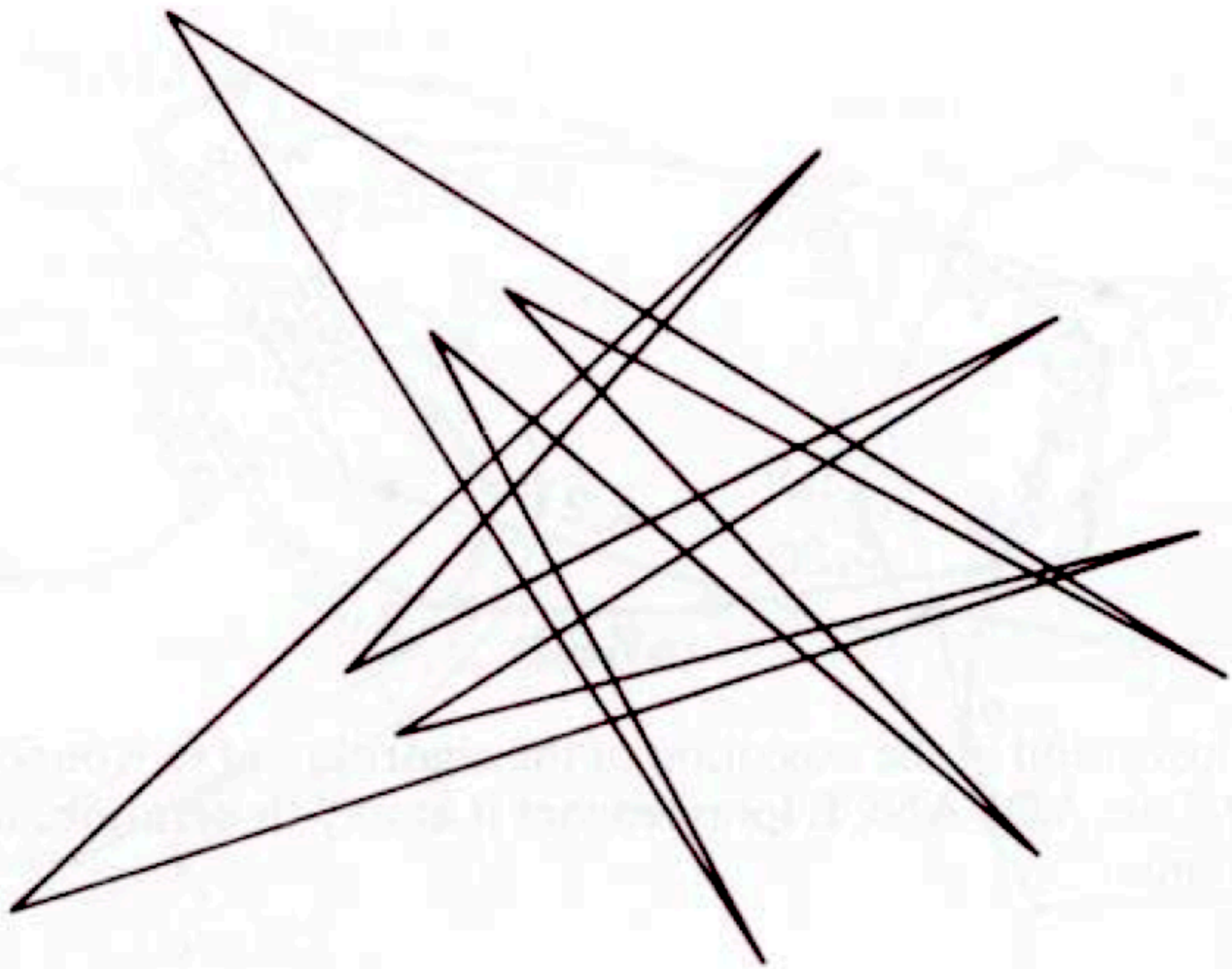
- **Determine pairs of intersecting objects?**



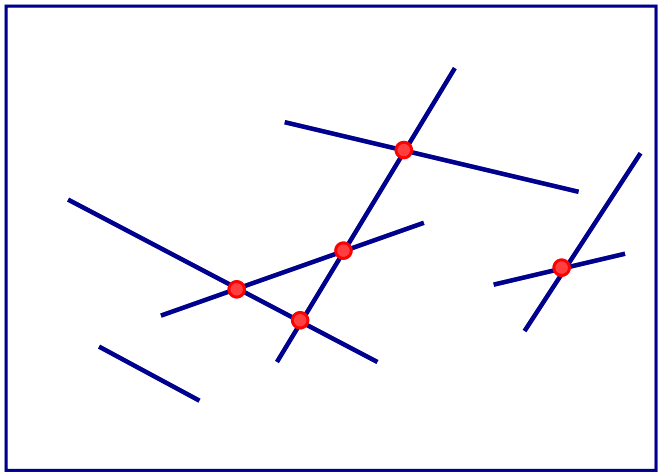
- **Complex shapes formed by boolean operations: intersect, union, diff.**
- **Collision detection in robotics and motion planning.**
- **Visibility, occlusion, rendering in graphics.**
- **Map overlay in GISs: e.g. road networks on county maps.**


P
Q
P∩Q


Q
Inside a slab each polygon forms a trapezoid
P

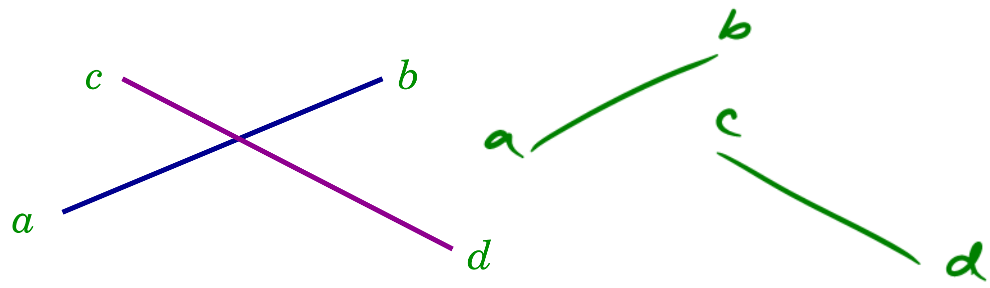


Line Segment Intersection



- **The most basic problem: intersections among line segments in R^2.**
- **General enough to capture polygons, 2D projections of 3D scenes.**
- **Naive algorithm: Check all pairs. $O(n^2)$.**
- **If k intersections, then ideal will be $O(n \log n + k)$ time.**
- **We will describe a $O((n + k) \log n)$ solution. Also introduce a new technique : plane sweep.**

Primitive Operation



- **How to decide if two line segments ab and cd intersect?**
- **Write the equations of each segment in parametric form:**

$$
\begin{aligned}
p(s) &= (1-s)a + sb && \textbf{for } 0 \leq s \leq 1 \\
q(t) &= (1-t)c + sd && \textbf{for } 0 \leq t \leq 1
\end{aligned}
$$

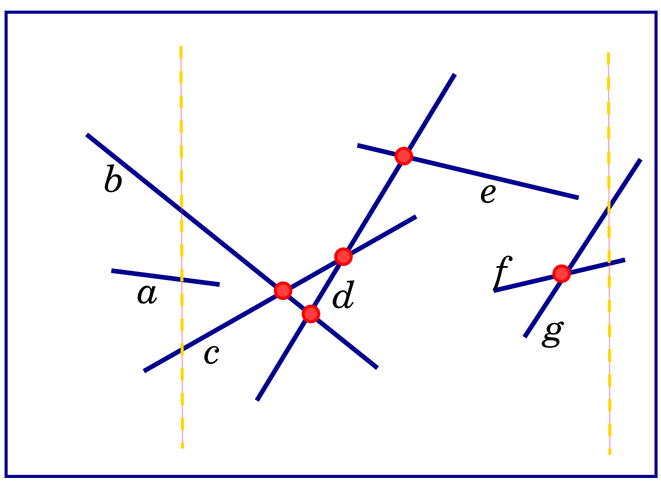
- **An intersection occurs if for some values of s, t, we get $p(s) = q(t)$.**
- **In terms of x, y , we get:**

$$
\begin{aligned}
(1-s)a_x + sb_x &= (1-t)c_x + td_x \\
(1-s)a_y + sb_y &= (1-t)c_y + td_y
\end{aligned}
$$

- **Solve for s, t and see if they lie in $[0, 1]$.**

Plane Sweep Algorithm

- **Input $S = \{s_1, s_2, \ldots, s_n\}$; each segment given by pair of endpoints.**
- **Report all intersecting segment pairs.**
- **We move an imaginary vertical line from left to right.**
- **Maintain vertical order of segments intersecting the sweep line; order changes only at discrete times.**

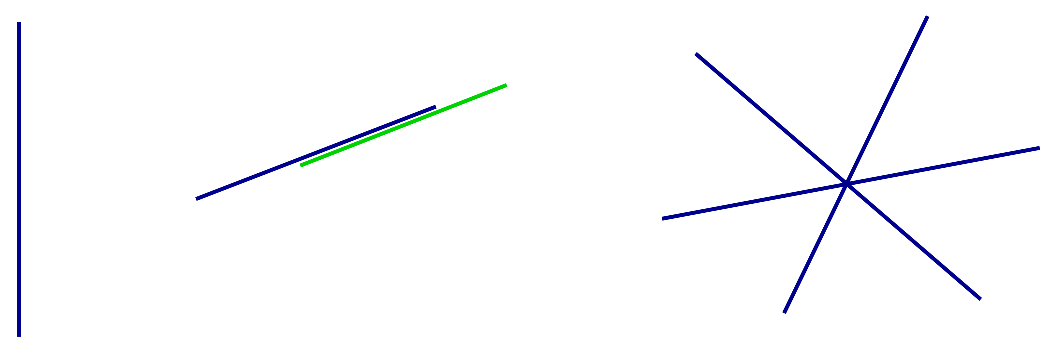


- **Intersections among S inferred by looking at localized information along sweep line.**

Simplifying Assumptions

- **In order to avoid dealing with technical special cases, which obscure the main ideas, we assume:**

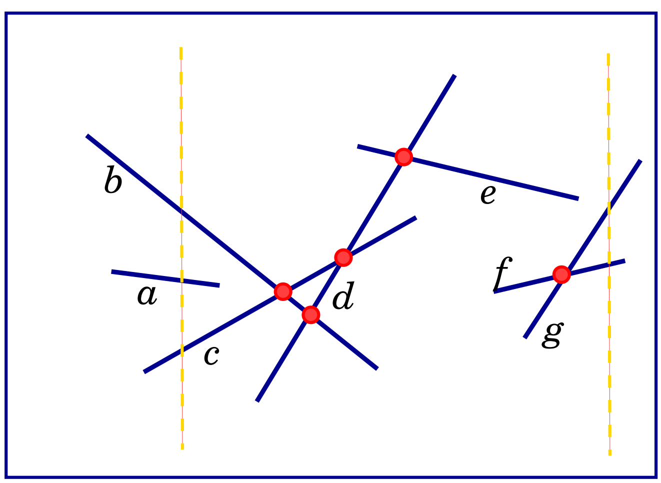
  1. **No segment is vertical.**
  2. **Any two segments intersect in at most one point.**
  3. **No three or more lines intersect in a common point.**



Data Structures

- **Sweep Line Status:** **Maintain the segments intersecting the sweep line ℓ in sorted order, from top to bottom.**

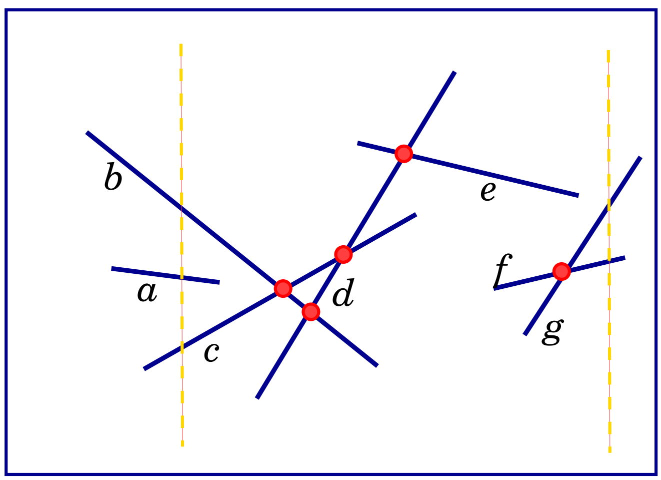
  1. **Balanced binary tree.**
  2. **Insert, delete, search in $O(\log n)$.**
  3. **The choice of the key? The y-position of $s \cap \ell$ changes as ℓ moves.**
  4. **Use "variable" key, the equation of the line: $y = mx + c$.**
  5. **Plugging in x fixes y coordinate.**
  6. **All order-comparisons among segments done for a fixed x-position of ℓ.**



Data Structures

- **Event Queue:** **Events represent instants when sweep line order changes.**

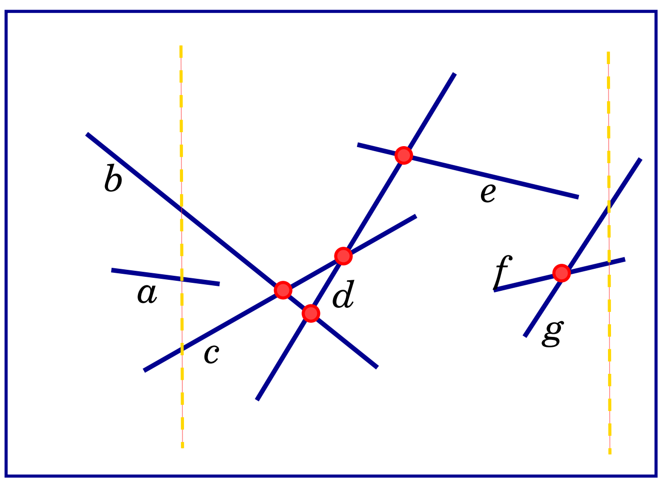
1. **While the y-coordinates of segments along ℓ change continuously, their ordering changes only at discrete steps.**



2. **Order changes when a segment begins, a segment ends, or two segments intersect.**
3. **Segments begin/end events known in advance; the intersection events generated dynamically.**
4. **Maintain events in x-sorted order, in a balanced binary tree.**

What's the Idea?

- **The algorithm requires knowing the intersection points (for event queue).**
- **But that's whole problem we are trying to solve!**
- **We don't need all intersections up front; only before the sweep line reaches them.**

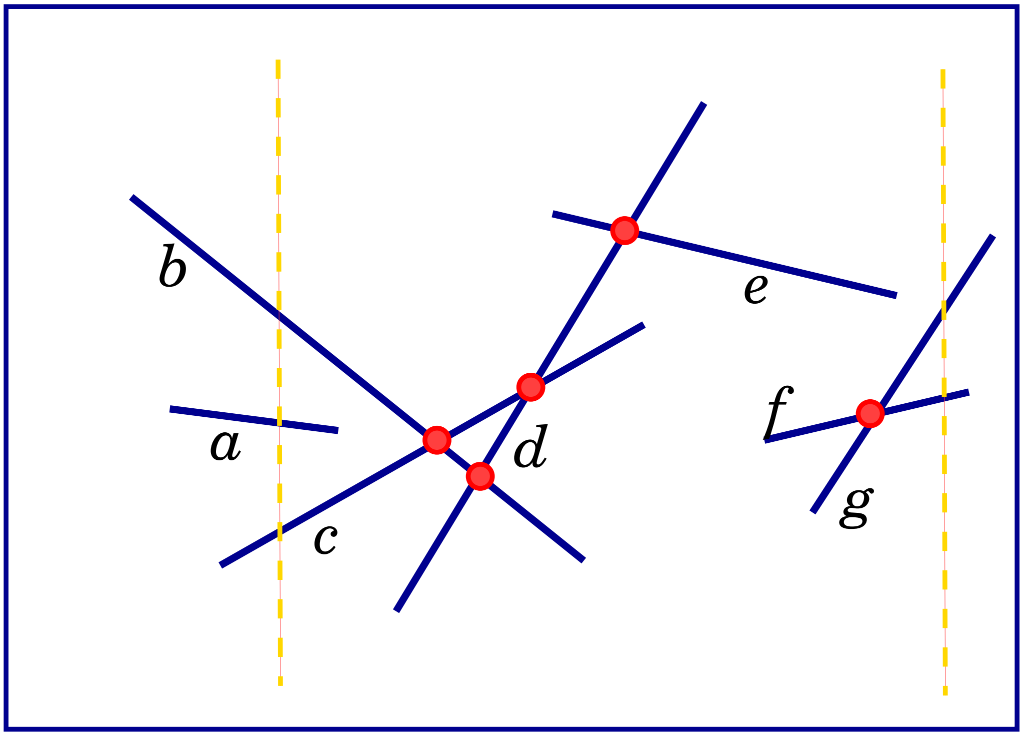


- **Plane sweep's idea is to maintain only the "most immediate" intersections.**
- **At any time, the Event Queue schedules only those intersections that are between two neighboring segments in the sweep line order.**

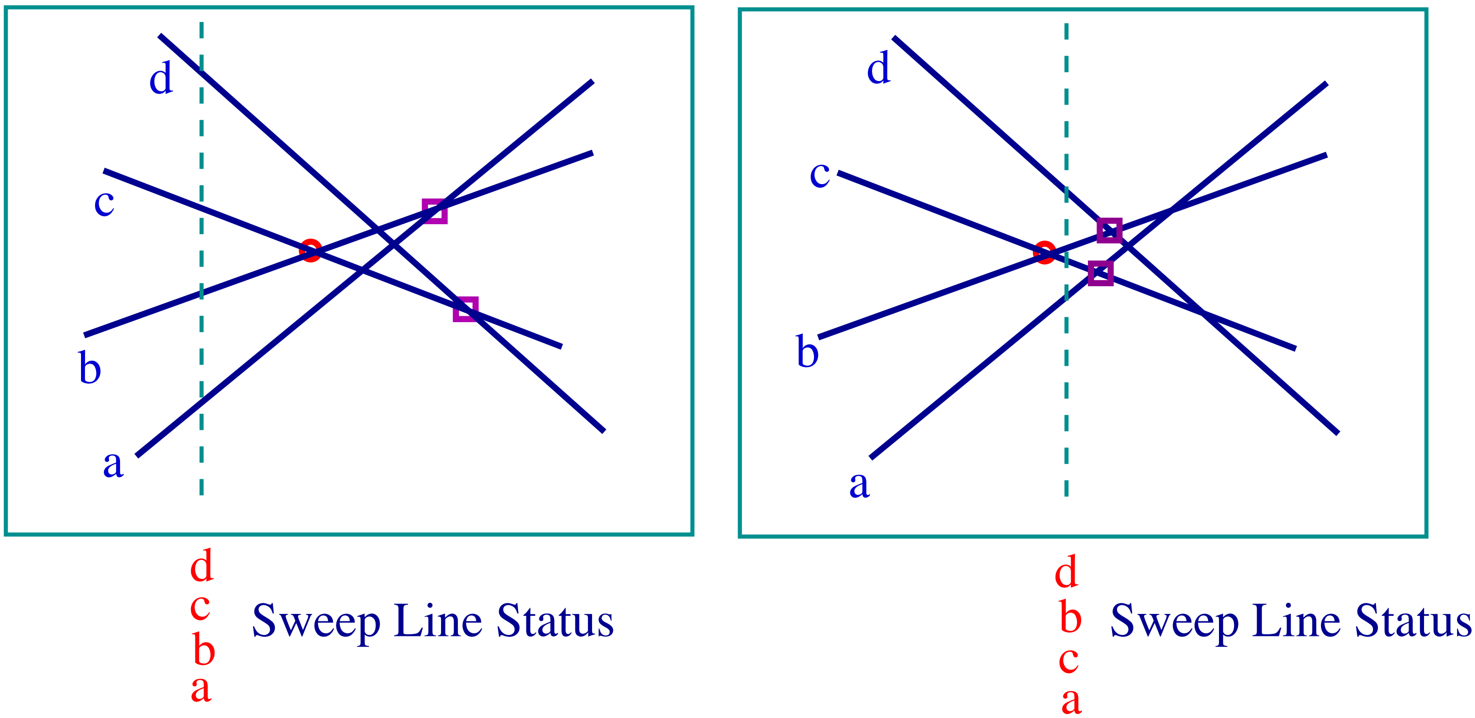
Algorithm

1. **Initialize Event Queue with endpoints of S, in sorted order.**
2. **While queue non-empty, extract the next event. Three cases:**
3. **[Left endpoint of a segment s_i]**
   - **Insert s_i into sweep line status tree;**
   - **If s_i intersects its above or below neighbors, add those intersections to Event Queue.**
4. **[Right endpoint of a segment s_i]**
   - **Delete s_i from sweep line status tree;**
   - **If s_i's neighbors intersect, add that intersection to Event Queue.**
5. **[Intersection of s_i and s_j]**
   - **Swap the order of s_i and s_j;**
   - **Delete intersection events involving s_i and s_j from the Event Queue.**
   - **Possibly add new intersection events between s_i, s_j and their new neighbors.**

Illustration

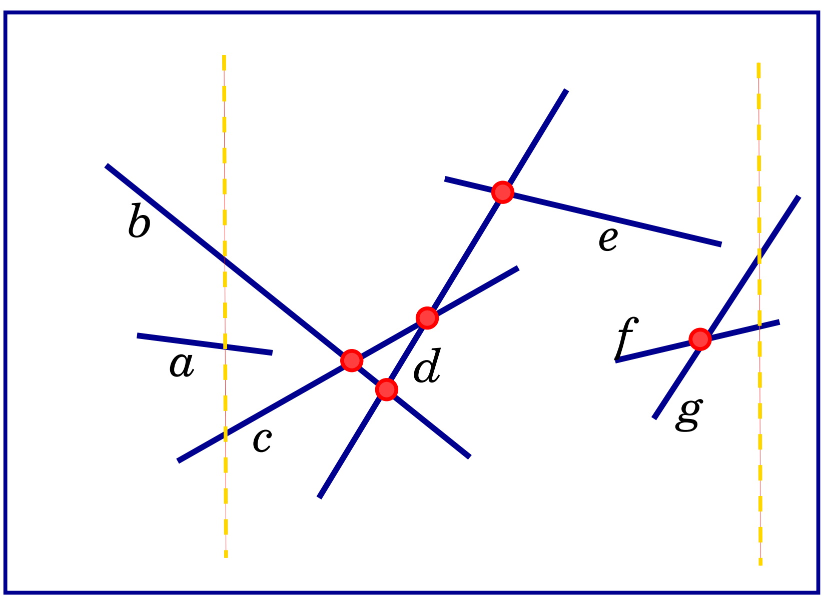


Illustration

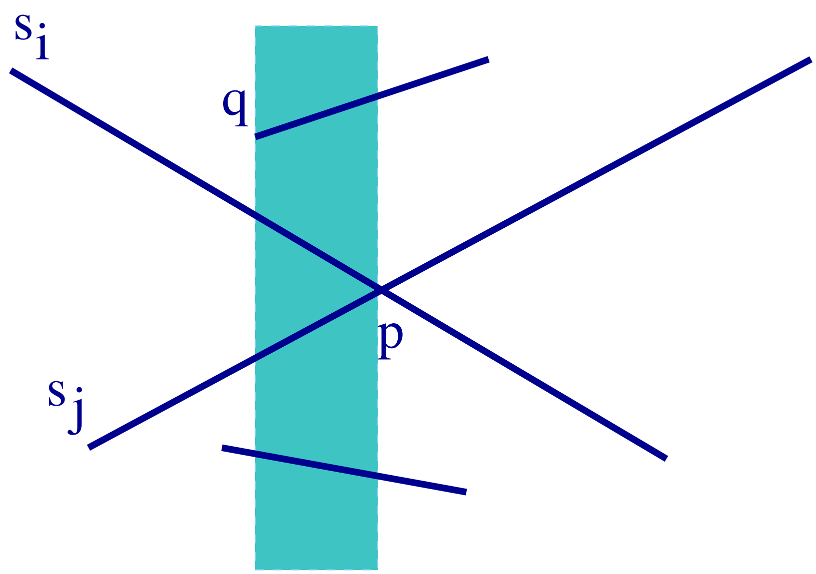


Correctness

1. **Algorithm only checks intersections between segments that are adjacent along sweep line at some point.**
2. **The algorithm obviously doesn't report false intersections.**
3. **But can it miss intersections?**
4. **No. If segments s_i and s_j intersect at point p, then s_i and s_j are neighbors just before the sweep line reaches p.**



Proof



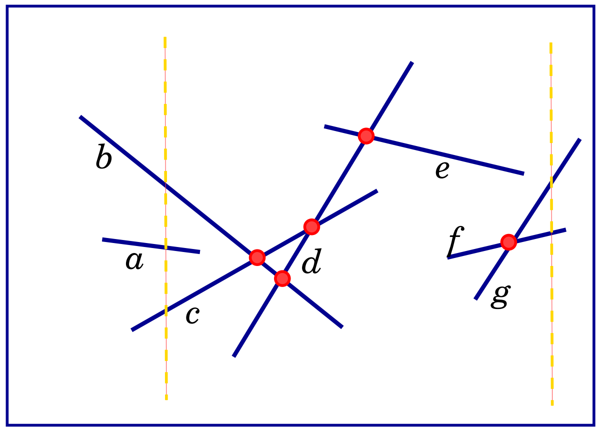
- **No three or more segments intersect at one point, so only s_i and s_j intersect at p.**

- **For sweep line placed just before p, there cannot be any segment between s_i and s_j; otherwise, there must be another event before p.**

- **Let q be the event before p. Then, the order of segments along sweep line after q and before p must remain unchanged.**

- **Thus, s_i and s_j are adjacent in the sweep line status tree when p is processed.**

Complexity



- **Number of events processed is $2n + k$.**
- **Number of events scheduled and descheduled can be larger.**
- **But each intersection processing creates at most 2 new events, and deletes at most 2 old events, so $O(k)$ events handled.**
- **Handling an event require $O(1)$ changes to the status tree, and $O(1)$ insert/delets in Event Queue.**
- **Thus, processing cost per event is $O(\log n)$.**
- **Time complexity is $O((n + k) \log(n + k))$.**