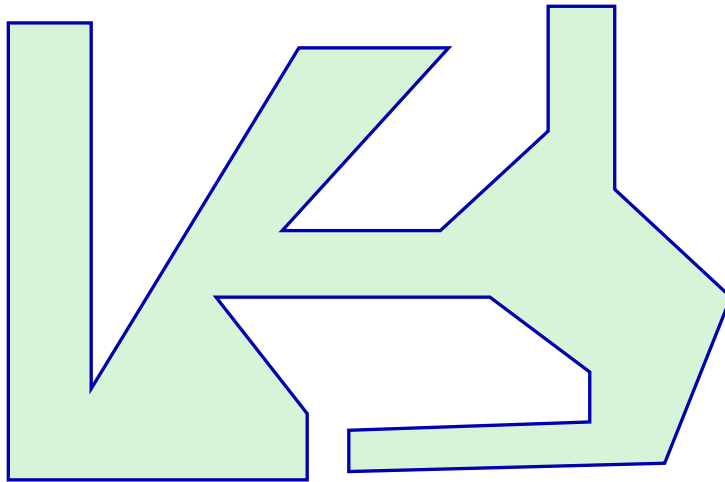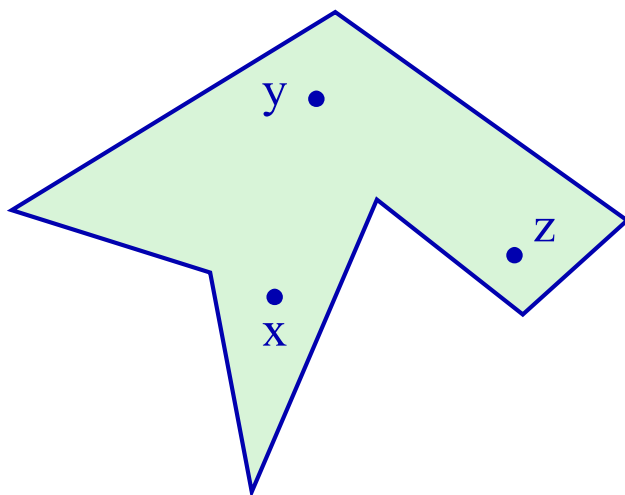# Art Gallery Theorem

- **The floor plan of an art gallery modeled as a simple polygon with n vertices.**

- **How many guards needed to see the whole room?**

- **Each guard is stationed at a fixed point, has 360° vision, and cannot see through the walls.**

- **Story: Problem posed to Vasek Chvatal by Victor Klee at a math conference in 1973. Chvatal solved it quickly with a complicated proof, which has since been simplified significantly using triangulation.**
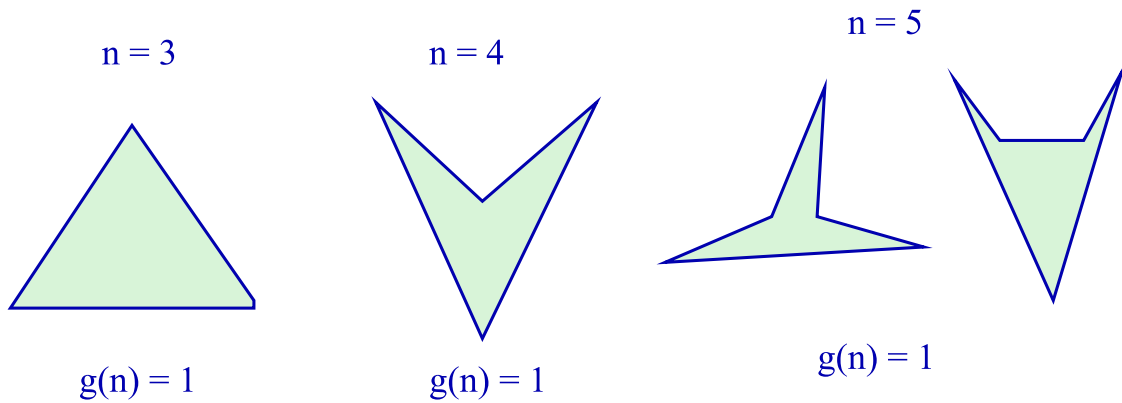
Formulation

- **Visibility**: p, q visible if $pq \in P$.

- y is visible from x and z. But x and z not visible to each other.

- $g(P) = $ **min. number of guards to see** P

- $g(n) = \max\limits_{|P|=n} g(P)$

- **Art Gallery Theorem** asks for bounds on function $g(n)$: what is the smallest $g(n)$ that always works for any n-gon?
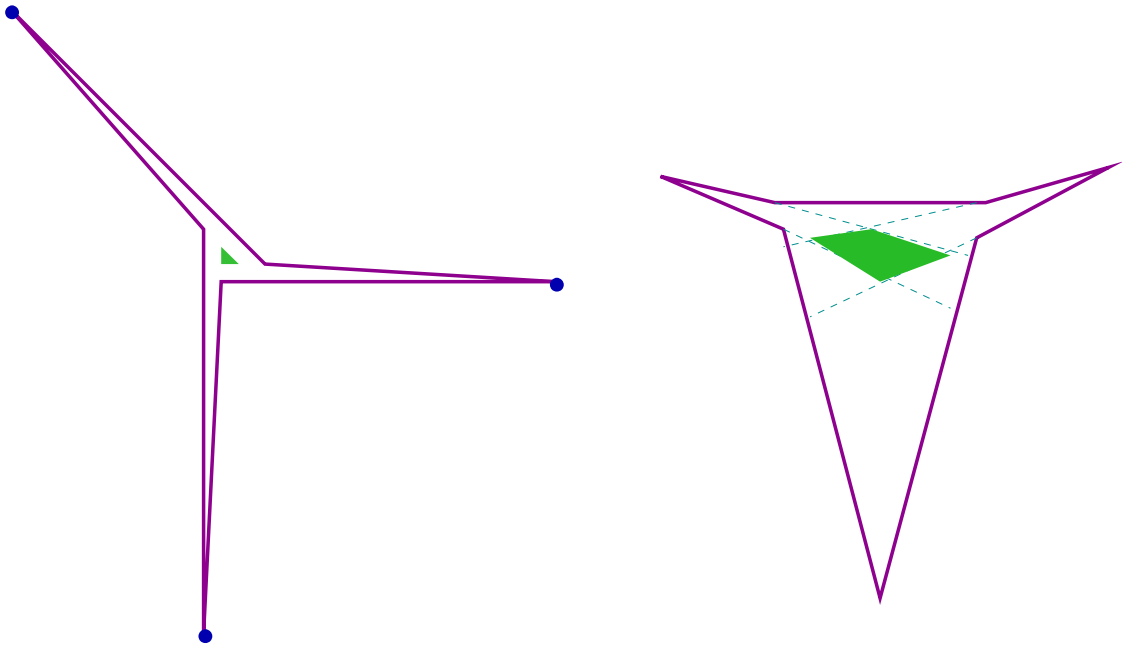
Trying it Out

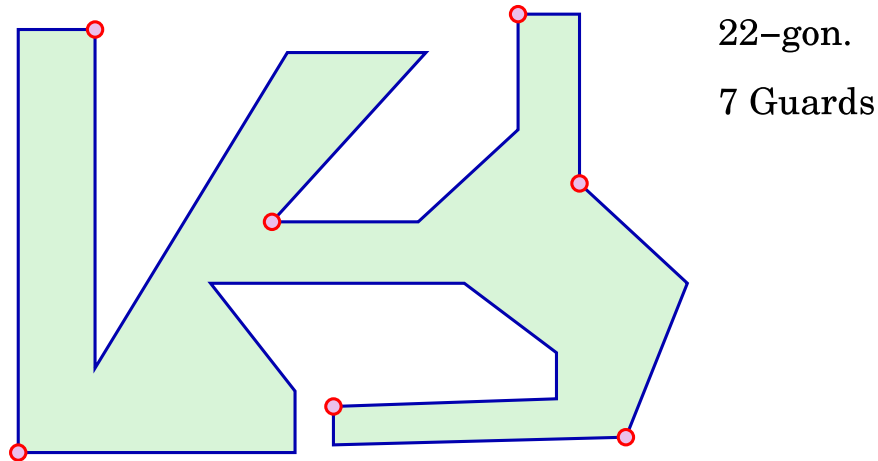1. **For $n = 3, 4, 5$, we can check that $g(n) = 1$.**

n = 3

n = 4

n = 5

g(n) = 1

g(n) = 1

g(n) = 1

2. **Is there a general formula in terms of n?**

Pathological Cases

1. **Fig. on left shows that seeing the boundary $\neq$ seeing the whole interior!**

2. **Even putting guards at every other vertex is not sufficient.**

3. **Fig. on right shows that putting guards on vertices alone might not give the best solution.**

Art Gallery Theorem

22–gon.

7 Guards

Theorem: $g(n) = \lfloor n/3 \rfloor$

1. **Every n-gon can be guarded with $\lfloor n/3 \rfloor$ vertex guards.**

2. **Some n-gons require at least $\lfloor n/3 \rfloor$ (arbitrary) guards.**
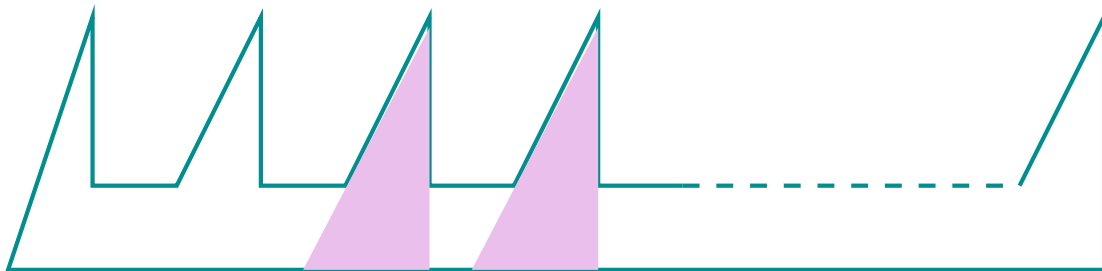
Necessity Construction

*Subhash Suri*   *UC Santa Barbara*

Fisk's Proof

Lemma: Triangulation graph can be 3-colored.

- P plus triangulation is a planar graph.

- 3-coloring means vertices can be labeled 1,2, or 3 so that no edge or diagonal has both endpoints with same label.

- Proof by Induction:

  1. Remove an ear.
  2. Inductively 3-color the rest.
  3. Put ear back, coloring new vertex with the label not used by the boundary diagonal.

Inductively 3−color

1   3

ear

2
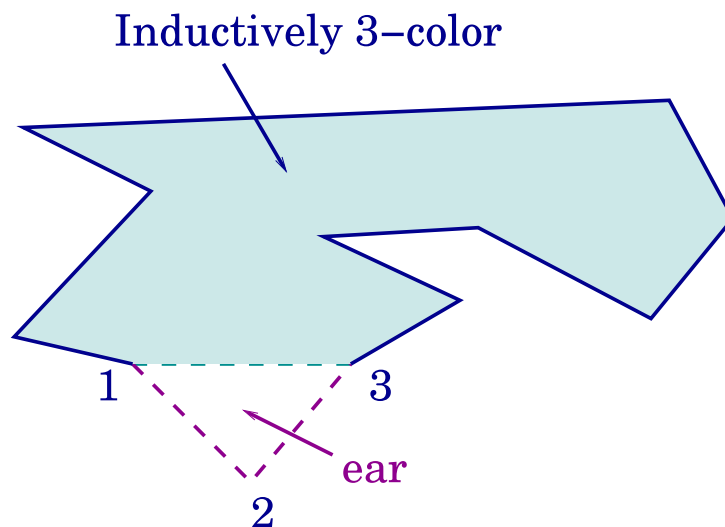
Proof
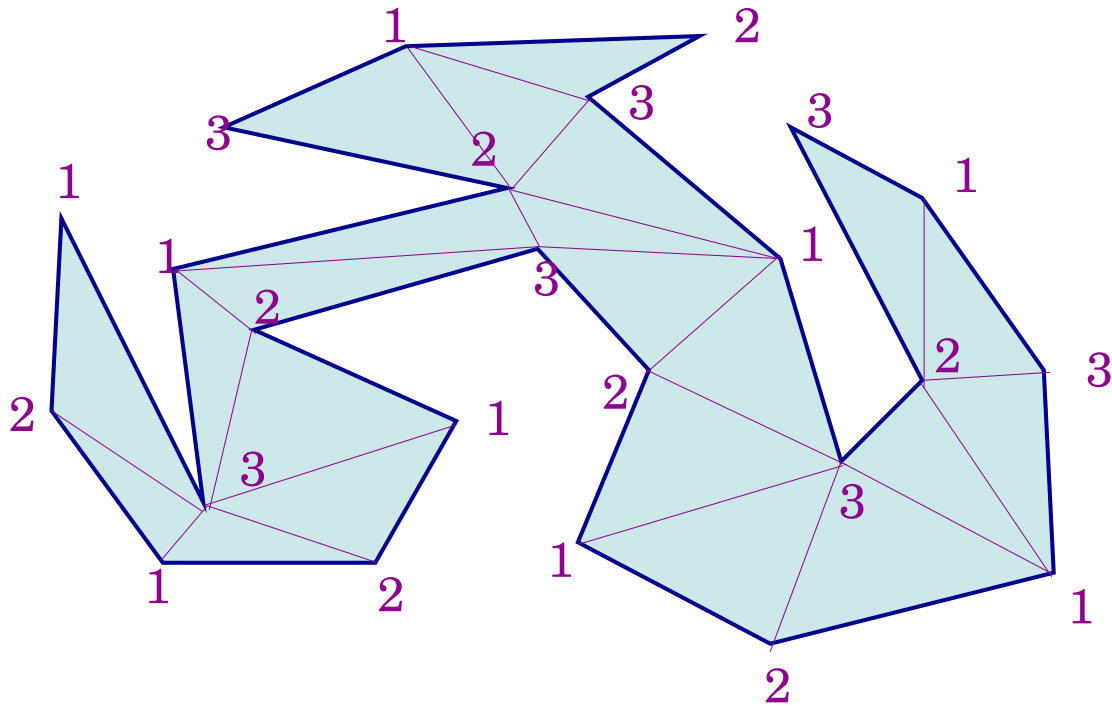
- **Triangulate P. 3-color it.**

- **Least frequent color appears at most $\lfloor n/3 \rfloor$ times.**

- **Place guards at this color positions—a triangle has all 3 colors, so seen by a gaurd.**

- **In example: Colors 1, 2, 3 appear 9, 8 and 7 times, resp. So, color 3 works.**