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Recognizing well covered graphs of families with special P_4-components*

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Abstract

A graph G is called well covered if every two maximal independent sets of G have the same number of vertices. In this paper we shall use the modular and primeval decomposition techniques to decide well coveredness of graphs such that, either all their P_4-connected components are separable or they belong to well known classes of graphs that, in some local sense, contain few P_4's. In particular, we shall consider the class of cographs, P_4-reducible, P_4-sparse, extended P_4-reducible, extended P_4-sparse graphs, P_4-extendible graphs, P_4-lite graphs, and P_4-tidy graphs.

1 Introduction

In this paper we are concerned with the so called well coveredness of a graph. To introduce this concept we need the following definitions. Let G be a graph. A subset I of G is an *independent* set if every pair of distinct vertices of I are not adjacent in G. An independent set is *maximal* if it is not properly contained in any other independent set of G. The maximum number of vertices in a maximal independent set of G is the *independence number*, $\alpha(G)$, of G. A graph G is called *well covered* if every two maximal independent sets of G have the same number of vertices. In other words, a graph G is well covered if every maximal independent set of G is a maximum independent set of G. The concept of well covered graph was introduced by Plummer [21]. Since the problem of finding the independence number of a general graph is $\mathcal{NP}$-complete, an interesting algorithm property of well covered graphs is that the greedy algorithm for producing a maximal independent set always produces a maximum independent set when applied to well covered graphs.

The problem of deciding if a graph is well covered is $co\mathcal{NP}$-complete. This was independently proved by Chvátal and Slater [4] and by Sankaranarayana and Stewart [26]. The problem remains $co\mathcal{NP}$-complete even when the input graph is $K_{1,4}$-free [3].

One of the main directions of the work on well covered graphs has been done towards exploring structural properties of some classes of graphs in order to characterize subclasses

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of well covered graphs. Some of these characterizations lead to polynomial algorithms that recognize if a graph of such classes is well covered. For example [2, 7, 9, 20, 22, 23, 24].

In this paper we consider some classes of graphs that have been characterized in terms of special properties of the unique primeval decomposition tree associated to each graph of the class. The primeval decomposition tree of any graph can be computed in linear time [1] and therefore it is the natural framework for finding polynomial time algorithms of many problems.

We shall characterize well coveredness for graphs such that, either all their P_4-connected components are separable or they belong to well known classes of graphs with few P_4's having non-separable P_4-connected components of special type. In particular we shall consider the class of cograph [5, 6], P_4-reducible [14], P_4-sparse [12, 16], extended P_4-reducible [10], extended P_4-sparse graphs [10], P_4-extendible graphs [15], P_4-lite graphs [13], P_4-tidy graphs [11].

In Section 2 we give some definitions and preliminary results. In Section 3 we prove some general results about the well coveredness of graphs. In Section 4 we use modular decomposition to describe an algorithm that decides, in linear time, if a graph is well covered under the hypothesis that all its P_4-connected components are separable. In Section 5 we use primeval decomposition to describe an algorithm that decides, in linear time, if a P_4-tidy graph is well covered. This result implies linear time algorithms for deciding well coveredness of any graph belonging to the above mentioned classes.

2 Preliminaries

2.1 Basic Notions

Throughout this paper, $G = (V, E)$ is a finite simple undirected graph with $|V| = n$ and $|E| = m$. The *complement graph* of $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $uv \in \overline{E}$ if and only if $uv \notin E$.

For a vertex $v \in V$ the *neighborhood* of v in G is $N(v) = \{v|uv \in E\}$. A *clique* of G is a set of pairwise adjacent vertices of G. Given a subset U of V, let $G[U]$ stand for the subgraph of G induced by U. Let P_n denote the chordless path on n vertices and $n-1$ edges. Let C_n denote the chordless cycle with n vertices. A graph is called *split graph* if its vertex set can be partitioned in a clique K and an independent set S. A split graph is a *spider* if and only if $|K| = |S| \geq 2$ and there exists a bijection f between S and K such that either $N(v) = \{f(v)\}$ for $v \in S$ *(thin legs)* or $N(v) = K \setminus \{f(v)\}$ for $v \in S$ *(thick legs)*. The simplest spider is a P_4. In a P_4 with vertices u, v, w, x and edges uv, vw, wx, the vertices v and w are called *midpoints* whereas the vertices u and x are called *endpoints*.

Let G and G' be two vertex disjoint graphs. The *parallel graph* $G \cup G'$ is defined by $V(G \cup G') = V(G) \cup V(G')$ and $E(G \cup G') = E(G) \cup E(G')$. The *serial graph* $G + G'$ is defined by $V(G + G') = V(G) \cup V(G')$ and $E(G + G') = E(G) \cup E(G') \cup \{vv'$ for each $v \in V(G)$ and $v' \in V(G')\}$.

2.2 Modular and primeval decomposition

A *module* of G is a set of vertices M of V such that each vertex in $V \setminus M$ is either adjacent to all vertices of M, or to none. The whole V and every singleton vertex are *trivial* modules. Whenever G has only trivial modules is called a *prime* graph. We say that M is a *strong* module if for any other module A the intersection $M \cap A$ is empty or equals either M or A. The unique partition of the vertex set of a graph G into maximal strong modules is used recursively to define its unique *modular decomposition tree* $T(G)$. The module M is *parallel (***P***)* if $G[M]$ is disconnected and its maximal strong submodules are the connected components of $G[M]$; M is *serial (***S***)* if $\overline{G}[M]$ is disconnected and its maximal strong submodules are the connected components of $\overline{G}[M]$; M is *neighbourhood (***N***)* if both $G[M]$ and $\overline{G}[M]$ are connected. Similarly, we say that G[M] is a parallel, serial or neighbourhood graph when M is respectively so. The leaves of $T(G)$ are the vertices of G and the internal nodes of $T(G)$ are modules labeled with P, S or N (for parallel, serial, or neighbourhood module, respectively). The modular decomposition tree of any graph can be computed in linear time [19, 25].

Let t be an internal node of the modular decomposition tree $T(G)$ of the graph G. We denote by $M(t)$ the module corresponding to t, which consists of the set of vertices of G associated with the subtree of $T(G)$ rooted at node t. Note that $M(t)$ is a strong module for every (internal or leaf) node t of $T(G)$. Let $t_1, t_2, \ldots, t_p$ be the children of the node t in $T(G)$. We denote by $G(t)$ the *representative graph* of node t defined as follows: $V(G(t)) = \{t_1, t_2, \ldots, t_p\}$, and $t_i t_j \in E(G(t))$ if there exists an edge $v_k v_l \in E(G)$ such that $v_k \in M(t_i)$ and $v_l \in M(t_j)$.

It is easy to see that if t is a node labelled by S, then $G(t)$ is a complete graph, if t is a node labelled by P, then $G(t)$ is an edgeless graph and if t is a node labelled by N, then $G(t)$ is a prime graph.

The structure of a neighborhood module have been further investigated by Jamison and Olariu [17] that introduced the notion of P_4-connected and separable P_4-connected graphs. Following their terminology, a graph G is *p-connected* (or, more extensively, *P_4-connected*) if, for each partition V_1, V_2 of V into two sets, there exists a chordless path of four vertices P_4 which contains vertices from V_1 and V_2. Such P_4 is a crossing between V_1 and V_2. A p-connected graph is called *separable* if its vertex set can be partitioned into two sets V_1 and V_2 in such a way that each crossing P_4 has its midpoints in V_1 and its endpoints in V_2.

The *p-connected components* of a graph G are the maximal induced p-connected subgraphs. Vertices of G that do not belong to any p-connected component of G are termed *weak vertices.*

The following theorem gives general structure for arbitrary graphs.

Theorem 1 *[17] For an arbitrary graph G exactly one of the following conditions is satisfied:*

1. *G is disconnected;*
2. *$\overline{G}$ is disconnected;*

3. *There is a unique proper separable p-connected component H of G with a partition (H_1, H_2) such that every vertex outside H is adjacent to all vertices in H_1 and to no vertex in H_2;*

4. *G is p-connected.*

This theorem implies a decomposition scheme for arbitrary graphs called *primeval decomposition*, that refines the modular decomposition in the following way. If M is a neighborhood module of G, then $G[M]$ is either a p-connected graph or it can be decomposed according to condition 3 of Theorem 1 and M, in this last case, is called a *decomposable neighborhood module*. Similarly we say that $G[M]$ is a *decomposable neighborhood graph*. The primeval decomposition tree $T(G)$ of the graph G is a unique labelled tree associated with the primeval decomposition of G in which the leaves of $T(G)$ are either the p-connected components or the weak vertices of G and an internal node is labelled by **P** for parallel module, or **S** for serial module, or **DN** for decomposable neighborhood module.

For separable p-connected components the following theorems hold:

Theorem 2 *[17] If a p-connected graph is separable then its partition is unique.*

Theorem 3 *[17] A p-connected graph is separable if and only if its representative graph is a split graph.*

Note that as a consequence of the above theorems, the partition (K, S) of the split representative graph of a separable p-connected graph is unique.

The theorem 3 implies the following:

Lemma 1 *The representative graph G' of a decomposable neighborhood graph G is a split graph.*

Proof. Let H be the unique proper separable p-connected component of G. Then $G \setminus H$ is a non-empty module of G, by condition 3 of Theorem 1. By Theorem 2 the representative graph H' of H is a split graph with partition (K, S). Then the representative graph G' of G is also a split graph with partition (K', S'), $K' = K$ and $S' = S \cup \{u\}$, where u is the representative vertex of $G \setminus H$ and it is connected to every vertex $v \in K$ and to none of S. ■

Note that by choosing S' of maximum cardinality (as in the proof of Lemma 1), the split partition (K', S') of G' is also unique.

3 Some general results about the well coveredness of graphs

In this section we will give some general results about the well coveredness of graphs and present an algorithm that decides, in linear time, if a graph is well covered under the hypothesis that all its P_4-connected components are separable.

The next lemma plays an important role in the following results.

Lemma 2 *If G is a well covered graph, then each module M of G is well covered.*

Proof. Assume G is a well covered graph. If the module M is a single vertex, then M is trivially well covered. Suppose on the contrary that there exists a module M that is not well covered. Then there exist maximal independent sets I_1 and I_2 in M with $|I_1| \neq |I_2|$. Let us consider the set $V' = V \setminus (M \cup N(v))$, where v is any vertex of M. If $V' = \emptyset$, then every vertex of $G[V \setminus M]$ is adjacent to v and therefore to every other vertex of M, by definition of module. Then I_1 and I_2 are maximal independent sets of different cardinalities in G. This contradicts the well coveredness of G. If $V' \neq \emptyset$, and I is a maximal independent set of $G[V']$, by definition of module $I \cup I_1$ and $I \cup I_2$ are maximal independent sets of different cardinalities in G. This contradicts the well coveredness of G. ■

In the following we will associate to each vertex t_i of the representative graph $G(t)$ a weight $w(t_i)$ equal to the independence number of its corresponding module $M(t_i)$. If I is an independent set of $G(t)$ we will denote $w(I) = \sum_{t_i \in I} w(t_i)$ the weight of set I.

Lemma 3 *A graph G is well covered if and only if all its strong maximal modules are well covered and all the maximal independent sets of its weighted representative graph G' have the same weight.*

Proof. If a strong maximal module of G is not well covered, then G is not well covered by Lemma 2. Let us assume that every strong maximal module of G is well covered. Therefore, if a vertex v of G' has weight $w(v)$, all the maximal independent sets of the corresponding module of G have cardinality $w(v)$. Let I be a maximal independent set of G'. By definition of module, it is easy to see that by replacing each vertex $v \in I$ with any maximal independent set of the corresponding module of G, we obtain a maximal independent set of G. Moreover, all these sets have cardinality $w(I)$. Now, consider all the maximal independent sets of G'. If all of them have the same weight then all the maximal independent sets of G have the same cardinality and G is well covered. Therefore G is well covered if and only if all its strong maximal modules are well covered and all maximal independent sets of G' have the same weight. ■

Theorem 4 *A parallel graph G is well covered if and only if every maximal strong module of G is well covered. The independence number of G is the sum of the independence number of its maximal strong modules.*

Proof. Let $G = G_1 \cup G_2 \cup \ldots \cup G_p$, $p \geq 2$, be a parallel graph. The representative graph G' of G is an edgeless graph with p vertices. Since there is a unique independent set in G', then if every maximal strong module of G is well covered, by Lemma 3, G is well covered. Moreover, the independence number of G is clearly the sum of the independence number of its maximal strong modules. ■

Theorem 5 *A serial graph G is well covered if and only if every maximal strong module of G is well covered and all of them have the same independence number. The independence number of G is the same of the independence number of its maximal strong modules.*

Proof. Let $G = G_1 + G_2 + \ldots + G_p$, $p \geq 2$, be a serial graph. Its representative graph G' is isomorphic to K_p, where each vertex v of G' has weight $w(v)$ equal to the independence number of the corresponding maximal strong module. Every vertex of K_p is a maximal independent set. Therefore, by Lemma 3, G is well covered if and only if every maximal strong module of G is well covered and all of them have the same independence number. ■

In the next theorem we will consider a separable p-connected graph G. Recall that its corresponding representative graph G' is a split graph with a unique partition (K, S), by theorems 2 and 3.

Theorem 6 *Let G be a separable p-connected graph. Let G' be its corresponding weighted representative split graph with partition (K, S). G is well covered if and only if every maximal strong module of G is well covered and, in G', $w(v) = \sum_{z \in N(v) \cap S} w(z)$ for every $v \in K$. Furthermore the independence number of G is equal to $\sum_{u \in S} w(u)$.*

Proof. Let G be a separable p-connected graph and G' its corresponding weighted representative split graph with partition (K, S). By Theorem 2, G' is a split graph with a unique partition (K, S). Then the maximal independent sets of G' are either S or the sets $(S \cup \{v\}) \setminus (N(v) \cap S)$, where v is any vertex of K. The maximal independent sets of G' have weight either $\sum_{u \in S} w(u)$ or $w(v) + \sum_{u \in S} w(u) - \sum_{z \in N(v) \cap S} w(z)$. Therefore, by Lemma 3, G is well covered if and only if all its maximal strong modules are well covered and for each $v \in K$ we have $w(v) = \sum_{z \in N(v) \cap S} w(z)$. Thus the independence number of G is equal to $\sum_{u \in S} w(u)$. ■

Theorem 7 *Let G be a decomposable neighborhood graph. Let G' be the weighted representative split graph of G with partition (K', S'). G is well covered if and only if every maximal strong module of G is well covered and, in G', $w(v) = \sum_{z \in N(v) \cap S'} w(z)$ for every $v \in K'$. Furthermore the independence number of G is equal to $\sum_{u \in S'} w(u)$.*

Proof. The proof is the same as in the previous theorem. ■

Theorems 6 and 7 imply the following result.

Theorem 8 *If G is a well covered decomposable neighborhood graph and H the unique proper separable p-connected component of G, then H is not well covered.*

Proof. Let G be a well covered decomposable neighborhood graph and let H be the proper separable p-connected component of G. The representative graph of H is a split graph H', with partition (K, S) by Theorem 2. The representative graph of G, by Lemma 1, is also a split graph G', with partition $(K, S \cup \{u\})$, where u is the representative vertex of $G \setminus H$ and it is connected to every vertex $v \in K$ and to none of S. Let us assume that H is also well covered. Then, by Lemma 3 and Theorem 6, every maximal independent set of H' has weight equal to $\sum_{z \in S} w(z)$. Since every vertex $v \in K$ is connected to the vertex u, in G', the set $(\{v\} \cup S) \setminus (N(v) \cap S))$ is a maximal independent set of both H' and G'. But if G is well covered, by Lemma 3 and Theorem 7, all the maximal independent sets of G' have weight $w(u) + \sum_{z \in S} w(z)$, a contradiction. ■

4 The well coveredness of graphs that have only separable p-connected components

Now using the results of the last section, we describe an algorithm for deciding if a graph G is well covered under the hypothesis that every p-connected component of G is separable. In the affirmative case, the algorithm will also compute the independence number of G.

Our algorithm is based on the modular decomposition of G. Let $M_1, M_2, \ldots, M_k$ be the maximal strong modules of G. If G is a parallel graph, then by Theorem 4, G is well covered if and only if each $G[M_i]$ is well covered. Next, for each module M_i, if $G[M_i]$ is a serial graph, then by Theorem 5, $G[M_i]$ is well covered if and only if each module of $G[M_i]$ is well covered and all of them have the same independence number. If both $G[M_i]$ and $\overline{G}[M_i]$ are connected, then $G[M_i]$ is either a decomposable neighborhood graph or a p-connected graph by Theorem 1. Under the hypothesis that every p-connected components of G is separable, the representative graph of $G[M_i]$ is a split graph by Theorem 2 and Lemma 1. Then it is enough to verify if the representative graph of $G[M_i]$ is a split graph. In the negative case, G contains a p-connected components which is not separable. In the affirmative case, we consider the weighted representative graph of $G[M_i]$. By theorems 6 and 7, $G[M_i]$ is well covered if and only if if every maximal strong module of $G[M_i]$ is well covered and, for every $v \in K$, we have $w(v) = \sum_{u \in N(v) \cap S} w(u)$. The independence number of $G[M_i]$ is equal to $\sum_{u \in S} w(u)$. For each new submodule, we can now proceed recursively.

We can now give a formal description of a recursive Boolean procedure for deciding if a graph G is well covered, under the hypothesis that its p-connected components are separable and, in the affirmative case, computing its independence number. As we have already pointed out, our algorithm is based on the modular decomposition tree $T(G)$ of G. For each node t in $T(G)$, we denote by $ch(t)$ the set of its children. Every internal node t of $T(G)$ is labelled by **P** for parallel module, or **S** for serial module, or **N** for neighborhood module and every leaf has $label(t) \in \{1, 2, \ldots, n\}$, where $n = |V(G)|$. For each node t the algorithm computes $s(t)$ that is the minimum label of the leaves of the subtree of $T(G)$ rooted at node t and the independence number $\alpha(t)$ of $G[M(t)]$.

procedure WellCovered$(G, T(G), t, \alpha(t), s(t))$
Input: the graph G and its modular decomposition tree $T(G)$ labelled as above.
Output: TRUE and the independence number $\alpha(G)$ of G, if G is well covered, or FALSE otherwise.

if t is a leaf, then $\alpha(t) = 1$ and $s(t) = label(t)$
else

if $label(t) = P$ or $label(t) = S$, then
for $i = 1, \ldots, |ch(t)|$ do
WellCovered$(G[M(t_i)], T(G[M(t_i)]), t_i, \alpha(t_i), s(t_i))$
if $label(t) = P$, then
$\alpha(t) = \sum_{i=1,\ldots,|ch(t)|} \alpha(t_i)$ and $s(t) = \min_{i=1,\ldots,|ch(t)|} s(t_i)$
else
if $label(t) = S$ and $\alpha(t_1) = \alpha(t_2) = \cdots = \alpha(t_{|ch(t)|})$, then $\alpha(t) = \alpha(t_1)$
else return FALSE and STOP
else ($label(t) = N$)
Construct the weighted representative graph $G'(t)$ with
$V(G'(t)) = \{s(t_i)\}_{i=1,\ldots,|ch(t)|}\}$
if $G'(t)$ is not a split graph, then
return FALSE and STOP (G does not satisfy the hypothesis)
else
find the partition K and S of $V(G'(t))$
if $w(s(t_i)) = \sum_{s(t_j) \in N(s(t_i)) \cap S} w(s(t_j))$ for every $s(t_i) \in K$, then
$\alpha(t) = \sum_{s(t_j) \in S} w(s(t_j))$ and $s(t) = \min_{i=1,\ldots,|ch(t)|} s(t_i)$
else return FALSE and STOP
return TRUE and $\alpha(t)$.

Theorem 9 *The algorithm to decide well coveredness of a graph G, whose p-connected components are separable, is correct and runs in linear time.*

Proof. The correctness follows by the above discussion. Let $n = |V(G)|$ and $m = |E(G)|$. The modular decomposition tree $T(G)$ can be constructed in $O(n+m)$ and $|V(T(G))| < 2n$. The traversal of $T(G)$ and the computation of $\alpha(t)$ and $s(t)$ for each node t of $T(G)$ can be done in $O(|V(T(G))|)$. Constructing the representative graphs of all the N-nodes of G can be done in $O(n + m)$. In fact the total number number of edges of all representative graphs is less or equal than m and the total number of vertices is O(n), since the vertex set of the representative graph of two different N-nodes have at most one vertex in common. Finally verifying that all of them are split graphs can also be done in $O(n + m)$ time [18]. Therefore the whole algorithm runs in linear time. ■

5 The well coveredness of graphs with few P_4's

In the last section, we have described an algorithm that given a graph G with the property that all p-connected components are separable, it decides if G is well covered. Now we are going to consider special classes of graphs which are obtained by forbidding in the primeval decomposition the presence of p-connected components or restricting the p-connected components, not necessarely separable, to be some particular graphs.

The graphs we shall consider are graphs which in a local sense contain only a restricted number of P_4's, and they have been extensively studied.

It all started with the class of *cographs*, which is a class of graphs where no P_4 is allowed to exist. In particular they are characterized by having only serial and parallel nodes in their modular decomposition tree. These graphs have been investigated independently by many authors and many nice structural properties are known [5, 6], which have motivated researchers to define classes of graphs obtained as extension of cographs.

Hoáng [12] introduced the class of P_4-sparse graphs, which is the class such that no set of five vertices induces more than one P_4. Its p-connected components are spiders [16].

Jamison and Olariu [14, 15, 13] introduced the class of P_4-reducible graphs, P_4-extendible and P_4-lite. The P_4-reducible graphs are the graphs such that no vertex belongs to more than one P_4, and its p-connected components are P_4's. The P_4-extendible are graphs where each p-connected component consists of at most five vertices. Each p-connected component is either P_5 or $\overline{P_5}$ or C_5, or P_4 with one vertex eventually substituted by a homogeneous set with cardinality two. The P_4-lite are graphs such that every induced subgraph with at most six vertices either contains at most two P_4's or is isomorphic to a spider. The p-connected components of a P_4-lite graph are either a spider (possibly with one vertex replaced by a homogeneous set of cardinality 2) or one of the graphs P_5, $\overline{P_5}$.

Giakoumakis and Vanherpe [10] studied structural and algorithmic properties of *extended P_4-reducible* and *extended P_4-sparse* graphs. These classes are obtained from P_4-reducible and P_4-sparse graphs, respectively, by also allowing C_5's as p-connected components.

Another generalization of the previously mentioned graph classes are the *P_4-tidy graphs*. They were studied by Giakoumakis et al. in [11]. A graph is P_4-tidy if for every P_4 there exists at most one vertex outside which, together with three of its vertices, induces a P_4. The structure of the p-connected components of P_4-tidy graphs can be described as follows:

Theorem 10 *[11] A p-connected component of a P_4-tidy graph is either isomorphic to a spider (possibly with one vertex replaced by a homogeneous set of cardinality 2) or to one of the graphs P_5, $\overline{P_5}$ and C_5.*

The P_4-tidy graphs strictly contain the classes of cographs, P_4-reducible, P_4-sparse, P_4-extendible, extended P_4-reducible, extended P_4-sparse, and P_4-lite graphs.

We recall from [11] the Figure 1 that shows the previous classes partially ordered by inclusion.

Theorem 11 *A p-connected tidy graph G is well covered if and only if it is a C_5, or $\overline{P_5}$, or a thin spider with a vertex possibly substituted by a K_2.*

Proof. The permitted p-connected tidy graphs by Theorem 10 are either isomorphic to a spider (possibly with one vertex replaced by a homogeneous set of cardinality 2) or to one of the graphs P_5, $\overline{P_5}$ or a C_5. Clearly C_5 and $\overline{P_5}$ are well covered while P_5 is not. If G is a spider with at most one vertex possibly replaced by a homogeneous set of cardinality 2, it is easy to see that the conditions for well coveredness of Theorem 6 are verified if and only if G is a thin spider with a vertex possibly substituted by a K_2. ∎

Theorem 12 *Every decomposable neighborhood P_4-tidy graph G is not well covered.*

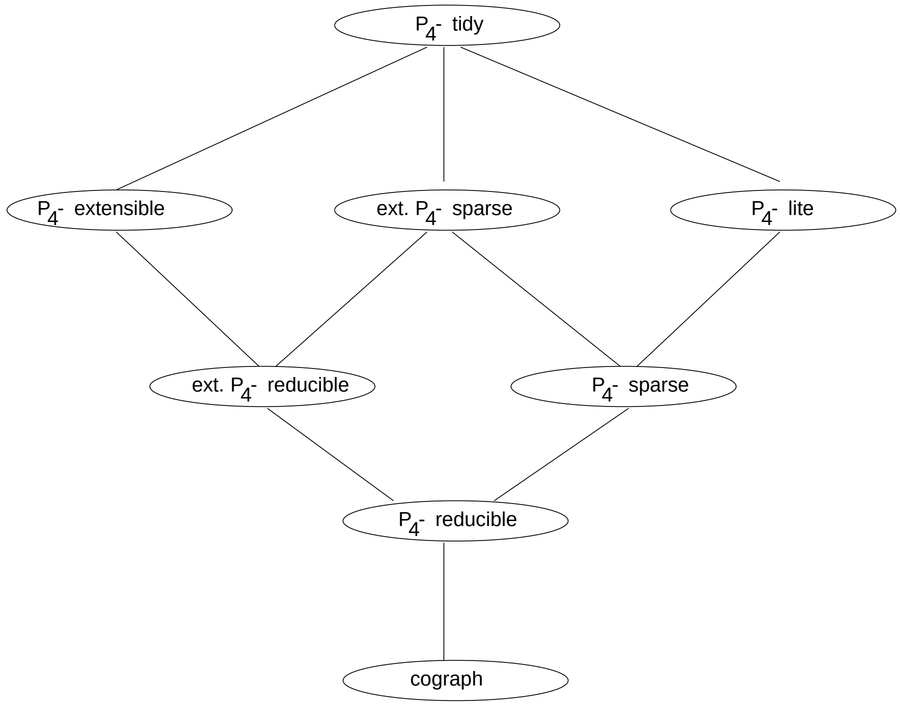

Figure 1: A Hasse diagram.

Proof. The only possible separable p-connected component H is a spider with at most one vertex possibly replaced by a homogeneous set of cardinality 2. If H is a thin spider with a vertex possibly substituted by a K_2 then H is well covered and G is not well covered by Theorem 8. In the remaining cases it is easy to verify that the conditions for well coveredness of Theorem 7 are never satisfied. ∎

Theorem 13 *Let G be a P_4-tidy graph G and let $T(G)$ be its unique primeval decomposition tree. Then G is well covered if and only if in $T(G)$ the following conditions hold:*

(i) *every strong maximal submodule of a parallel module of $T(G)$ is well covered,*

(ii) *every strong maximal submodule of a serial module of $T(G)$ is well covered and all of them have the same independence number,*

(iii) *$T(G)$ does not contain any strong maximal decomposable neighborhood module,*

(iv) *$T(G)$ contains only p-connected components isomorphic to C_5, or $\overline{P_5}$, or a thin spider with a vertex possibly substituted by a K_2.*

Proof. It follows by Lemma 2 and Theorems 1, 4, 5, 11, 12. ∎

We can now describe an algorithm that decides if a P_4-tidy graph G is well covered and, in the affirmative case, computes its independence number. Our algorithm is based on the primeval decomposition tree $T(G)$ of G. For each node t in $T(G)$, we shall denote by $ch(t)$ the set of its children. Associated to every node t of $T(G)$ there is a label as follows:

$$label(t) = \begin{cases} P \text{ for parallel node} \\ S \text{ for serial node} \\ ND \text{ for decomposable neighborhood node} \\ W \text{ for weak vertex} \\ P_5 \text{ for the } p\text{-connected component } P_5 \\ P_5C \text{ for the } p\text{-connected component } \overline{P_5} \\ C_5 \text{ for the } p\text{-connected component } C_5 \\ SP(SPC) \text{ for the thin (thick) spider} \\ SP1(SPC1) \text{ for the thin (thick) spider with a vertex replaced by } K_2 \\ SP2(SPC2) \text{ for the thin (thick) spider with a vertex replaced by } \overline{K_2}. \end{cases}$$

Associated to every leaf of $T(G)$ there is also the number of vertices n_{pc} of the corresponding p-connected component or 1 if it is a weak vertex.

A description of the recursive procedure is given below:

procedure WellCovered$(T(G), t, \alpha(t))$
Input: the primeval decomposition tree T of a P_4-tidy graph G, labelled as above.
Output: FALSE if G is not well covered or TRUE otherwise. In the last case $\alpha(t)$ contains the independence number of G.

if $label(t) = ND$, then return FALSE and STOP
else
  if $label(t) = P$ or $label(t) = S$, then
    for $i = 1, \ldots, |ch(t)|$ do WellCovered$(T(G[M(t_i)]), t_i, \alpha(t_i))$
    if $label(t) = P$, then $\alpha(t) = \alpha(t_1) + \alpha(t_2) + \cdots + \alpha(t_{|ch(t)|})$
    else
    if $label(t) = S$ and $\alpha(t_1) = \alpha(t_2) = \cdots = \alpha(t_{|ch(t)|})$, then $\alpha(t) = \alpha(t_1)$
    else return FALSE and STOP
  else
    if $label(t) = W$, then $\alpha(t) = 1$
    else if $label(t) = P_5C$ or $label(t) = C_5$, then $\alpha(t) = 2$
      else if $label(t) = SP$ or $label(t) = SP1$, then $\alpha(t) = \lfloor \frac{n_{pc}}{2} \rfloor$
        else return FALSE and STOP
return TRUE and $\alpha(t)$.

Theorem 14 *The algorithm to decide if a P_4-tidy graph is well covered is correct and runs in linear time.*

Proof. The correctness follows by Theorem 13. Let $n = |V(G)|$ and $m = |E(G)|$. The primeval decomposition tree $T(G)$ can be constructed in $O(n+m)$ and $|V(T(G))| < 2n$. Verifying if each leaf of $T(G)$ is a permitted p-connected component can be done in $O(n)$. Finally the traversal of $T(G)$ can be done in $O(|V(T(G))|)$. Therefore the algorithm runs in linear time. ■

Finally we would like to observe that the above algorithm can be used to decide if a graph belonging to any of the classes mentioned in this section is well covered. In fact, the primeval decomposition tree of graphs belonging to any of those classes is characterized either by the absence of p-connected components (the class of cographs) or by the absence of some type of p-connected components that are present in a P_4-tidy graph. In particular, we can also apply the algorithm of Section 3 for cographs (which have no p-connected components) and for P_4-sparse graphs and P_4-reducible graphs (which have only separable p-connected components).

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