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Matching Signatures and Pfaffian Graphs

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Abstract

We prove that every 4-Pfaffian that is not Pfaffian essentially has a unique signature matrix. We also give a simple composition Theorem of $2r$-Pfaffian graphs from r Pfaffian spanning subgraphs. We apply these results and exhibit a graph that is 6-Pfaffian but not 4-Pfaffian. This is a counter-example to a conjecture of Norine [5], which states that if a graph G is k-Pfaffian but not $(k-1)$-Pfaffian then k is a power of four.

1 Introduction

Let G be a graph. Let $\{1, 2, \ldots, n\}$ be the set of vertices of G. For u and v adjacent vertices of G, we denote the edge joining u and v by uv or vu. Let D be an orientation of G. If D has an edge directed from u to v then we denote that directed edge by uv. Let $M := \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$ be a perfect matching of D. Then the *sign of M in D*, denoted $\mathrm{sgn}(M, D)$, is the sign of the permutation

$$\pi_D(M) := \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_k & v_k \end{pmatrix}.$$

A change in the order of the enumeration of the edges of the perfect matching changes the number of inversions by an even number. Therefore, the sign of the permutation remains unchanged. We conclude that the sign of a perfect matching is well-defined.

Let k be a positive integer, let $\boldsymbol{D} := (D_1, D_2, \ldots, D_k)$ be a k-tuple of orientations of G. We say that $\boldsymbol{D}$ is a *k-orientation* of G. For each perfect matching M of G, we may consider the k-tuple

$$\mathbf{sgn}(M, \boldsymbol{D}) := (\mathrm{sgn}(M, D_1), \mathrm{sgn}(M, D_2), \ldots, \mathrm{sgn}(M, D_k)),$$

called the *signature vector of M relative to $\boldsymbol{D}$*. We denote by $\mathcal{M}(G)$, or simply $\mathcal{M}$, if G is understood, the set of perfect matchings of G. The *signature matrix of $\mathcal{M}$ relative to $\boldsymbol{D}$* is the matrix

$$\mathbf{sgn}(\mathcal{M}, \boldsymbol{D}) := (\mathbf{sgn}(M, \boldsymbol{D}) : M \in \mathcal{M}).$$

If the system $\mathbf{sgn}(\mathcal{M}, \boldsymbol{D})\ \boldsymbol{x} = \mathbf{1}$ has a solution then we say that $\boldsymbol{D}$ is *Pfaffian* and, for any solution $\boldsymbol{\alpha}$ of that system, we say that $(\boldsymbol{D}, \boldsymbol{\alpha})$ is a *Pfaffian k-pair*. We say that G

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is *k-Pfaffian* if it has a Pfaffian k-orientation. We remark that relabelling the vertices of graph G either changes the signs of all perfect matchings relative to D or does not change the sign of any perfect matching of G relative to D. Consequently, the property of G being k-Pfaffian does not depend on the particular enumeration of the vertices of G. We define the *Pfaffian number* of a graph G, denoted $\mathrm{pf}(G)$, to be the minimum k such that G is k-Pfaffian. Galluccio and Loebl [1] and, independently, Tesler [6], proved the following remarkable result:

Theorem 1.1
If G is embedable on an orientable surface of genus g then $\mathrm{pf}(G) \leq 4^g$.

In 2008, Norine [5] stated the following conjecture:

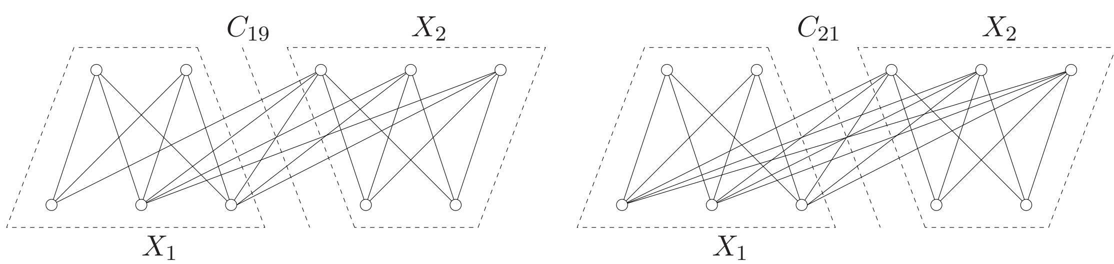
Conjecture 1.2
The Pfaffian number of a graph is always a power of four.

In fact, in 1967, Kasteleyn [2, page 99] stated a similar belief: "If the genus of the graph is g the number of Pfaffians required is 4^g".

Norine proved the following result:

Theorem 1.3 (Norine [5])
Every 3-Pfaffian graph is Pfaffian and every 5-Pfaffian graph is 4-Pfaffian.

By Theorem 1.3, a counter-example to Conjecture 1.2 must have Pfaffian number six or more. In this paper we show that graph G_{19}, depicted in Figure 1(a), is 6-Pfaffian, but not 4-Pfaffian.



(a) G_{19} (b) G_{21}

Figure 1: Graphs G_{19} and G_{21}.

Let us describe how graph G_{19} may be obtained. Figure 1(b) shows graph G_{21}. This graph is obtained from two disjoint copies, G_1 and G_2, of $K_{3,2}$, with sets of vertices X_1 and X_2, by joining every vertex of the majority part of G_1 to each vertex of the majority part of G_2. Those added edges span a $K_{3,3}$ and constitute a tight cut of G_{21}, which we denote by C_{21}. Graph G_{19} is obtained from G_{21} by removing two adjacent edges of C_{21}: the resulting tight cut is denoted C_{19}.

We now give a brief description of the outline of the proof. For every graph G, let $\mathcal{M}(G)$ denote the set of perfect matchings of G. Let r be a positive integer, $G_1, G_2, \ldots, G_r$ Pfaffian spanning subgraphs of G. We say that $G_1, G_2, \ldots, G_r$ is an *r-decomposition* of a graph G if there are r sets $S_1, S_2, \ldots, S_r$ of edges of G such that:

- $\{\mathcal{M}(G_i) : i = 1, 2, \ldots r\}$ is a partition of $\mathcal{M}(G)$, and
- for each perfect matching M of G, $|M \cap S_i|$ is odd if and only if $M \in \mathcal{M}(G_i)$.

In Section 2 we prove that G_{21} is 6-Pfaffian and also that $G_{19} - e$ is 4-Pfaffian, for every edge e of G_{19}. We do this by showing the following fundamental result:

Theorem 2.7 (Composition)
If a graph has an r-decomposition then it is $2r$-Pfaffian.

We show that a graph obtained from G_{21} by removing six edges from C_{21} so that the resulting tight cut spans a P_4 is Pfaffian. From this and Theorem 2.7 it follows that G_{21} is 6-Pfaffian, because it is possible to cover the edges of $K_{3,3}$ with three P_4's. It also follows that for every edge e in C_{19}, $G_{19} - e$ is 4-Pfaffian, because it is possible to cover $K_{3,3}$ minus any three edges with two P_4's. We also show that $G_{21} - e$ is 4-Pfaffian for any edge e not in C_{21}. This establishes the fact that $G_{19} - e$ is 4-Pfaffian, for every edge e. In sum, we prove that G_{19} is 6-Pfaffian, and if not 4-Pfaffian, then it is a minimal non-4-Pfaffian graph.

In Section 3 we prove that G_{19} is not 4-Pfaffian. For this, we derive the following fundamental result, which says that the signature matrix associated with a normal Pfaffian 4-pair of a non-Pfaffian graph is essentially unique:

Theorem 3.2 (Uniqueness of Signature Matrices)
Let G be a non-Pfaffian graph, $(\boldsymbol{D}, \boldsymbol{\alpha})$ a normal Pfaffian 4-pair of G. Then, $\boldsymbol{\alpha} = \mathbf{1/2}$.

We prove that G_{19} cannot possibly satisfy the property stated in Theorem 3.2. We deduce that G_{19} is 6-Pfaffian and minimal non-4-Pfaffian. Indeed, we believe that G_{19} is the smallest counter-example to Conjecture 1.2.

2 Composition of Pfaffian Graphs

In this section we prove that graph G_{21} is 6-Pfaffian. Consequently, G_{19} is also 6-Pfaffian. We also prove that $G_{19} - e$ is 4-Pfaffian, for every edge e. For this we need to establish some pre-requisites. First we establish a relation involving the Pfaffian numbers of the two C-contractions of a graph G and the Pfaffian number of G, where C is a tight cut of G. We then prove the Composition Theorem. This is a fundamental result that is used to prove that G_{21} is 6-Pfaffian and also that $G_{19} - e$ is 4-Pfaffian.

2.1 Edge Cuts, Similarity, Normal Pairs

Let G be a graph, X a set of vertices of G. We denote by $\partial(X)$ the (edge-)cut C consisting of those edges having one end in X, the other end in $\overline{X}$. The sets X and $\overline{X}$ are called *shores* of C. We say that two orientations D and D' of G are *similar* if the set of edges of G on which D and D' disagree is a cut of G. We say that two k-orientations $\boldsymbol{D}$ and $\boldsymbol{D'}$ of G are *similar* if there is a permutation f on $\{1, 2, \ldots, k\}$ such that $\boldsymbol{D}_i$ and $\boldsymbol{D}_{f(i)}$ are similar, for $i = 1, 2, \ldots, k$. For a directed graph D and a subset S of $E(D)$, let $D \otimes S$ denote the directed graph obtained from D by the reversal of the edges of S. The proof of the following result is straightforward:

Lemma 2.1
Let D be a directed graph, M a perfect matching of D, and $C := \partial(X)$ a cut of D. Then, $\mathrm{sgn}(M, D) = \mathrm{sgn}(M, D \otimes C)$ *if and only if $|X|$ is even.*

Corollary 2.2
Let $\boldsymbol{D}$ and $\boldsymbol{D'}$ be two similar k-orientations of G. Then, $\boldsymbol{D}$ is Pfaffian if and only if $\boldsymbol{D'}$ is Pfaffian.

Corollary 2.3
Let $(\boldsymbol{D}, \boldsymbol{\alpha})$ be a Pfaffian k-pair of a graph G. Then, G has a Pfaffian k-pair $(\boldsymbol{D'}, \boldsymbol{\alpha'})$ such that $\boldsymbol{D}$ and $\boldsymbol{D'}$ are similar and $\boldsymbol{\alpha}'_i = |\boldsymbol{\alpha}_i|$ for $i = 1, 2, \ldots, k$.

A Pfaffian k-pair $(\boldsymbol{D}, \boldsymbol{\alpha})$ of G is *normal* if $\boldsymbol{\alpha} > \mathbf{0}$.

Corollary 2.4
Every graph G has a normal Pfaffian $\mathrm{pf}(G)$*-pair.*

2.2 Cut Contractions and Tight Cuts

The graph obtained from X by contracting X to a single new vertex x and by removing any resulting loops is denoted $G/X \to x$. The graphs $G/X \to x$ and $G/\overline{X} \to \overline{x}$ are called *C-contractions of G*. Assume further that G has a perfect matching. Cut C is *tight in G* if every perfect matching of G has precisely one edge in C. Little and Rendl [3] proved the following important result:

Theorem 2.5
Let C be a tight cut of a graph G. Then, G is Pfaffian if and only if both C-contractions of G are Pfaffian.

From Theorem 2.5 we deduce that if both C-contractions of G are Pfaffian and C is a tight cut then G is also Pfaffian, We need a generalization of this result for k-Pfaffian graphs. Theorem 2.5 does not extend naturally to k-Pfaffian graphs. Indeed, G_{21} is not 4-Pfaffian, yet both C_{21}-contractions of G_{21} are equal to $K_{3,3}$ up to multiple edges, whence 4-Pfaffian.

Theorem 2.6
Let C be a tight cut of a graph G, let G' and G'' denote the two C-contractions of G. Then, $\mathrm{pf}(G) \leq \mathrm{pf}(G') \cdot \mathrm{pf}(G'')$.

<u>Proof</u>: Let X denote the shore of C such that $G' = G/\overline{X} \to \overline{x}$ and $G'' := G/X \to x$. Let $(\boldsymbol{D'}, \boldsymbol{\alpha'})$ be a Pfaffian $\mathrm{pf}(G')$-pair of G'. Adjust notation so that contraction vertex $\overline{x}$ has the highest label. Likewise, let $(\boldsymbol{D''}, \boldsymbol{\alpha''})$ be a Pfaffian $\mathrm{pf}(G'')$-pair of G'', and adjust notation so that contraction vertex x has minimum label, equal to 1.

Let e_1 and e_2 denote any two multiple edges of G'. Denote by $\boldsymbol{D'} - e_2$ the $\mathrm{pf}(G')$-orientation of $G' - e_2$ obtained by deleting the edge e_2 from each orientation of $\boldsymbol{D'}$. The pair $(\boldsymbol{D'} - e_2, \boldsymbol{\alpha'})$ is also a Pfaffian $\mathrm{pf}(G')$-pair. Therefore, an extension of this pair to G', obtained by orienting e_2 in the same direction of e_1 in $\boldsymbol{D'_i} - e_2$, for $i = 1, 2, \ldots, \mathrm{pf}(G')$, is also a Pfaffian $\mathrm{pf}(G')$-pair. So, we may choose $(\boldsymbol{D'}, \boldsymbol{\alpha'})$ such that every pair of multiple edges of G' has the same direction in each orientation of $\boldsymbol{D'}$. These observations imply that, for an orientation $\boldsymbol{D'_i}$ of $\boldsymbol{D'}$, the set S of edges of C directed away from contraction vertex $\overline{x}$ are part of a cut C' disjoint with $C - S$. We reverse the orientation of the edges of cut C' in $\boldsymbol{D'_i}$, thereby obtaining a similar orientation. We may thus assume that in $\boldsymbol{D'_i}$ all the edges of C enter contraction vertex $\overline{x}$. This conclusion holds for $i = 1, 2, \ldots, \mathrm{pf}(G')$. Likewise, we may assume that each edge of cut C leaves x in $\boldsymbol{D''_j}$, for $j = 1, 2, \ldots, \mathrm{pf}(G'')$. Define

$$\boldsymbol{D}_{ij} := \boldsymbol{D'_i} \cup \boldsymbol{D''_j} \quad \text{and} \quad \boldsymbol{\alpha}_{ij} := \boldsymbol{\alpha'_i}\boldsymbol{\alpha''_j} \quad \text{for } i = 1, 2, \ldots, \mathrm{pf}(G') \text{ and } j = 1, 2, \ldots, \mathrm{pf}(G'').$$

Label the vertices of G as follows: the vertices of X inherit their labels from G'; the vertices of $\overline{X}$ inherit their labels from G'', but are increased by $|X| - 1$. This clearly produces a labeling $1, 2, \ldots, |V(G)|$ of G. We assert that under this labeling, $(\boldsymbol{D}, \boldsymbol{\alpha})$ is a Pfaffian $\mathrm{pf}(G')\cdot\mathrm{pf}(G'')$-pair of G. For this, let M be a perfect matching of G. Then, $M' := M \cap E(G')$ is a perfect matching of G' and $M'' := M \cap E(G'')$ is a perfect matching of G''. The number of inversions of the permutation associated with M in $\boldsymbol{D}_{ij}$ is equal to the sum of the number of inversions of the permutations associated with M' in $\boldsymbol{D'_i}$ and M'' in $\boldsymbol{D''_j}$, for $i = 1, 2, \ldots, \mathrm{pf}(G')$ and $j = 1, 2, \ldots, \mathrm{pf}(G'')$. Thus, $\mathrm{sgn}(M, \boldsymbol{D}_{ij}) = \mathrm{sgn}(M, \boldsymbol{D'_i})\cdot\mathrm{sgn}(M, \boldsymbol{D''_j})$. Consequently,

$$\sum_{i,j} \boldsymbol{\alpha}_{ij}\, \mathrm{sgn}(M, \boldsymbol{D}_{ij}) = \sum_i \boldsymbol{\alpha'_i}\, \mathrm{sgn}(M', \boldsymbol{D'_i}) \sum_j \boldsymbol{\alpha''_j}\, \mathrm{sgn}(M'', \boldsymbol{D''_j}) = \sum_i \boldsymbol{\alpha'_i}\, \mathrm{sgn}(M', \boldsymbol{D'_i}) = 1.$$

This conclusion holds for each perfect matching M of G. We deduce that, as asserted, $(\boldsymbol{D}, \boldsymbol{\alpha})$ is a Pfaffian $\mathrm{pf}(G') \cdot \mathrm{pf}(G'')$-pair of G. □

2.3 Composition of Pfaffian Orientations

Theorem 2.7 (Composition)
If a graph has an r-decomposition then it is $2r$-Pfaffian.

<u>Proof</u>: Let G be a graph, assume that G has an r-decomposition. Let $G_1, G_2, \ldots, G_r$ be Pfaffian spanning subgraphs of G, let $S_1, S_2, \ldots, S_r$ be sets of edges of G such that:

- $\{\mathcal{M}(G_i) : i = 1, 2, \ldots r\}$ is a partition of $\mathcal{M}(G)$, and
- for each perfect matching M of G, $|M \cap S_i|$ is odd if and only if $M \in \mathcal{M}(G_i)$.

Let us use the same labeling for each graph G_i and also for graph G. For $i = 1, 2, \ldots, r$, let D_i be a Pfaffian orientation of G_i. Adjust notation, by replacing D_i, if necessary, by $D_i \otimes \partial(v)$, for some vertex v of G, so that every perfect matching of G_i has sign equal to one in D_i. Let D'_i be an arbitrary extension of D_i to an orientation of G. Let $D''_i := D'_i \otimes S_i$. Let

$$\boldsymbol{D} := (D'_1, D'_2, \ldots, D'_r, D''_1, D''_2, \ldots, D''_r),$$
$$\boldsymbol{\alpha}_1 := \boldsymbol{\alpha}_2 := \ldots := \boldsymbol{\alpha}_r = 1/2 \quad \text{and} \quad \boldsymbol{\alpha}_{r+1} := \boldsymbol{\alpha}_{r+2} := \ldots := \boldsymbol{\alpha}_{2r} = -1/2.$$

We assert that $(\boldsymbol{D}, \boldsymbol{\alpha})$ is a Pfaffian $2r$-pair of G. For this, let M be a perfect matching of G. By hypothesis, M is a perfect matching of precisely one of the graphs G_i, say G_k. By hypothesis, $|M \cap S_i|$ is odd if and only if $i = k$. For $i \neq k$, as $|M \cap S_i|$ is even, it follows that M has equal signs in D'_i and in D''_i. Consequently, $\boldsymbol{D} \cdot \boldsymbol{\alpha} = 1/2[\mathrm{sgn}(M, D'_k) - \mathrm{sgn}(M, D''_k)]$. As $|M \cap S_k|$ is odd, we have that

$$\mathrm{sgn}(M, D'_k) = \mathrm{sgn}(M, D_k) = 1 \quad \text{and} \quad \mathrm{sgn}(M, D''_k) = -\mathrm{sgn}(M, D_k) = -1.$$

Consequently, $\boldsymbol{D} \cdot \boldsymbol{\alpha} = 1$. This conclusion holds for each perfect matching M of G. As asserted, G is $2r$-Pfaffian. □

Corollary 2.8
Let C be a tight cut of a graph G. Let $\{C_1, C_2, \ldots, C_r\}$ be a partition of C. Assume that $G_i := G - (C - C_i)$ is Pfaffian, for $i = 1, 2, \ldots, r$. Then, G is $2r$-Pfaffian.

2.4 Graph G_{21} is 6-Pfaffian and graph $G_{19} - e$ is 4-Pfaffian

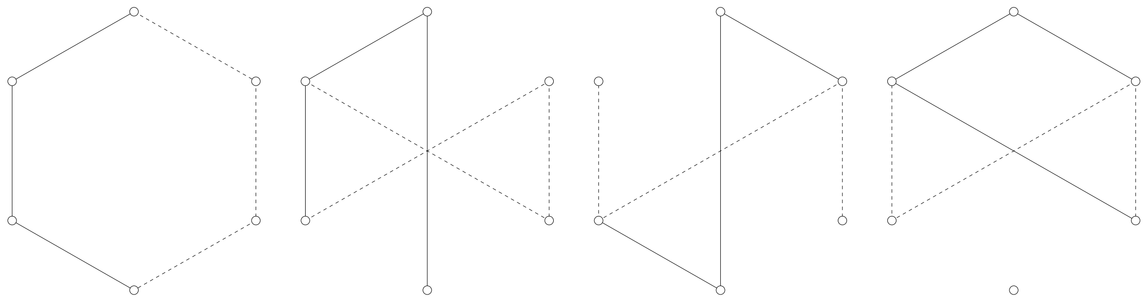
We now derive the next result as a straightforward consequence of Corollary 2.8.

Theorem 2.9
Let R be a (possibly empty) subset of tight cut C_{21} of G_{21}. If $C_{21} - R$ may be covered by r edge-disjoint P_4's then $G_{21} - R$ is $2r$-Pfaffian.

Proof: Assume that $C_{21} - R$ is covered by r P_4's. Let $C_1, \ldots, C_r$ denote the set of edges of the r P_4's. By Corollary 2.8, it suffices to show that $G_i := G_{21} - (C - C_i)$ is Pfaffian, for $i = 1, \ldots, r$. For this, note that the C_i-contractions of G_i are equal to $K_{3,3} - e$, up to multiple edges. As $K_{3,3} - e$ is planar, it is Pfaffian. Therefore, both C_i-contractions of G_i are Pfaffian. Moreover, C_i is tight in G_i. We deduce that G_i is Pfaffian. The assertion holds. □

Theorem 2.10
Graph G_{21} is 6-Pfaffian and graph $G_{19} - e$ is 4-Pfaffian.

Proof: Note that $G_{21}[C_{21}]$ is $K_{3,3}$ (see Figure 1(b)). Let R be any set of three edges of $K_{3,3}$. As indicated in Figure 2, $K_{3,3} - R$ is the union of two P_4's.



(a) R is a matching (b) precisely two edges of R are adjacent (c) R spans P_4 (d) R spans a star

Figure 2: Decomposition of $K_{3,3} - R$ in two P_4's, where R is a set of three edges.

In particular, if R is the set of edges of a P_4 of $K_{3,3}$, we deduce that $K_{3,3}$ is the union of three P_4's. By Theorem 2.9, graph G_{21} is 6-Pfaffian. For any edge e of C_{19}, the cut $C_{19} - e$ spans a $K_{3,3}$ minus three edges. In this case, $G_{19} - e$ is 4-Pfaffian, by Theorem 2.9. Finally, if e is an edge of G_{21} that does not lie in C_{21}, then, up to multiple edges, one of the C_{21}-contractions of $G_{21} - e$ is $K_{3,3} - e$, the other C-contraction is $K_{3,3}$. As $K_{3,3}$ is 4-Pfaffian and $K_{3,3} - e$ is Pfaffian, it follows that $G_{21} - e$ is 4-Pfaffian. We deduce that $G_{19} - e$ is also 4-Pfaffian. This conclusion holds for each edge e of G_{19}. The assertion holds. □

3 4-Pfaffian Graphs

In this section we prove that graph G_{19} is not 4-Pfaffian. We do this by proving Theorem 3.2, and then applying it to G_{19} to show, by contradiction, that it is not 4-Pfaffian.

3.1 Uniqueness of Signature Matrices

In order to prove Theorem 3.2, we need the following result, due to Norine [5]:

Lemma 3.1
Let G be a graph, $(D_1, D_2, \ldots, D_r)$ a family of orientations of G, r odd. Then, there is an orientation D of G such that the inequality below holds for every perfect matching M of G:

$$\mathrm{sgn}(M, D) = \mathrm{sgn}(M, D_1)\,\mathrm{sgn}(M, D_2)\ \ldots\ \mathrm{sgn}(M, D_r).$$

Theorem 3.2 (Uniqueness of Signature Matrices)
Let G be a non-Pfaffian graph, $(\boldsymbol{D}, \boldsymbol{\alpha})$ a normal Pfaffian 4-pair of G. Then, $\boldsymbol{\alpha} = \mathbf{1/2}$.

<u>Proof</u>: Consider the signature matrix $\boldsymbol{S} := \mathbf{sgn}(\mathcal{M}, \boldsymbol{D})$ of $\boldsymbol{D}$. Then, $\boldsymbol{S} \cdot \boldsymbol{\alpha} = \mathbf{1}$.

Lemma 3.3
For any two distinct rows $\boldsymbol{S}_i$ and $\boldsymbol{S}_j$ of $\boldsymbol{S}$, there is a column $\boldsymbol{S}_\ell$ and a column $\boldsymbol{S}_m$ of $\boldsymbol{S}$ such that $\boldsymbol{S}_{i\ell} = 1$, $\boldsymbol{S}_{im} = -1$, $\boldsymbol{S}_{j\ell} = 1$ and $\boldsymbol{S}_{jm} = 1$.

Proof: Clearly, $(\boldsymbol{S}_i + \boldsymbol{S}_j) \cdot \boldsymbol{\alpha} = 2$. By hypothesis, $\boldsymbol{\alpha} > \boldsymbol{0}$. Therefore, $\boldsymbol{S}_i + \boldsymbol{S}_j$ must have a positive entry. We deduce that there is a column ℓ such that $\boldsymbol{S}_{i\ell} = 1$ and $\boldsymbol{S}_{j\ell} = 1$. Likewise, $(\boldsymbol{S}_i - \boldsymbol{S}_j) \cdot \boldsymbol{\alpha} = 0$. By hypothesis, $\boldsymbol{S}_i$ and $\boldsymbol{S}_j$ are distinct. Therefore, $\boldsymbol{S}_i - \boldsymbol{S}_j$ must have a negative entry. We deduce that there is a column m such that $\boldsymbol{S}_{im} = -1$ and $\boldsymbol{S}_{jm} = 1$. □

By hypothesis, G is not Pfaffian. Therefore, $\boldsymbol{S}$ must have two distinct rows. By Lemma 3.3, every row of $\boldsymbol{S}$ must have two entries equal to 1 and at least one entry equal to -1. We may thus call a row of $\boldsymbol{S}$ *single* if it has precisely one entry equal to -1, *double* otherwise.

We assert that $\boldsymbol{S}$ has at least one single row. For this, note first that there are at most $\binom{4}{2} = 6$ distinct double rows. But those six types appear in three complementary pairs. By Lemma 3.3, no two members of a complementary pair may occur in $\boldsymbol{S}$. We deduce that $\boldsymbol{S}$ has at most three distinct double rows. By hypothesis, G is not Pfaffian (and consequently not 3-Pfaffian, by Theorem 1.3), therefore for every solution $\boldsymbol{x}$ of $\mathbf{Sx} = \mathbf{1}$, no entry of $\boldsymbol{x}$ is equal to zero. Thus, the rank of matrix $\boldsymbol{S}$ is equal to four. We deduce that $\boldsymbol{S}$ must have at least one single row, as asserted. Adjust notation so that (i) row $\boldsymbol{S}_1$ is single and (ii) $\boldsymbol{S}_{11}$ is its single entry equal to -1. By Lemma 3.3, for every row $\boldsymbol{S}_i$ distinct from $\boldsymbol{S}_1$, its entry $\boldsymbol{S}_{i1}$ is equal to 1.

We now prove that $\boldsymbol{S}$ must have two distinct single rows. For this, assume, to the contrary, that any single row of $\boldsymbol{S}$ is equal to $\boldsymbol{S}_1$. Then, for any row $\boldsymbol{S}_i$ we have $P_i := \boldsymbol{S}_{i2}\boldsymbol{S}_{i3}\boldsymbol{S}_{i4} = 1$. By Lemma 3.1, G has an orientation D that is Pfaffian, a contradiction. As asserted, $\boldsymbol{S}$ has two distinct single rows. Adjust notation so that (i) row $\boldsymbol{S}_2$ is single and (ii) $\boldsymbol{S}_{22}$ is its single entry equal to -1. By Lemma 3.3, for every row $\boldsymbol{S}_i$ distinct from $\boldsymbol{S}_2$, its entry $\boldsymbol{S}_{i2}$ is equal to 1.

We now prove that $\boldsymbol{S}$ has three distinct single rows. For this, note that any double row of $\boldsymbol{S}$ must have the two negative entries in columns 3 and 4 and is thus unique. The rank of $\boldsymbol{S}$ is equal to four. Therefore, $\boldsymbol{S}$ must have a single row distinct from both $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$. Adjust notation so that (i) row $\boldsymbol{S}_3$ is single and (ii) $\boldsymbol{S}_{33}$ is its single entry equal to -1. By Lemma 3.3, for every row $\boldsymbol{S}_i$ distinct from $\boldsymbol{S}_3$, its entry $\boldsymbol{S}_{i3}$ is equal to 1.

The rank of $\boldsymbol{S}$ is equal to four. Therefore, $\boldsymbol{S}$ must have rows distinct from $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ and $\boldsymbol{S}_3$. Let $\boldsymbol{R}$ be one such row of $\boldsymbol{S}$. By Lemma 3.3, $\boldsymbol{R}_1 = \boldsymbol{R}_2 = \boldsymbol{R}_3 = 1$ and $\boldsymbol{R}_4 = -1$. We conclude that $\boldsymbol{S}$ has four distinct rows, each of which is single. The submatrix of $\boldsymbol{S}$ consisting of those four rows is a permutation of $\boldsymbol{J} - 2\boldsymbol{I}$, where $\boldsymbol{J}$ denotes the 4×4 matrix consisting solely of entries equal to 1 and $\boldsymbol{I}$ denotes the 4×4 identity matrix. The determinant of matrix $\boldsymbol{J} - 2\boldsymbol{I}$ is non-null. This implies that $\boldsymbol{\alpha} = \boldsymbol{1/2}$, as asserted. Consequently, every row of $\boldsymbol{S}$ is single. □

3.2 Graph G_{19} is not 4-Pfaffian

For directed graph D, a cycle Q of even length of D is *evenly oriented* if the number of forward edges of Q is even, *oddly oriented* otherwise. The following result appears in the book by Lovász and Plummer [4, Lemma 8.3.1]:

Lemma 3.4

Let M_1 and M_2 be two perfect matchings of a directed graph D, and let k denote the number of evenly oriented M_1, M_2-alternating cycles. Then, $\mathrm{sgn}(M_1, D) \cdot \mathrm{sgn}(M_2, D) = (-1)^k$.

Theorem 3.5

Graph G_{19} is not 4-Pfaffian.

Proof: Refer to Figure 1 on page 2. Let us first prove that G_{19} is not Pfaffian. For this, note that C_{19} is tight and each C_{19}-contraction of G_{19} is, up to multiple edges, equal to $K_{3,3}$, in turn non-Pfaffian. Therefore, by Theorem 2.5, G_{19} is not Pfaffian.

Assume, to the contrary, that G_{19} is 4-Pfaffian. Let $(\boldsymbol{D}, \boldsymbol{\alpha})$ be a normal Pfaffian 4-pair of G. Let $\boldsymbol{S}$ be the signature matrix $\mathbf{sgn}(\mathcal{M}(G), \boldsymbol{D})$. By the Theorem on the Uniqueness of Signature Matrices, we have that $\boldsymbol{\alpha} = \mathbf{1/2}$, whence every row of $\boldsymbol{S}$ contains precisely one entry equal to -1. Moreover, every column of $\boldsymbol{S}$ contains one entry equal to -1. The set $\mathcal{M}$ of the perfect matchings of G is thus partitioned in four non-null classes, $\mathcal{M}_i$, $i = 1, 2, 3, 4$, such that $\mathcal{M}_i$ is the set of those perfect matchings of G that have sign -1 in $\boldsymbol{D}_i$ (and sign 1 in all the other three orientations in $\boldsymbol{D} - \boldsymbol{D}_i$). Let us now derive some properties of $\boldsymbol{D}$. Recall first that $G_{19} - C_{19}$ is the union of two disjoint $K_{3,2}$'s, $G_i := G_{19}[X_i]$, $i = 1, 2$ (see Figure 1). The proof of the following auxiliary result is easily done by induction:

Lemma 3.6

In every orientation of $K_{3,2}$, the number of evenly oriented cycles is odd.

Lemma 3.7

Let Q be a quadrilateral in $G_{19} - C_{19}$ that is evenly oriented in $\boldsymbol{D}_i$. Then, Q is evenly oriented in precisely one more orientation $\boldsymbol{D}_j$ of G, $j \neq i$. Moreover, every perfect matching of G that contains two edges in Q lies in $\mathcal{M}_i \cup \mathcal{M}_j$.

Proof: Let M be a perfect matching of G that contains two edges in Q. Let $N := M \triangle Q$. The signs of M and N in $\boldsymbol{D}_i$ are distinct. Therefore, one of M and N has sign -1 in $\boldsymbol{D}_i$, the other has sign 1 in $\boldsymbol{D}_i$. Consequently, there exists an integer j distinct from i such that one of M and N has sign -1 in $\boldsymbol{D}_i$ and sign 1 in $\boldsymbol{D}_j$, the other has sign 1 in $\boldsymbol{D}_i$ and sign -1 in $\boldsymbol{D}_j$. We deduce that Q is evenly oriented in $\boldsymbol{D}_j$ as well. For any orientation $\boldsymbol{D}_k$ in $\boldsymbol{D} - \boldsymbol{D}_i - \boldsymbol{D}_j$, the signs of M and N in $\boldsymbol{D}_k$ are both equal to 1. Therefore, Q is oddly oriented in $\boldsymbol{D}_k$. We deduce that Q is oddly oriented in the two orientations of $\boldsymbol{D} - \boldsymbol{D}_i - \boldsymbol{D}_j$ and evenly oriented in $\boldsymbol{D}_i$ and in $\boldsymbol{D}_j$. Finally, we have already seen that M lies in $\mathcal{M}_i \cup \mathcal{M}_j$. This conclusion holds for each perfect matching M of G that contains two edges in Q. □

Corollary 3.8

For each shore X_i of C, at most one of the three cycles in $G_{19}[X_i]$ is oddly oriented in every orientation in $\boldsymbol{D}$.

Proof: Let r denote the number of cycles of $G_{19}[X_i]$ that are evenly oriented in some orientation in $\boldsymbol{D}$. By Lemma 3.7, each such cycle is evenly oriented in precisely two orientations.

A simple counting argument then shows that the number of pairs $(Q, \boldsymbol{D}_j)$ such that Q is a cycle of $G_{19}[X_i]$ that is evenly oriented in $\boldsymbol{D}_j$ is equal to $2r$. Every orientation contains at least one evenly oriented cycle in $G_{19}[X_i]$. We deduce that $2r \geq 4$, whence $r \geq 2$. As asserted, at most one of the three quadrilaterals of $G_{19}[X_i]$ is oddly oriented in every orientation in $\boldsymbol{D}$. □

Let x_1 and x_2 denote the two *universal* vertices of X_1, that is, the two vertices of degree five in G_{19} that lie in X_1. For $i = 1, 2$, subgraph $G_{19}[X_1]$ of G_{19} has two cycles, Q_1 and Q_2, such that Q_i contains x_i but does not contain both x_1 and x_2. By the Corollary, at least one of Q_1 and Q_2 is evenly oriented in some orientation in $\boldsymbol{D}$. Adjust notation so that x is a universal vertex of X_1, Q is a cycle of $G_{19}[X_1] - x$ that is evenly oriented in $\boldsymbol{D}_1$. Adjust notation so that Q is evenly oriented in $\boldsymbol{D}_2$ as well. Then, Q is oddly oriented in $\boldsymbol{D}_3$ and in $\boldsymbol{D}_4$.

Let Q' denote a cycle in $G_{19}[X_2]$ that is evenly oriented in $\boldsymbol{D}_3$. Then, Q' is also evenly oriented in $\boldsymbol{D}_j$, for some j in $\{1, 2, 4\}$, but oddly oriented in the two orientations in $\boldsymbol{D} - \boldsymbol{D}_3 - \boldsymbol{D}_j$.

Let e be the edge of $G_{19} - V(Q) - V(Q')$. That edge exists, because the vertex x in $X_1 - V(Q)$ is universal. Let M be a perfect matching of G_{19} that contains edge e. Then, M contains two edges in Q and two edges in Q'. Let $N := M \triangle E(Q)$. By the Lemma, $\{M, N\} \subset \mathcal{M}_1 \cup \mathcal{M}_2$. Again, by the Lemma, $\{M, N\} \subset \mathcal{M}_3 \cup \mathcal{M}_j$, which implies that M and N lie both in $\mathcal{M}_j$. But M and N have distinct signs in $\boldsymbol{D}_1$, whence cannot both lie in the same class $\mathcal{M}_j$. We have deduced a contradiction from the hypothesis that G_{19} is 4-Pfaffian. □

4 Remarks

Norine has shown that for every graph G, if $\mathrm{pf}(G) \leq 5$ then $\mathrm{pf}(G) \in \{1, 4\}$. We have shown that G_{19} is a graph that is 6-Pfaffian but not 4-Pfaffian. This is a counter-example to Conjecture 1.2. We believe it to be a minimum counter-example. We would like to pose two Conjectures:

CONJECTURE 4.1
For every graph G, if $\mathrm{pf}(G) > 1$ then $\mathrm{pf}(G)$ is even and $\mathrm{pf}(G) \geq 4$. Moreover, for every even integer $k \geq 4$ there exists a graph G whose Pfaffian number is k.

CONJECTURE 4.2
Let G be a graph, $(\boldsymbol{D}, \boldsymbol{\alpha})$ a normal Pfaffian $\mathrm{pf}(G)$-pair of G. If $\mathrm{pf}(G)$ is even then $\boldsymbol{\alpha} = \mathbf{1/2}$.

It is not difficult to show that the smallest counter-example G to Conjecture 4.2 is minimal $\mathrm{pf}(G)$-Pfaffian.

CONJECTURE 4.3
Let G be a graph. If G is minimal $\mathrm{pf}(G)$-Pfaffian and $\mathrm{pf}(G)$ is even then graph G has a $\frac{\mathrm{pf}(G)}{2}$-decomposition.

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List of Assertions

Theorem 2.6 5

Let C be a tight cut of a graph G, let G' and G'' denote the two C-contractions of G. Then, $\mathrm{pf}(G) \leq \mathrm{pf}(G') \cdot \mathrm{pf}(G'')$.

Theorem 2.7 {`thm:composition`} 5

(Composition) If a graph has an r-decomposition then it is $2r$-Pfaffian.

Corollary 2.8 {`cor:tight-cut-partition`} 6

Let C be a tight cut of a graph G. Let $\{C_1, C_2, \ldots, C_r\}$ be a partition of C. Assume that $G_i := G - (C - C_i)$ is Pfaffian, for $i = 1, 2, \ldots, r$. Then, G is $2r$-Pfaffian.

Theorem 2.9 {`thm:C-R`} 6

Let R be a (possibly empty) subset of tight cut C_{21} of G_{21}. If $C_{21} - R$ may be covered by r edge-disjoint P_4's then $G_{21} - R$ is $2r$-Pfaffian.

Theorem 2.10 {`thm:positive`} 6

Graph G_{21} is 6-Pfaffian and graph $G_{19} - e$ is 4-Pfaffian.

Lemma 3.1 {`lem:composicao-de-impar-orientacoes`} 7

Let G be a graph, $(D_1, D_2, \ldots, D_r)$ a family of orientations of G, r odd. Then, there is an orientation D of G such that the inequality below holds for every perfect matching M of G:

$$\mathrm{sgn}(M, D) = \mathrm{sgn}(M, D_1)\, \mathrm{sgn}(M, D_2)\ \ldots\ \mathrm{sgn}(M, D_r).$$

Theorem 3.2 {`thm:J-2I`} 7

(Uniqueness of Signature Matrices) Let G be a non-Pfaffian graph, $(\boldsymbol{D}, \boldsymbol{\alpha})$ a normal Pfaffian 4-pair of G. Then, $\boldsymbol{\alpha} = \mathbf{1/2}$.

Lemma 3.3 {`lem:mais-mais-mais-menos`} 8

For any two distinct rows $\boldsymbol{S}_i$ and $\boldsymbol{S}_j$ of $\boldsymbol{S}$, there is a column $\boldsymbol{S}_\ell$ and a column $\boldsymbol{S}_m$ of $\boldsymbol{S}$ such that $\boldsymbol{S}_{i\ell} = 1$, $\boldsymbol{S}_{im} = -1$, $\boldsymbol{S}_{j\ell} = 1$ and $\boldsymbol{S}_{jm} = 1$.

Lemma 3.4 {`lem:evenly-oriented`} 9

Let M_1 and M_2 be two perfect matchings of a directed graph D, and let k denote the number of evenly oriented M_1, M_2-alternating cycles. Then, $\mathrm{sgn}(M_1, D) \cdot \mathrm{sgn}(M_2, D) = (-1)^k$.

Theorem 3.5 {`thm:not-4-Pfaffian`} 9

Graph G_{19} is not 4-Pfaffian.

Lemma 3.6 9

In every orientation of $K_{3,2}$, the number of evenly oriented cycles is odd.

Lemma 3.7 {`lem:4-orientacao-k32`} 9

Let Q be a quadrilateral in $G_{19} - C_{19}$ that is evenly oriented in $\boldsymbol{D}_i$. Then, Q is evenly oriented in precisely one more orientation $\boldsymbol{D}_j$ of G, $j \neq i$. Moreover, every perfect matching of G that contains two edges in Q lies in $\mathcal{M}_i \cup \mathcal{M}_j$.