A PTAS for the disk cover problem of geometric objects

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Abstract

We present PTASes for the disk cover problem: given geometric objects and a finite set of disk centers, minimize the total cost for covering those objects with disks under a polynomial cost function on the disks' radii. We describe the first FPTAS for covering a line segment when the disk centers form a discrete set, and a PTAS for when a set of geometric objects, described by polynomial algebraic inequalities, must be covered. The latter result holds for any dimension.

Keywords: Disk Cover, Approximation Algorithm, Computational Geometry, Wireless Networks, Sensor Coverage

1. Introduction

Motivated by applications to wireless networks [1, 2] and sensor coverage [3], we are interested in covering sets of planar geometric objects by disks. Given a point (disk center) c in the Euclidean plane and a non-negative number r (radius), a disk $D(c, r)$ is the set of points at distance at most r from c. We generically refer to a geometric object as *well-behaved* if it can be described by a system of algebraic polynomial inequalities with degrees bounded by a constant. Henceforth, we restrict ourselves to these geometric objects and omit the term well-behaved. Given a geometric object g, we denote by $\wp(g)$ the set of points on g. Similarly, if G is a set of geometric objects, we let $\wp(G) = \bigcup_{g \in G} \wp(g)$. Moreover, we say that a collection of

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geometric objects G is covered by another collection of geometric objects G' if $\wp(G) \subseteq \wp(G')$.

The *Minimum Geometric Disk Cover Problem*, MGDCP, can be stated as follows: on the Euclidean plane, let $\mathcal{L}$ be a set of geometric objects, $\mathcal{C} = (c_1, \ldots, c_n)$ be a list of points (disk centers), and $f(r)$ be a function that assigns a non negative cost to a disk of radius r. We want to determine a collection of non-negative numbers $S = (r_1, \ldots, r_n)$, which defines a list of disks, $\mathcal{D} = (D(c_1, r_1), \ldots, D(c_n, r_n))$, so that: $\mathcal{D}$ covers $\mathcal{L}$ and the total cost of these disks, $f(S) = \sum_{r \in S} f(r)$, is minimized. In this work, we restrict the cost functions to be polynomials $f(r) = r^{\kappa}$, for some positive constant $\kappa \geq 1$. An instance of the MGDCP is described by a tuple $I = (\mathcal{C}, \mathcal{L}, f)$. The special case of this problem where $\mathcal{L}$ is composed of only one line segment is denoted by MLSDCP.

Many practical problems can be modeled as instances of the MGDCP, including the problem of positioning base station antennas to provide wireless network access over a city area, or positioning sensors to oversee a farming field.

Related Works

In [2], Li et al. consider the problem of covering a line segment with wireless sensors of adjustable ranges, positioned on the segment itself. The objective is to find a range assignment with minimum cost. This problem is referred to as *Min-Cost Linear Coverage* (MCLCP) and it has two variants. In the *discrete* case, the coverage range for each sensor v must belong to a given finite set $R(v)$, while in the *continuous* case, the range must be chosen from a given continuous interval. For the discrete variant, where the costs of the sensors may be arbitrary, Li et al. present a polynomial-time exact algorithm. For the continuous case, where the cost of each sensor is given by a function $f(r) = r^{\kappa}$, they present an FPTAS for $\kappa = 1$ and a 2-approximation algorithm for any constant $\kappa > 1$.

Lev-Tov and Peleg, in [1], address the problem called *Minimum Sum of Radii Cover* (MSRCP), in whose formulation, two sets of points X and Y are given and the objective is to choose the radii R_i for the disks centered at the points $x_i \in X$ so as to cover all the points in Y with minimum total cost. In this case, the cost is given by $\sum_{x_i \in X} R_i$. They consider the 1-dimensional version of the problem, and present a polynomial-time exact algorithm as well as a linear time 4-approximation algorithm. For the 2-dimensional version, they describe a PTAS.

Bilò et al. consider, in [4], a variation of the MSRCP, where the centers of the disks may be at the points of the set to be covered, i.e., $X = Y$, provided the number of disks is within a given bound k. The cost of a disk of radius r is $f(r) = r^\alpha$, where $\alpha \geq 1$ is a constant. The objective is to cover all the points of Y with minimum total cost, using at most k disks. They show that for any $\alpha \geq 2$ the problem is NP-hard. They also describe PTASes for the 2-dimensional problem, for any constant $\alpha \geq 1$. In [5], Alt et al. improve the result of [4] by showing that the problem is already NP-hard for any superlinear cost function, i.e, $f(r) = r^\alpha$ with $\alpha > 1$.

Gibson et al. [6] consider the version where $\alpha = 1$ of the variation of the MSRCP addressed in [4], giving an exact polynomial-time algorithm, under the assumption that two candidate solutions can be compared efficiently, in polynomial time.

Abu-Affash et al. [7] consider the disk multicover problem, where the objective is to cover each point of a given set Y with disks centered at points given by a set X at least k times. They consider the cost of a disk given by a function $f(r) = r^\alpha$, with $\alpha = 2$ where r is the radius assigned to the disk. They present an algorithm with a $23.02 + 63.91(k-1)$ approximation guarantee. For the continuous case, in which the points of Y are given by a polygon, they present a $63.94 + 177.64(k-1)$ approximation algorithm, and when $k = 1$, using the algorithm in [8], they obtain a 25-approximation algorithm.

Bar-Yehuda and Rawitz [9] consider the polygon multicover problem and present a $(3^\alpha k + \varepsilon)$-approximation algorithm, with $\alpha \geq 1$. They assume rational coordinates and the running time is polynomial on the input size.

Results

We present PTASes for special cases of the problem of covering geometric objects with disks of variable radii and discrete centers, whose costs are given by a polynomial function. More precisely, we present the first FPTAS for covering a single line segment with disks restricted to a discrete set of centers, improving upon the previous 2-approximation given by Li et al. [2], subject to a cost function $f(r) = r^\kappa$ with $\kappa > 1$. We also present a PTAS for the problem of covering objects described by a system of algebraic polynomial inequalities, which may comprise polygons and more complex objects whose boundaries may be described by a number of inequalities of degrees bounded by a constant. For the particular case where $\kappa = 1$, and with the assumption that it is possible to compare two sums of square roots of integers in polynomial time, we present an FPTAS for the problem of covering such complex objects. These algorithms can be extended to solve the corresponding d-dimensional versions of those problems.

2. Line Segment Disk Cover

In this section, before we present an FPTAS, denoted $\mathcal{A}_\varepsilon$, for MLSDCP, we need to describe the *Min-Cost Linear Coverage* Problem (MCLCP). Let l be a line segment, $C = \{c_1, \ldots, c_n\} \subset l$ be a set of disk centers, and let R_i be a set of μ possible radii values for disks centered at $c_i \in C$. To simplify the formulation, we consider $0 \in R_i$. Let P_{ij}, which can be arbitrary, be the cost to assign the radius r_j to the disk centered at c_i. These elements comprise an instance of the discrete version of MCLCP. We want to determine a list $S = (r_1, \ldots, r_n)$ of radii $r_i \in R_i$ for $1 \leq i \leq n$, so that the disks defined by S cover the segment l and the sum of the costs of the chosen disks is minimum. Li et al. [2] present an exact algorithm, which we denote by $\mathcal{A}_{\text{MCLC}}$, with time $O(\mu^2 n^2)$ to solve this version of MCLCP. The algorithm $\mathcal{A}_{\text{MCLC}}$ will be used as a subroutine for $\mathcal{A}_\varepsilon$ described below.

Basically, the algorithm $\mathcal{A}_\varepsilon$ generates a discretized set of values that can be assigned to the radii of an instance of MLSDCP, transforming it into an instance of MCLCP, which is solved using $\mathcal{A}_{\text{MCLC}}$.

Surely, the degenerate case where the line segment is a single point and a center $c \in C$ coincides with that point has as optimal solution a disk with radius zero centered at c. Hence, we assume without loss of generality that the instance to be solved is non-degenerate.

Firstly, assume that $\varepsilon < 4\kappa$, let $\xi = \varepsilon/(4\kappa)$ and $\ell = \|a, b\|_1$, where a and b are the extremes of l. The algorithm defines several radii for disks at each center, all of which are between $r_{\min}$ and $r_{\max}$, where $r_{\min} = \xi\ell/(2n)$ and $r_{\max} = \max_{c \in \mathcal{C}, p \in l} \|c, p\|_1$. Note that any radius in a solution can be bounded by $r_{\max}$ since any point p in the line segment is distant from any center by at most $\max_{c \in \mathcal{C}, p \in l} \|c, p\|_2 \leq \max_{c \in \mathcal{C}, p \in l} \|c, p\|_1$. Both $r_{\min}$ and $r_{\max}$ have representations that are computable in polynomial time. Let $R = \{r_{\min}, r_{\min}(1+\xi)^1, r_{\min}(1+\xi)^2, \ldots, r_{\min}(1+\xi)^{m-1}\}$ a set of radius values, where m is the smallest integer such that $r_{\min}(1+\xi)^{m-1} \geq r_{\max}$.

Using the discretization given by R, there will be m possible radii for each center, defining nm possible disks. Each of these disks that intersects the line segment l is replaced by a disk that covers the same portion of l, but centered on the line segment itself, at the midpoint of the covered portion, and with radius equal to half of the length of the covered portion. The cost of these disks is set to be equal to the cost of the original disk. An instance of the MCLCP created from the discretized instance MLSDCP consists of: the set C' made up of the centers of those replacement disks; for each center $c'_i \in C'$, a set R'_i containing the radius of the created disks centered at c'_i; and a set P'_{ij} with the costs of the disks centered at c'_i with radius $r'_j \in R'_i$. A solution S' of such instance is obtained through the algorithm $\mathcal{A}_{\text{MCLC}}$. A solution S for the discretized instance MLSDCP can easily be obtained from the solution S' using the corresponding disks.

We now prove that the algorithm $\mathcal{A}_\varepsilon$ is an FPTAS.

Lemma 2.1. *The number $m = |R|$ is polynomial in the input size and in $1/\varepsilon$.*

PROOF. As m is the smallest integer such that $r_{\min}(1+\xi)^{m-1} \geq r_{\max}$, we have that $m = \lceil \log_{1+\xi}(r_{\max}/r_{\min}) \rceil + 1$. Therefore,

$$\begin{aligned} m = \lceil \log_{1+\xi}(r_{\max}/r_{\min}) \rceil + 1 &\leq \frac{\log(r_{\max}/r_{\min})}{\log(1+\xi)} + 2 \\ &= \frac{\log(2n\|c,s\|_1/(\xi\ell))}{\log(1+\xi)} + 2 \qquad (1) \\ &\leq \frac{\log(2n) + \log(4\kappa/\varepsilon) + \log(\|c,s\|_1/\ell)}{\varepsilon/4\kappa} + 2 \qquad (2) \\ &= O(\frac{\kappa}{\varepsilon}(\log(n) + \log(\kappa/\varepsilon) + \log(\|c,s\|_1/\ell)), \end{aligned}$$

where, in (1), $c \in \mathcal{C}$ and s is a point of the geometric object to be covered such that $r_{\max} = \|c,s\|_1$, and (2) is valid because $\log(1+\xi) \geq \xi = \varepsilon/4\kappa$ when $0 \leq \xi \leq 1$. □

Lemma 2.2. *If S^* is an optimal solution for an instance of* MLSDCP*, then $f(S^*) \geq nf(\ell/(4n))$.*

PROOF. Denote by RMLSDCP the relaxed version of the problem MLSDCP where we define a list of n radii $(r_1, \ldots, r_n)$ of disks to cover the segment, and each disk may be centered on any point of the Euclidean plane. It is clear that any solution to the problem MLSDCP is also a solution to RMLSDCP. An optimal solution for an instance of the problem RMLSDCP is given by n disks with centers equally spaced over the segment, each disk with radius $\|a,b\|_2/(2n)$. Note that, since f is a convex function, this solution attains minimum cost. As $\|a,b\|_2 \geq \|a,b\|_1/2 = \ell/2$, the result follows. □

Theorem 2.1. *Algorithm $\mathcal{A}_\varepsilon$ is an FPTAS for the* MLSDCP and runs in $O(n^2m^2)$ time.

PROOF. Let $I = (\mathcal{C}, \{l\}, f)$ be an instance of MLSDCP. Let $S^* = (r_1^*, \dots, r_n^*)$ be an optimal solution for I and $S = (r_1, \dots, r_n)$ the solution obtained by the algorithm $\mathcal{A}_\varepsilon$. Let $S' = (r_1', \dots, r_n')$ be a solution obtained from S^*, by rounding up each radius r_i^* to the smallest radius $r_i' \in R$ such that $r_i' \geq r_i^*$. As S' is obtained from S^* by increasing some of the radii, it is clear that S' is also a feasible solution.

Let $B = \{i \in \{1, \dots, n\} : r_i' > r_{\min}\}$ be the set of (indices of) disks in S' with radius larger than $r_{\min}$, and $T = \{1, \dots, n\} \setminus B$ be the set of disks in S' with radius equal to $r_{\min}$. Since S is an optimal solution when the radii are restricted to the set R, we have that $f(S) \leq f(S')$. So, it remains to prove that S' has cost within a $1 + \varepsilon$ factor of the cost of the solution S^*. That is,

$$
\begin{aligned}
f(S) \leq f(S') &= \sum_{i \in B} f(r_i') + \sum_{i \in T} f(r_{\min}) \\
&\leq \sum_{i \in B} f((1+\xi) r_i^*) + \sum_{i \in T} f\left(\frac{\xi \ell}{4n}\right) && (3) \\
&\leq (1+\xi)^\kappa \sum_{i \in B} f(r_i^*) + \xi^\kappa n f\left(\frac{\ell}{4n}\right) && (4) \\
&\leq (1+\xi)^\kappa f(S^*) + \xi^\kappa f(S^*) && (5) \\
&= \left(\left(1 + \frac{\varepsilon/4}{\kappa}\right)^\kappa + \left(\frac{\varepsilon/4}{\kappa}\right)^\kappa\right) f(S^*) \\
&\leq (e^{\varepsilon/4} + \varepsilon/4) f(S^*) && (6) \\
&\leq (1 + 2(\varepsilon/4) + \varepsilon/4) f(S^*) && (7) \\
&\leq (1+\varepsilon) f(S^*),
\end{aligned}
$$

where (3) is valid because f is a non-decreasing function and $r_i' \leq (1+\xi) r_i^*$, (4) holds since $f(r) = r^\kappa$, (5) follows from Lemma 2.2, (6) is valid since $(1 + x/\kappa)^\kappa \leq e^x$ when $|x| \leq \kappa$ and $\kappa \geq 1$, and the validity of (7) follows from the fact that $e^x \leq 1 + 2x$ when $0 \leq x \leq 1$.

By Lemma 2.1, the algorithm generates a set R with a polynomial number, m, of possible radii. For each disk that intersects the segment, we may consider its intersection region as the intersection of the line segment with another disk centered on the segment l. Therefore, we obtain at most nm

distinct disks centered on l. The algorithm $\mathcal{A}_{\text{MCLC}}$ is then applied on the transformed instance with disks on the segment. From the time complexity of algorithm $\mathcal{A}_{\text{MCLC}}$, we can conclude that the time complexity of algorithm $\mathcal{A}_\varepsilon$ is $O(n^2m^2)$, which is polynomial on the length of the input and on $1/\varepsilon$. Thus, the algorithm $\mathcal{A}_\varepsilon$ is an FPTAS for MLSDCP. $\square$

Instances of the continuous variant of MCLCP can be solved using the algorithm $\mathcal{A}_\varepsilon$ with a minor change. The instances of MCLCP have, for each center $c_i \in C$, a continuous interval $I_i = [g_i, h_i]$ of values from which the radii of the disks may be chosen. To overcome this (continuity) requirement, it is necessary to establish, for each center c_i, the possible values of the discretized set R within the limits of those intervals before applying the algorithm $\mathcal{A}_{\text{MCLC}}$. This can be done by taking, for each center $c_i \in C$, the set of possible ranges $R'_i = (R \cap I_i) \cup \{g_i, h_i\}$. This change does not affect the previous results, since the proof of Theorem 2.1 remains unchanged.

Thus, this algorithm improves the 2-approximation result established in [2] for the continuous variant of MCLCP when the cost of each disk is given by a function $f(r) = r^\kappa$, with $\kappa > 1$.

3. Geometric Disk Cover

In this section, we present a PTAS for MGDCP, where the objects to be covered are well-behaved in the sense described in Section 1. The algorithm, which we denote by $\mathcal{B}_\varepsilon$, uses as a subroutine, an algorithm for the Min-Size k-Clustering Problem, (MSCP), which can be stated as follows. Given a set X of points in d-dimensional space, let $\mathcal{F}$ be a set of fixed costs F_p for each point $p \in X$. Let $f(r) = r^\alpha$ be a function, where $\alpha \geq 1$ is constant, and let k be an integer constant. The MSCP consists of choosing the radii r_i for at most k disks centered at points $p_i \in X$ so that all the points of X are covered with minimum total cost. The cost is given by $\sum_{p_i \in X | r_i > 0}(f(r_i) + F_{p_i})$. Bilò et al. [4] present a PTAS for the MSCP with time complexity $O(n^{(\frac{\alpha}{\varepsilon})^{O(d)}})$. In particular, for $d = 2$, they give a PTAS with complexity $O(n^{\frac{\alpha^4}{\varepsilon^6}})$. We denote the PTAS presented in [4] by $\mathcal{B}_\xi^{\text{MSC}}$.

The algorithm $\mathcal{B}_\varepsilon$, described below, is similar to the algorithm $\mathcal{A}_\varepsilon$. The idea is to generate an instance of MSCP and apply the algorithm $\mathcal{B}_\xi^{\text{MSC}}$ to obtain a solution that is later transformed to a solution of the instance of the MGDCP.

Given an instance $I = (\mathcal{C}, \mathcal{L}, f)$ of the MGDCP, let $\xi = \varepsilon/(8\kappa)$, $r_{\min} = \xi \cdot \max_{o \in \mathcal{L}}\{\max_{p,q \in o}\{\|p, q\|_1\}\}/2n$, and $r_{\max} = \max_{c \in \mathcal{C},\, o \in \mathcal{L}}\{\max_{p \in o}\{\|c, p\|_1\}\}$. The value $r_{\min}$ is chosen in this way because an optimal solution has a lower bound given by the covering of the diameter of the largest object. The degenerate case where all the objects to be covered are points can be solved by the algorithm $\mathcal{B}_\xi^{\text{MSC}}$ directly. Hence, we may assume that the instance to be solved is non-degenerate.

Let $R = \{r_{\min}, r_{\min}(1+\xi)^1, r_{\min}(1+\xi)^2, \ldots, r_{\min}(1+\xi)^{m-1}\}$ be a set of radii values, where m is the smallest integer such that $r_{\min}(1+\xi)^{m-1} \geq r_{\max}$. Note that, analogous to the Lemma 2.1, the number of possible radii $m = |R|$ is polynomial in the input size and in $1/\varepsilon$.

Consider the set of all circles centered at each point $c \in \mathcal{C}$ with all possible radii in R. These circles define an arrangement whose total combinatorial complexity is quadratic on the number of circles (see [10]) and, therefore, the total number of regions is polynomial in n and m.

Let $\mathcal{R}$ be the set of regions that intersect at least one geometric object given in the input. Each region can be described as a solution of a system of polynomial inequalities. For each of these systems, we can add the inequalities that bound the geometric objects. Using the algorithm presented in [11] to solve systems of polynomial inequalities, it is possible to obtain, for each region $h \in \mathcal{R}$, a point $p(h)$ that belongs to a geometric object.

Define an instance of the MSCP as follows: let $\mathcal{P} = \{p(h) : h \in \mathcal{R}\}$ and let $\mathcal{X} = \mathcal{C} \cup \mathcal{P}$. To obtain the set $\mathcal{F}$, for each point $p_i \in \mathcal{X}$ define its fixed cost F_i, as $F_i = 0$ if $p_i \in \mathcal{C}$, and $F_i = \infty$ (or a sufficiently large value) if $p_i \in \mathcal{P}$. By defining the fixed costs in this way, we prevent the disks centered at points $p_i \in \mathcal{P}$ from being part of a solution.

As a final step, we obtain a solution $\hat{S} = (\hat{r}_1, \ldots, \hat{r}_n)$ by applying algorithm $\mathcal{B}_\xi^{\text{MSC}}$ and generate a solution $S = (r_1, \ldots, r_n)$ for instance I, with $r_i = (1+\xi)\hat{r}_i$ when $\hat{r}_i \geq r_{\min}$, and $r_i = r_{\min}$ otherwise.

To prove that algorithm $\mathcal{B}_\varepsilon$ runs in polynomial time, we may proceed as in the previous section, using that algorithm $\mathcal{B}_\xi^{\text{MSC}}$ has polynomial complexity.

Lemma 3.1. *The time cost to determine a representative point for each region in $\mathcal{R}$ is polynomial.*

Proof. The time cost we seek is given by the complexity of the algorithm from [11]. That algorithm runs in time $b((mn+\gamma)\delta)^{d^2}$, where b is the maximum length in bits of the representation of the coefficients of the polynomials in the system, δ is the maximum degree of these polynomials, γ is the number of polynomials in the system that describes the objects and d is the dimension of the space. With b, γ, δ and d given as constants, the total time is polynomial in mn. This algorithm is executed for each region, whose number is also polynomial in mn, resulting in a polynomial total time to determine the set $\mathcal{P}$ of representative points. $\square$

Theorem 3.1. *The algorithm $\mathcal{B}_\varepsilon$ is a PTAS for* MGDCP.

Proof. Let $I = (\mathcal{C}, \mathcal{L}, f)$ be an instance for the problem MGDCP. Let $S^* = (r_1^*, \ldots, r_n^*)$ be an optimal solution for I and $S = (r_1, \ldots, r_n)$ be the solution obtained by the algorithm $\mathcal{B}_\varepsilon$, which covers all the points in $\mathcal{P}$. Let $\hat{S}^*$ be an optimal solution for the problem, MSCP, of covering the points in $\mathcal{P}$, and $\hat{S}$ be the solution obtained by algorithm $\mathcal{B}_\xi^{\text{MSC}}$.

Note that there is a mapping of each point in $\mathcal{P}$ to a region in $\mathcal{R}$, and that each region of $\mathcal{R}$ is defined by intersections of circles. Each circle comes from a set of concentric circles, with radii increasing by a factor $(1+\xi)$.

The solution S, obtained from $\hat{S} = (\hat{r_1}, \ldots, \hat{r_n})$ by increasing each radius $\hat{r_i}$ by a factor of $(1+\xi)$ and rounding up the resulting radii that remain smaller than $r_{\min}$ to $r_{\min}$, covers all the regions in $\mathcal{R}$. Let $B = \{i \in \{1, \ldots, n\} : r_i = (1+\xi)\hat{r_i} > r_{\min}\}$ be the set of (indices of) disks in S with radius larger than $r_{\min}$, and $T = \{1, \ldots, n\} \setminus B$ be the set of disks in S with radius rounded up to $r_{\min}$. As each point in $\mathcal{P}$ belongs to at least one geometrical object, we have $f(\hat{S}^*) \le f(S^*)$. Therefore,

$$
\begin{aligned}
f(S) &= \sum_{i \in B} f((1+\xi)\hat{r_i}) + \sum_{i \in T} f(r_{\min}) \\
&\le (1+\xi)^\kappa f(\hat{S}) + n f(r_{\min}) \\
&\le (1+\xi)^\kappa (1+\xi) f(\hat{S}^*) + \xi^\kappa f(S^*) && (8) \\
&\le \left(1 + \frac{\varepsilon/8}{\kappa}\right)^{\kappa+1} f(S^*) + \left(\frac{\varepsilon/8}{\kappa}\right)^\kappa f(S^*) \\
&\le \left((e^{\varepsilon/8})(1+\varepsilon/8) + \left(\frac{\varepsilon/8}{\kappa}\right)^\kappa\right) f(S^*) && (9) \\
&\le \left(\left(1 + 2\left(\frac{\varepsilon}{8}\right)\right)\left(1 + \frac{\varepsilon}{8}\right) + \frac{\varepsilon}{8}\right) f(S^*) && (10) \\
&\le (1+\varepsilon) f(S^*),
\end{aligned}
$$

where (8) is valid because the solution $\hat{S}$ is obtained by the algorithm $\mathcal{B}_{\xi}^{\text{MSC}}$, which is a PTAS, (9) follows from $(1 + x/\kappa)^{\kappa} \leq e^{x}$ when $|x| \leq \kappa$ and $\kappa \geq 1$, and the validity of (10) follows from the fact that $e^{x} \leq 1+2x$ when $0 \leq x \leq 1$.

By Lemma 3.1 the time complexity to determine the representative points in $\mathcal{P}$ is polynomial. Thus, the instance solved with the algorithm $\mathcal{B}_{\xi}^{\text{MSC}}$ has polynomial size, resulting in a polynomial total time.

Thus, the algorithm $\mathcal{B}_{\varepsilon}$ is a PTAS for the problem MGDCP. □

After obtaining the set $\mathcal{P}$ of representative points, if the instance meets the restrictions assumed by the work of Gibson et al. [6], where the cost of the disks are such that $\alpha = 1$ and with the assumption that two candidate solutions can be compared in polynomial time, the algorithm of [6] can be used instead of the algorithm $\mathcal{B}_{\xi}^{\text{MSC}}$, obtaining an exact solution in polynomial time for covering the set $\mathcal{P}$. In this case, we have obtained an FPTAS since the total time is polynomial in the size of the input and in $1/\varepsilon$.

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References

[1] N. Lev-Tov, D. Peleg, Polynomial time approximation schemes for base station coverage with minimum total radii, Comp. Net. 47 (4) (2005) 489–501. doi:10.1016/j.comnet.2004.08.012.

[2] M. Li, X. Sun, Y. Zhao, Minimum-cost linear coverage by sensors with adjustable ranges, in: Cheng, Yu et al. (Ed.), Wireless Algorithms, Systems, and Applications, Vol. 6843 of LNCS, Springer-Verlag, 2011, pp. 25–35.

[3] A. Agnetis, E. Grande, P. Mirchandani, A. Pacifici, Covering a line segment with variable radius discs, Comput. Oper. Res. 36 (5) (2009) 1423–1436. doi:10.1016/j.cor.2008.02.013.

[4] V. Bilò, I. Caragiannis, C. Kaklamanis, P. Kanellopoulos, Geometric clustering to minimize the sum of cluster sizes, in: G. Brodal, S. Leonardi (Eds.), Algorithms – ESA, Vol. 3669 of LNCS, Springer-Verlag, 2005, pp. 460–471.

[5] H. Alt, E. Arkin, H. Brönnimann, J. Erickson, S. Fekete, C. Knauer, J. Lenchner, J. Mitchell, K. Whittlesey, Minimum-cost coverage of point sets by disks, in: Proc. of the 22nd Ann. Symp. on Comp. Geom., SoCG, ACM, New York, 2006, pp. 449–458. doi:10.1145/1137856.1137922.

[6] M. Gibson, G. Kanade, E. Krohn, I. Pirwani, K. Varadarajan, On clustering to minimize the sum of radii, SIAM Journal on Computing 41 (1) (2012) 47–60. doi:10.1137/100798144.

[7] A. K. Abu-Affash, P. Carmi, M. J. Katz, G. Morgenstern, Multi cover of a polygon minimizing the sum of areas, in: Proceedings of the 5th international conference on WALCOM: algorithms and computation, Springer-Verlag, Berlin, 2011, pp. 134–145.
URL `dl.acm.org/citation.cfm?id=1966169.1966190`

[8] A. Freund, D. Rawitz, Combinatorial interpretations of dual fitting and primal fitting, in: R. Solis-Oba, K. Jansen (Eds.), Approximation and Online Algorithms, Vol. 2909 of LNCS, Springer-Verlag, 2004, pp. 317–318. doi:10.1007/978-3-540-24592-6_11.

[9] R. Bar-Yehuda, D. Rawitz, A note on multicovering with disks, Comput. Geom. 46 (3) (2013) 394–399. doi:10.1016/j.comgeo.2012.10.006.

[10] D. Halperin, Arrangements, in: J. E. Goodman, J. O'Rourke (Eds.), Handbook of Discrete and Computational Geometry, CRC Press LLC, Boca Raton, FL, 2004, Ch. 24, pp. 529–562.

[11] D. Y. Grigor'ev, N. N. Vorobjov Jr, Solving systems of polynomial inequalities in subexponential time, Journal of Symbolic Computation 5 (1–2) (1988) 37 – 64. doi:10.1016/S0747-7171(88)80005-1.