

A One-Dimensional Bin Packing Problem with Shelf Divisions [★]

E. C. Xavier ^{a,*} F. K. Miyazawa ^a

^a*Instituto de Computação — Universidade Estadual de Campinas, Avenida Albert Einstein, 1251, Caixa Postal 6176 — 13084-971 — Campinas-SP — Brazil, Fax: (+55)(19)3521-5847*

Abstract

Given bins of size B , non-negative values d and Δ , and a list L of items, each item $e \in L$ with size s_e and class c_e , we define a shelf as a subset of items packed inside a bin with total items size at most Δ such that all items in this shelf have the same class. Two subsequent shelves must be separated by a shelf division of size d . The size of a shelf is the total size of its items plus the size of the shelf division. The Class Constrained Shelf Bin Packing Problem (CCSBP) is to pack the items of L into the minimum number of bins, such that the items are divided into shelves and the total size of the shelves in a bin is at most B . We present hybrid algorithms based on the First Fit (Decreasing) and Best Fit (Decreasing) algorithms, and an APTAS for the problem CCSBP when the number of different classes is bounded by a constant C .

Key words: Approximation algorithms, bin packing, shelf packing
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* Corresponding author: Instituto de Computação — Universidade Estadual de Campinas, Caixa Postal 6176 — 13084-971 — Campinas-SP — Brazil, Fax: (+55)(19)3521-5847

Email addresses: eduardo.xavier@gmail.com (E. C. Xavier),
fkf@ic.unicamp.br (F. K. Miyazawa).

1 Introduction

In this paper we present approximation algorithms for a class constrained bin packing problem when the items must be separated by non-null shelf divisions. We denote this problem by *Class Constrained Shelf Bin Packing Problem* (CCSBP).

An instance for the CCSBP problem is a tuple $I = (L, s, c, d, \Delta, B)$, where L is a list of items, s and c are size and class functions over L , d is the size of the shelf division, Δ is the maximum size of a shelf and B is the size of the bins. Given a sublist of items $L' \subseteq L$ we denote by $s(L')$ the sum of the sizes of the items in L' , i.e, $s(L') = \sum_{e \in L'} s_e$. A *shelf packing* \mathcal{P} of an instance I for the CCSBP problem is a set of bins $\mathcal{P} = \{P_1, \dots, P_k\}$, where the items packed in a bin $P_i \in \mathcal{P}$ are partitioned into shelves $\{N_1^i, \dots, N_{q_i}^i\}$ such that for each shelf N_j^i we have that $s(N_j^i) \leq \Delta$, all items in N_j^i are of the same class and $\sum_{j=1}^{q_i} (s(N_j^i) + d) \leq B$. Without loss of generality we consider that $0 < s_e \leq \Delta$ and $c_e \in \mathbb{Z}^+$ for each $e \in L$.

The CCSBP problem is to find a shelf packing of the items of L into the minimum number of bins. This problem is *NP*-hard since it is a generalization of the bin packing problem: in this case consider that the instance has just one class, $\Delta = B$ and $d = 0$. We note that the term shelf is used under another context in the literature for the 2-D strip packing problem. In this case, packings are two staged packings divided into levels.

There are many practical applications for the CCSBP problem even when there is only one class of items. For example, when the items to be packed must be separated by non-null shelf divisions (inside a bin) and each shelf has a limited capacity. In Figure 1 we can see an example of a shelf packing of items into one bin, with $B = 60$, $\Delta = 17$, $d = 3$ and all items of the same class. The CCSBP problem is also adequate when some items cannot be stored in a same shelf (like foods and chemical products). In most of the cases, the sizes of the shelf divisions have non-negligible width. Although these problems are very common in practice, to our knowledge this is the first paper that presents approximation results for them.

An interesting application for the CCSBP problem was introduced by Ferreira et al. [4] in the iron and steel industry. In this problem, we have raw material rolls that must be cut into final rolls grouped by certain properties after two cutting phases. The rolls obtained after the first phase, called primary rolls, are submitted to different processing operations (tensioning, tempering, laminating, hardening etc.) before the second phase cut. Due to technological limitations, primary rolls have a maximum allowable width and each cut has a trimming process that generates a loss in the roll width. Each processing

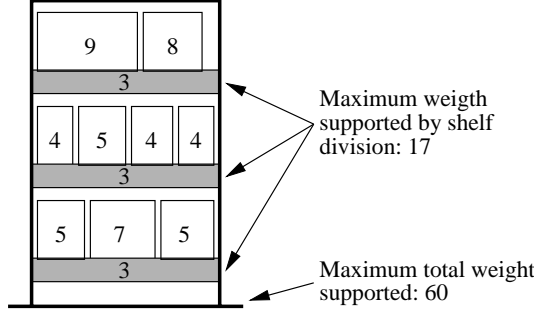


Fig. 1. Example of a shelf packing of items into one bin.

operation has a high cost which implies items to be grouped before doing it, where each group corresponds to one shelf.

Given an algorithm \mathcal{A} , and an instance I for the CCSBP problem, we denote by $\mathcal{A}(I)$ the number of bins used by algorithm \mathcal{A} to pack the instance I and by $\text{OPT}(I)$ the number of bins used in an optimal solution. The algorithm \mathcal{A} is an α -approximation, if $\mathcal{A}(I)/\text{OPT}(I) \leq \alpha$, for any instance I . In this case, we also say that \mathcal{A} has an absolute performance bound α . In bin packing problems, it is also usual to use the asymptotic worst case analysis. We say that \mathcal{A} has an asymptotic performance bound α if there is a constant β such that $\mathcal{A}(I) \leq \alpha \text{OPT}(I) + \beta$ for any instance I .

Given an algorithm A_ε , for some $\varepsilon > 0$, and an instance I for some problem P we denote by $A_\varepsilon(I)$ the value of the solution returned by algorithm A_ε when executed on instance I . We say that A_ε , for $\varepsilon > 0$, is an asymptotic polynomial time approximation scheme (APTAS) for the problem CCSBP if there exist constants t and K such that $A_\varepsilon(I) \leq (1 + t\varepsilon)\text{OPT}(I) + K$ for any instance I .

Results: In this paper we present hybrid algorithms for the CCSBP problem, based on the First Fit (Decreasing) and Best Fit (Decreasing) algorithms for the bin packing problem. When the number of different classes is bounded by a constant, we show that the hybrid versions of the First Fit and Best Fit algorithms have an asymptotic performance bound of 3.4 and the hybrid versions of the First Fit Decreasing and Best Fit Decreasing algorithms have an asymptotic performance bound less than 2.445. In the case where the number of different classes is part of the input, we show that the hybrid versions of the First Fit and Best Fit algorithms have an absolute performance bound of 4 and the hybrid version of the First Fit Decreasing algorithm has an absolute performance bound of 3. At last, for the case when the number of classes is bounded by a constant, we present an APTAS for the CCSBP problem.

Related Work: A special case of the CCSBP problem is the Bin Packing problem, which is one of the most studied problems in the literature. Some of the most famous algorithms for the bin packing problem are the algorithms FF, BF, FFD and BFD, with asymptotic performance bounds $17/10$, $17/10$,

11/9 and 11/9, respectively. Fernandez de la Vega and Lueker [3] presented an APTAS for the bin packing problem. Dawande et al. [2], presented approximation schemes for a class constrained version of the bin packing (CCBP), where bins can have different sizes and each bin is used to pack items of at most k different classes, and the number of different classes in the input instance is bounded by a constant. Shachnai and Tamir [7], presented a polynomial time approximation scheme for a dual version of the problem CCBP also for the case where the number of different classes in the input instance is bounded by a constant. Shachnai and Tamir [8], considered a special case of an online class constrained bin packing problem. In this case all items have the same size and must be packed without knowledge of the next subsequent items of the input. We refer the reader to Coffman et al. [1] for a survey on approximation algorithms for bin packing problems. In [11], we consider the knapsack version of the CCSBP problem, where each item e has also a value v_e . The objective is to find a shelf packing of a subset S of the items in just one knapsack (bin) of size K , such that the total value of the items in S is maximum. We also give a PTAS for this problem. We remark that, despite of the similarity of the problems, the techniques and algorithms used in this paper are not related to the ones used in the knapsack version of the problem. Practical approaches for the CCSBP problem were considered by Ferreira et al. [4], that introduced the problem in the iron and steel industry. Recently, the problem was considered by Hoto et al. [5] and Marques and Arenales [6]. Hoto et al. [5], considered the cutting stock version of the problem where a demand of items must be attended by the minimum number of bins. They use a column generation strategy. In [6] exact and heuristic algorithms are presented for a knapsack version of the problem.

1.1 Notation

On the forthcoming sections we use the following notation: Let $I = (L, s, c, d, \Delta, B)$ be an instance for the CCSBP problem. We denote by $n = |L|$ the number of items in this instance. For any integer t , we denote by $[t]$ the set $\{1, \dots, t\}$. We assume that each class belongs to the set $[C]$. We assume that C is bounded by a constant, unless otherwise stated. We denote by $\text{OPT}_s(I)$ the minimum number of non-null shelves in an optimal packing of I , and by $\text{OPT}(I)$ the number of bins in this optimal solution. Given a packing $\mathcal{P} = \{P_1, \dots, P_k\}$, we denote by $|\mathcal{P}| = k$ the number of bins used in this packing, and by $N_s(\mathcal{P})$ the number of shelves used in all bins of \mathcal{P} . Given an algorithm \mathcal{A} we denote by $\mathcal{A}(I)$ the number of bins used by the algorithm \mathcal{A} to pack the instance I .

1.2 Simple Lower Bounds

The following facts present lower bounds for the number of bins used in any optimum solution for the CCSBP problem.

Fact 1 For any instance $I = (L, s, c, d, \Delta, B)$, we have

$$\text{OPT}(I) \geq \frac{s(L)}{\lceil B/(d + \Delta) \rceil \Delta}.$$

PROOF. Since $\lceil B/(d + \Delta) \rceil$ is an upper bound for the number of totally filled shelves in a bin, the total items size in a bin is at most $\lceil B/(d + \Delta) \rceil \Delta$. \square

Fact 2 For any instance $I = (L, s, c, d, \Delta, B)$, we have

$$\text{OPT}(I) \geq \frac{s(L) + \text{OPT}_s(I)d}{B} \geq \frac{s(L) + \lceil s(L)/\Delta \rceil d}{B}.$$

PROOF. The statement holds since $\lceil s(L)/\Delta \rceil$ is a lower bound for the number of shelves used in any packing. \square

2 Hybrid versions of the First Fit and Best Fit Algorithms

In this section we present hybrid versions of the First Fit (Decreasing) and Best Fit (Decreasing) algorithms, for the classic bin packing problem, to the CCSBP problem. Without loss of generality, we assume that all bins have capacity 1.

We briefly describe how these algorithms work for the classic bin packing problem. The First Fit (FF) and the Best Fit (BF) algorithms pack the items of a given list $L = (e_1, \dots, e_m)$ in the order given by L . Assume that the items e_1, \dots, e_{i-1} have been packed into bins B_1, B_2, \dots, B_k , each bin with capacity 1. To pack the next item e_i , the algorithm FF (resp. BF) finds the smallest index j , $1 \leq j \leq k$, such that $s(B_j) + s(e_i) \leq 1$ (resp. $s(B_j)$ is maximum given that $s(B_j) + s(e_i) \leq 1$). If the algorithm FF (resp. BF) finds such a bin, the item e_i is packed into the bin B_j . Otherwise, the item e_i is packed into a new bin B_{k+1} . This process is repeated until all the items of L have been packed.

The First Fit Decreasing (FFD) (resp. Best Fit Decreasing (BFD)) algorithm first sorts the items of L in non-increasing order of size and then apply the algorithm FF (resp. BF). The following result holds (see [1,9]).

Theorem 3 *For any instance I for the bin packing problem, we have*

$$\text{FF}(I) \leq \frac{17}{10} \text{OPT}(I) + 1, \quad \text{BF}(I) \leq \frac{17}{10} \text{OPT}(I) + 1,$$

$$\text{FFD}(I) \leq \frac{11}{9} \text{OPT}(I) + 3, \quad \text{BFD}(I) \leq \frac{11}{9} \text{OPT}(I) + 3$$

$$\text{and } \text{FFD}(I) \leq \frac{3}{2} \text{OPT}(I).$$

Now we can present the hybrid algorithms for the problem CCSBP.

Algorithms SFF, SBF, SFFD and SBFD: Given an instance $I = (L, s, c, d, \Delta, B)$, the algorithm SFF (resp. SBF, SFFD and SBFD) uses the algorithm FF (resp. BF, FFD and BFD) to pack all items of a same class into shelves of size Δ . The algorithm considers the size of each generated shelf as the total items size in the shelf plus the size of the shelf division d . The set of generated shelves are then packed into bins of size B using the algorithm FF (resp. BF, FFD and BFD).

Given an instance $I = (L, s, c, d, \Delta, B)$ for the CCSBP problem, we denote by $\text{OPT}_\Delta(I)$ the minimum number of shelves of size Δ needed to pack L , where all items in a shelf have the same class. Clearly $\text{OPT}_\Delta(I)$ is a lower bound for the number of shelves used in any optimal solution. That is, $\text{OPT}_\Delta(I) \leq \text{OPT}_s(I)$.

Theorem 2.1 *Let I be an instance for the CCSBP problem. If the number of classes in I is bounded by C then*

$$\text{SFF}(I) \leq \left(3 + \frac{2}{5}\right) \text{OPT}(I) + 2C, \quad \text{SBF}(I) \leq \left(3 + \frac{2}{5}\right) \text{OPT}(I) + 2C,$$

$$\text{SFFD}(I) \leq \left(2 + \frac{4}{9}\right) \text{OPT}(I) + 6C \quad \text{and} \quad \text{SBFD}(I) \leq \left(2 + \frac{4}{9}\right) \text{OPT}(I) + 6C.$$

PROOF. Let \mathcal{A}' be an algorithm in $\{\text{FF}, \text{BF}, \text{FFD}, \text{BFD}\}$ such that

$$\mathcal{A}'(I') \leq \alpha \text{OPT}(I') + \beta$$

for any instance I' of the classic bin packing problem and let \mathcal{A} be the corresponding algorithm in $\{\text{SFF}, \text{SBF}, \text{SFFD}, \text{SBFD}\}$. Let $I = (L, s, c, d, \Delta, B)$ be an instance for the CCSBP problem. Consider the packing \mathcal{P} produced by the algorithm \mathcal{A} for the instance I . We consider that $|\mathcal{P}| > 1$, otherwise \mathcal{P} is optimum. On average, all bins in \mathcal{P} are filled by at least $1/2$ (including shelf divisions), since the algorithms pack the shelves in such a way that any pair of bins have total contents size greater than 1. We can conclude the following:

$$\begin{aligned} \mathcal{A}(I)(1/2) &\leq s(L) + N_s(\mathcal{P})d \\ &\leq s(L) + (\alpha\text{OPT}_\Delta(I) + C\beta)d \end{aligned} \tag{1}$$

$$\leq s(L) + (\alpha\text{OPT}_s(I) + C\beta)d \tag{2}$$

$$\begin{aligned} &\leq \alpha(s(L) + d\text{OPT}_s(I)) + C\beta d \\ &\leq \alpha\text{OPT}(I) + C\beta, \end{aligned} \tag{3}$$

where (1) holds from Theorem 3, and (3) follows from Fact 2 and the fact that $d \leq 1$. \square

Notice that any algorithm for the classic bin packing problem can be easily extended to an algorithm with the same asymptotic performance bound and that produces bins that on average are filled by at least half of its capacities. Therefore, the following result can be easily derived as a generalization of the previous theorem.

Corollary 2.2 *Given an algorithm \mathcal{A}' for the bin packing problem, where $\mathcal{A}'(L) \leq \alpha\text{OPT}(L) + \beta$ for any instance L , then there exists an algorithm \mathcal{A} for the CCSBP such that*

$$\mathcal{A}(I) \leq 2\alpha\text{OPT}(I) + 2\beta C,$$

for any instance I of the CCSBP problem.

This result shows that when the number of classes is bounded by a constant we can obtain, using an APTAS for the bin packing problem, algorithms for the CCSBP problem with asymptotic performance bound as close to 2 as desired (although with high time complexity and with high value of β).

Now we consider that the number of different classes is not bounded by a constant. Notice that if a given algorithm \mathcal{A}' for the bin packing problem has absolute performance bound α , then we can derive an algorithm \mathcal{A} for the CCSBP problem with absolute performance bound 2α , even if the number of different classes of items is given as part of the input. Using the fact that algorithms FF, BF and FFD have absolute performance bound 2, 2 and $3/2$ respectively, we can obtain the following result.

Corollary 2.3 *Let I be an instance for the CCSBP problem, then*

$$\text{SFF}(I) \leq 4\text{OPT}(I), \quad \text{SBF}(I) \leq 4\text{OPT}(I) \quad \text{and} \quad \text{SFFD}(I) \leq 3\text{OPT}(I),$$

even if the number of different classes is not bounded by a constant.

From the practical point of view, the size of the shelf division d is not so large compared with Δ . The next theorem shows that if d is a small fraction

of Δ , we can obtain a better performance bound for the Best and First Fit strategies.

Theorem 2.4 *Let $I = (L, s, c, d, \Delta, B)$ be an instance for the CCSBP problem. If the number of classes in I is bounded by C and $d = \frac{\Delta}{r}$, $r \geq 1$, we have*

$$\begin{aligned} \text{SFF}(I) &\leq \left(2 + \frac{14}{5r}\right) \text{OPT}(I) + 2C, & \text{SBF}(I) &\leq \left(2 + \frac{14}{5r}\right) \text{OPT}(I) + 2C, \\ \text{SFFD}(I) &\leq \left(2 + \frac{8}{9r}\right) \text{OPT}(I) + 6C, & \text{SBFD}(I) &\leq \left(2 + \frac{8}{9r}\right) \text{OPT}(I) + 6C, \\ \text{and } \text{SFFD}(I) &\leq \left(2 + \frac{2}{r}\right) \text{OPT}(I). \end{aligned}$$

PROOF. Let \mathcal{A}' be an algorithm in $\{\text{FF}, \text{BF}, \text{FFD}, \text{BFD}\}$ such that

$$\mathcal{A}'(I') \leq \alpha \text{OPT}(I') + \beta$$

for any instance I' of the classic bin packing problem and let \mathcal{A} the corresponding algorithm in $\{\text{SFF}, \text{SBF}, \text{SFFD}, \text{SBFD}\}$. Let $I = (L, s, c, d, \Delta, B)$ be an instance for the CCSBP problem and \mathcal{P} the packing produced by the algorithm \mathcal{A} for the instance I .

We divide the proof in two cases, according to the values of $N_s(\mathcal{P})$ and $\text{OPT}_s(I)$.

CASE 1: $N_s(\mathcal{P}) < \text{OPT}_s(I)$. In this case, we have

$$\begin{aligned} \mathcal{A}(I)(1/2) &\leq s(L) + N_s(\mathcal{P})d \\ &< s(L) + \text{OPT}_s(I)d \\ &\leq \text{OPT}(I), \end{aligned} \tag{4}$$

where inequality (4) holds from Fact 2. That is,

$$\mathcal{A}(I) \leq 2\text{OPT}(I). \tag{5}$$

CASE 2: $N_s(\mathcal{P}) \geq \text{OPT}_s(I)$. In this case, we can follow the proof of Theorem 2.1 and obtain inequality (2). That is,

$$\mathcal{A}(I)(1/2) \leq s(L) + (\alpha \text{OPT}_s(I) + C\beta)d. \tag{6}$$

Since on average each shelf generated by the algorithm \mathcal{A} is filled by at least $\Delta/2$ (not including the shelf division), we have

$$\begin{aligned}
\text{OPT}(I) &\geq s(L) \\
&\geq N_s(\mathcal{P})(\Delta/2) \\
&\geq \text{OPT}_s(I)\Delta/2 = \text{OPT}_s(I)dr/2.
\end{aligned}$$

That is, $\text{OPT}_s(I)d \leq (2\text{OPT}(I))/r$. Therefore, from inequality (6), we have

$$\begin{aligned}
\mathcal{A}(I)(1/2) &\leq s(L) + (\alpha\text{OPT}_s(I) + C\beta)d. \\
&= s(L) + \text{OPT}_s(I)d + (\alpha - 1)\text{OPT}_s(I)d + C\beta d. \\
&\leq \text{OPT}(I) + \frac{\alpha - 1}{r}(2\text{OPT}(I)) + C\beta d. \\
&\leq \left(1 + \frac{2(\alpha - 1)}{r}\right)\text{OPT}(I) + C\beta d.
\end{aligned}$$

That is,

$$\mathcal{A}(I) \leq \left(2 + \frac{4(\alpha - 1)}{r}\right)\text{OPT}(I) + 2\beta C. \tag{7}$$

The theorem follows from inequalities (5), (7) and theorem 3. \square

The following proposition shows that the previous theorem presents an asymptotic performance bound that is tight for the algorithms SFF and SBF, when d is very small compared to Δ .

Proposition 2.5 *The asymptotic performance bound of the algorithms SFF and SBF is at least 2, even when there is only one class.*

PROOF. Let $I_n = (L, s, c, d, \Delta, B)$ be an instance with $L = (e_1, \dots, e_{2n})$, $\varepsilon = 1/n$, $d = \varepsilon/2$, $B = 1$, $\Delta = 1/2$ and $s(e_i) = 1/2 - \varepsilon$ when i is odd and $s(e_i) = \varepsilon$ otherwise. Notice that $d = \Delta/n$. Also assume that all items have a same class. The SFF and SBF algorithms applied over this instance generates n shelves, each one containing one item of size $1/2 - \varepsilon$ and one item of size ε . The final packing generated by these algorithms has n bins, each one containing one shelf. An optimal packing with $n/2 + 1$ bins can be obtained in such a way that $n/2$ bins have two shelves each, one shelf with an item of size $1/2 - \varepsilon$, and the other shelf with two items, one item of size $1/2 - \varepsilon$ and another of size ε . The last bin contains the remaining items of size ε . \square

3 An Asymptotic Polynomial Time Approximation Scheme

In this section we present an APTAS for the CCSBP problem when the number of different classes is bounded by a constant C .

The algorithm is presented in Figure 2 and is denoted by ASBP_ϵ . It considers two cases: When $\epsilon \geq d + \Delta$, it uses an algorithm denoted by ASBP'_ϵ and in the other case, it uses an algorithm denoted by ASBP''_ϵ . Notice that the algorithm ASBP''_ϵ receives as input a rescaled instance so that the maximum shelf capacity is 1.

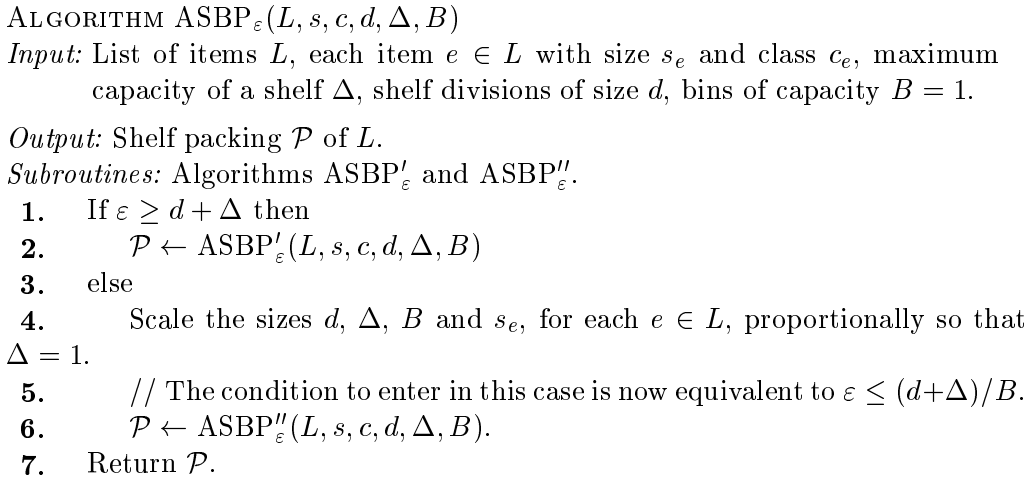


Fig. 2. Algorithm ASBP_ϵ .

The intuition to consider these two cases is that in the first case, we can pack shelves almost optimally because the maximum size of a shelf is bounded by ϵ , and then the bins can be filled by at least $(1 - \epsilon)$. In the second case, since $\epsilon < d + \Delta$, we can bound by a constant the number of shelves used in each bin of an optimal solution. Then an enumeration step can be done to guess the shelves that are used in an optimal solution and an almost optimal shelf packing can be generated for large items. Small items are packed later using a linear programming strategy. In the following two subsections we show that algorithms ASBP'_ϵ and ASBP''_ϵ are APTAS.

3.1 The algorithm ASBP'_ϵ

In this section we show that the algorithm ASBP'_ϵ is an APTAS for its corresponding case. This algorithm uses two subroutines: One is the FF algorithm and the other is an APTAS for the one dimensional bin packing problem presented by Fernandez de la Vega and Lueker [3]. We consider the version of this APTAS presented by Vazirani [10], which we denote by FL_ϵ , for which

the following statement holds.

Theorem 3.1 *For any $\varepsilon > 0$, there exists a polynomial time algorithm FL_ε to pack a list of items L , each item $e \in L$ with size $s_e \in [0, \Delta]$, into bins of capacity Δ such that $\text{FL}_\varepsilon(L) \leq (1 + \varepsilon) \text{OPT}_\Delta(L) + 1$, where $\text{OPT}_\Delta(L)$ is the minimum number of bins of capacity Δ to pack L .*

The algorithm ASBP'_ε is presented in Figure 3. Given an instance I , the algorithm ASBP'_ε first packs all items of the instance into bins of size Δ using the algorithm FL_ε . The algorithm ASBP'_ε considers each one of these bins of size Δ as a shelf, where the size of a shelf is its total items size plus the size d of a shelf division. The algorithm ASBP'_ε packs these shelves into bins of size 1 using the algorithm FF.

ALGORITHM $\text{ASBP}'_\varepsilon(L, s, c, \Delta, d, B)$

Input: List of items L , each item $e \in L$ with size s_e and class c_e , maximum capacity of a shelf Δ , shelf divisions of size d , bins of capacity $B = 1$ and $\varepsilon \geq d + \Delta$.

Output: Shelf packing \mathcal{P} of L .

Subroutines: Algorithms FL_ε and FF.

1. Let L_c be the set of items of class c in L .
 2. For each class $c \in [C]$ let \mathcal{P}_Δ^c be the packing of L_c obtained by the algorithm FL_ε using bins of capacity Δ .
 3. Let \mathcal{P}_Δ be the union of the packings \mathcal{P}_Δ^c , for each $c \in [C]$.
 4. Consider each bin $D \in \mathcal{P}_\Delta$ as a shelf with size $\sum_{e \in D} s_e + d$.
 5. Let S be the set of shelves obtained from \mathcal{P}_Δ .
 6. Let \mathcal{P} be the packing obtained with the algorithm FF to pack the shelves of S into unit bins.
 7. Return \mathcal{P} .
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Fig. 3. Algorithm ASBP'_ε where $\varepsilon \geq d + \Delta$.

The following statement holds for the algorithm ASBP'_ε .

Lemma 3.2 *The algorithm ASBP'_ε , is an APTAS for the CCSBP problem when the given instance I is such that $B = 1$ and $\varepsilon \geq d + \Delta$.*

PROOF. In step 2, the algorithm obtains a packing \mathcal{P}_Δ^c of items of class c in L (items in L_c) into bins of capacity Δ using the algorithm FL_ε . By Theorem 3.1, we have

$$|\mathcal{P}_\Delta^c| \leq (1 + \varepsilon) \text{OPT}_\Delta(L_c) + 1. \quad (8)$$

The algorithm then considers each bin in \mathcal{P}_Δ as a shelf and obtains a shelf packing \mathcal{P} using the algorithm FF to pack these shelves into unit bins. Since

$\varepsilon \geq d + \Delta$, all bins of \mathcal{P} , except perhaps the last, must be filled by at least $1 - \varepsilon$. So,

$$\begin{aligned}
(\text{ASBP}'_\varepsilon(L) - 1)(1 - \varepsilon) &\leq s(L) + d|\mathcal{P}_\Delta| \\
&\leq s(L) + d \sum_{c=1}^C ((1 + \varepsilon)\text{OPT}_\Delta(L_c) + 1) \\
&\leq (1 + \varepsilon)(s(L) + d \sum_{c=1}^C \text{OPT}_\Delta(L_c)) + dC \\
&\leq (1 + \varepsilon)(s(L) + d\text{OPT}_s(I)) + dC \\
&\leq (1 + \varepsilon)\text{OPT}(I) + C
\end{aligned} \tag{9}$$

where inequality (9) is valid from Fact 2. Also notice that $d < 1$. Therefore, for any $0 < \varepsilon < 1/3$ we have

$$\begin{aligned}
\text{ASBP}'_\varepsilon(L) &\leq \frac{1 + \varepsilon}{1 - \varepsilon}\text{OPT}(I) + \frac{C}{1 - \varepsilon} + 1 \\
&\leq (1 + 3\varepsilon)\text{OPT}(I) + \frac{3C}{2} + 1.
\end{aligned}$$

Notice that the running time of the algorithm ASBP'_ε only depends on the running times of algorithms FL_ε and FF , and the value of C . Let $T_{\text{FL}}(n, \varepsilon)$ and $T_{\text{FF}}(n, \varepsilon)$ be the running times of algorithms FL_ε and FF respectively. The running time of algorithm ASBP'_ε is $O(C T_{\text{FL}}(n, \varepsilon) + T_{\text{FF}}(n, \varepsilon))$. Since the algorithms FL_ε and FF have polynomial time complexity in n for fixed ε , the complexity time of algorithm ASBP'_ε is also polynomial in n for fixed ε . \square

3.2 The algorithm $\text{ASBP}''_\varepsilon$

Now, assume that the algorithm ASBP_ε obtains a shelf packing with the algorithm $\text{ASBP}''_\varepsilon$. Throughout this section, we consider that s_ε , d , Δ and B is the rescaled instance, such that $\Delta = 1$. Notice that, the equivalent condition to enter in this case is

$$\varepsilon < \frac{d + \Delta}{B} = \frac{d + 1}{B}. \tag{10}$$

Notice that the maximum number of shelves completely filled packed in a bin is at most $\lceil \frac{B}{d+\Delta} \rceil$ which from (10) is at most $\frac{1}{\varepsilon} + 1$. Observe that if there is any bin with more than $\frac{2}{\varepsilon} + 2$ shelves of a same class, it has at least two shelves of this class with total size at most Δ . In this case, these two shelves can be

combined into only one shelf. Without loss of generality we consider that each bin, in a solution for the CCSBP problem, contains at most $\frac{2}{\epsilon} + 2$ shelves of a same class.

In Figure 4 we present the algorithm ASBP''_{ϵ} . The algorithm first obtains a pair $(\mathcal{P}_1, \mathbb{P})$ where $\mathcal{P}_1 \cup \mathcal{P}'$, for each $\mathcal{P}' \in \mathbb{P}$, is a packing of big items (items with size at least ϵ^2). This pair is obtained by the subroutine A_{LR} . For each packing $\mathcal{P}_1 \cup \mathcal{P}'$, $\mathcal{P}' \in \mathbb{P}$, the algorithm ASBP''_{ϵ} uses the subroutine SMALL to pack the items with size less than ϵ^2 into the packing \mathcal{P}' . At least one of the generated packings uses at most $(1 + O(\epsilon))\text{OPT}(I) + O(1)$ bins as will be shown latter. The algorithm returns the packing with the smallest number of bins.

ALGORITHM $\text{ASBP}''_{\epsilon}(L, s, c, d, \Delta, B)$

Input: List of items L , each item $e \in L$ with size s_e and class c_e , maximum capacity of a shelf $\Delta = 1$, shelf divisions of size d , bins of capacity B and $\epsilon \leq (d + \Delta)/B$.

Output: Shelf packing \mathcal{P} of L .

Subroutines: Algorithms A_{LR} and SMALL .

1. Let G be the set of items $e \in L$ with size $s_e \geq \epsilon^2$ and S the set $L \setminus G$.
 2. Let $(\mathcal{P}_1, \mathbb{P})$ be a pair obtained from the algorithm A_{LR} applied over the list G .
 3. For each $Q \in \mathbb{P}$ do
 4. let \hat{Q} be the packing obtained using the algorithm SMALL to pack S into Q .
 5. Let \mathcal{P} be a packing $\mathcal{P}_1 \cup \hat{Q}$ where $Q \in \mathbb{P}$ and $|\hat{Q}|$ is minimum.
 6. Return \mathcal{P} .
-

Fig. 4. Algorithm ASBP''_{ϵ} .

In the next subsections we present the subroutines used by the algorithm ASBP''_{ϵ} . The first subroutine called A_{LR} is used to generate a set of packings of big items. On the next subsection we present an algorithm called SMALL used to pack small items (items with size smaller than ϵ^2) in the packings of the big items generated by the algorithm A_{LR} . In the last subsection we present the analysis of the algorithm ASBP''_{ϵ} .

3.2.1 Generating Packings for the Big Items

In this section, we present the algorithm A_{LR} used to pack items with size at least ϵ^2 of a given input instance I . This algorithm generates a set of packings such that at least one can be used to pack the small items, such that the resulting packing has size at most $(1 + O(\epsilon))\text{OPT}(I) + O(1)$. This algorithm uses the linear rounding technique, presented by Fernandez de la Vega and Lueker [3], and considers only items with size at least ϵ^2 . The algorithm A_{LR}

returns a pair $(\mathcal{P}_1, \mathbb{P})$, where \mathcal{P}_1 is a packing for a list of very big items and \mathbb{P} is a set of packings for the remaining items.

We use the following notation in the description of the linear rounding technique: Given two lists of items X and Y , let X_1, \dots, X_C and Y_1, \dots, Y_C be the partition of X and Y respectively in classes, where X_c and Y_c have only items of class c for each $c \in [C]$. We write $X \preceq Y$ if there is an injection $f_c : X_c \rightarrow Y_c$ for each $c \in [C]$ such that $s(e) \leq s(f(e))$ for all $e \in X_c$. Given two lists L_1 and L_2 we denote by $L_1 \parallel L_2$ the concatenation of these lists.

The algorithm A_{LR} uses three subroutines: A_{ALL} , SFF , and A_R . The algorithm SFF was presented in Section 2. In what follows we present the algorithms A_{ALL} and A_R .

Algorithm A_{ALL} : This is an algorithm used as subroutine to generate all possible packings with at most $\frac{2}{\varepsilon} + 2$ shelves of a same class, when the size of each item is bounded from below by a constant and the number of distinct sizes in each class is upper bounded by a constant t . The algorithm may generate empty shelves (used latter to pack small items). The following lemma guarantees the existence of such an algorithm.

Lemma 3.3 *Given an instance $I = (L, s, c, d, \Delta, B)$, with $\Delta = 1$, where the number of distinct items sizes in each class is at most a constant t , the number of different classes is bounded by a constant C and each item $e \in L$ has size $s_e \geq \varepsilon^2$, then there exists a polynomial time algorithm that generates all possible shelf packings of L with at most $\frac{2}{\varepsilon} + 2$ shelves of a same class in each bin.*

PROOF. The number of items in a shelf is bounded by $p = 1/\varepsilon^2$. Given a class, the number of different shelves for it is bounded by $r' = \binom{p+t+1}{p}$ and so, the number of different shelves is bounded by $r = Cr'$. Since the number of shelves in a bin is bounded by $q = C(\frac{2}{\varepsilon} + 2)$, the number of different bins is bounded by $u = \binom{q+r}{q}$. Notice that u is a (large) constant since all the values p, q, r and u depends only on ε, C and t which are constants.

Therefore, the number of all feasible packings is bounded by $\binom{n+u}{n}$, which is bounded by $(n+u)^u$, which in turn is polynomial in n . \square

Notice that the complexity time of the algorithm A_{ALL} is $O(n^{O(2C/\varepsilon)^{O(1/\varepsilon^2)^t}})$.

Algorithm A_R : Given two lists X and Y such that $X \preceq Y$ and a packing \mathcal{P}_Y of Y , there exists an algorithm, which we denoted by A_R (Replace), with

input (\mathcal{P}_Y, X) , that obtains a packing \mathcal{P}_X for X such that $|\mathcal{P}_X| = |\mathcal{P}_Y|$ as the next lemma guarantees.

Lemma 3.4 *If X and Y are two lists with $X \preceq Y$, then $\text{OPT}(X) \leq \text{OPT}(Y)$. Moreover, if \mathcal{P}_Y is a shelf packing of Y then there exists a polynomial time algorithm A_R that given \mathcal{P}_Y obtains a shelf packing \mathcal{P}_X of X such that $|\mathcal{P}_X| = |\mathcal{P}_Y|$.*

PROOF. The algorithm A_R sorts the lists X_c and Y_c for each $c \in [C]$ in non-increasing order of items size and then replaces in this order, each item of Y_c in the packing \mathcal{P}_Y by an item of X_c . The possible remaining items of Y_c are removed. \square

For any instance X , denote by \overline{X} the instance with precisely $|X|$ items with size equal to the size of the smallest item in X . Clearly, $\overline{X} \preceq X$.

The algorithm A_{LR} is presented in Figure 5. It consists in the following: Let G_1, \dots, G_C be the partition of the input list G into classes $1, \dots, C$ and let $n_c = |G_c|$ for each class c . The algorithm A_{LR} partition each list G_c into groups $G_c^1, G_c^2, \dots, G_c^{k_c}$. Let $G^1 = \cup_{c=1}^C G_c^1$. The algorithm generates a packing \mathcal{P}_1 of G^1 using

$$O(\varepsilon)\text{OPT}(I) + 1$$

bins and a set \mathbb{P} with polynomial number of packings for the items in $G \setminus G^1$. The packing \mathcal{P}_1 is generated by the algorithm SFF and the set of packings \mathbb{P} is generated using the algorithms A_{ALL} and A_R .

Denote by T_{ALL} , T_R and T_{SFF} the time complexity of algorithms A_{ALL} , A_R , and SFF respectively. The time complexity of steps 1–3 of algorithm A_{LR} is bounded by $O(n \log n)$. The overall time complexity of algorithm A_{LR} is $O(n \log n + T_{SFF} + T_{ALL} + T_{ALL}T_R)$. Since T_{ALL} , T_R and T_{SFF} have polynomial time complexity in n for fixed ε , the time complexity of algorithm A_{LR} is also polynomial in n for fixed ε .

The following statement holds for the packing \mathcal{P}_1 .

Lemma 3.5 *The packing \mathcal{P}_1 for the items in G^1 is such that $|\mathcal{P}_1| \leq 4\varepsilon \text{OPT}(I) + 1$.*

PROOF. First, consider the total items size packed in a bin T of some shelf packing. From Fact 1 any optimum solution must satisfy

ALGORITHM $A_{LR}(G)$

Input: List G with n items, each item $e \in G$ with size $s_e \geq \varepsilon^2$; maximum capacity of a shelf $\Delta = 1$; shelf divisions of size d and bins of capacity B .

Output: A pair $(\mathcal{P}_1, \mathbb{P})$, where \mathcal{P}_1 is a packing and \mathbb{P} is a set of packings, where $\mathcal{P}_1 \cup \mathcal{P}'$ is a packing of G for each $\mathcal{P}' \in \mathbb{P}$.

Subroutines: Algorithms A_{ALL} , SFF and A_R .

1. Partition G into lists G_c for each class $c = 1, \dots, C$ and let $n_c = |G_c|$.
 2. Partition each list G_c into $k_c \leq \lceil 1/\varepsilon^3 \rceil$ groups $G_c^1, G_c^2, \dots, G_c^{k_c}$, such that $G_c^1 \succeq G_c^2 \succeq \dots \succeq G_c^{k_c}$,
where $|G_c^j| = q_c = \lfloor n_c \varepsilon^3 \rfloor$ for all $j = 1, \dots, k_c - 1$,
and $|G_c^{k_c}| \leq q_c$. **3.** Let $G^1 = \cup_{c=1}^C G_c^1$.
 4. Let \mathcal{P}_1 be a packing of G^1 obtained by the algorithm SFF.
 5. Let \mathbb{Q} be the set of all possible packings obtained with the algorithm A_{ALL} over the list $(\overline{G_1^1} \parallel \dots \parallel \overline{G_1^{k_1-1}} \parallel \dots \parallel \overline{G_C^1} \parallel \dots \parallel \overline{G_C^{k_C-1}})$.
 6. Let \mathbb{P} be the set of packings obtained with the algorithm A_R over each pair $(\mathcal{Q}, G_1^2 \parallel \dots \parallel G_1^{k_1} \parallel \dots \parallel G_C^2 \parallel \dots \parallel G_C^{k_C})$, where $\mathcal{Q} \in \mathbb{Q}$.
 7. Return $(\mathcal{P}_1, \mathbb{P})$.
-

Fig. 5. Algorithm to obtain packings for items with size at least ε^2 .

$$\text{OPT}(I) \geq \frac{s(L)}{\lceil B/(d + \Delta) \rceil \Delta} = \frac{s(L)}{\lceil B/(d + 1) \rceil} \geq \frac{1}{2} \frac{s(L)}{B/(d + 1)}. \quad (11)$$

Notice that $\sum_{c=1}^C n_c = n$. The algorithm SFF packs at least $\lfloor B/(d + \Delta) \rfloor$ shelves in each bin, each shelf with at least one item. This means that each bin has at least $\lfloor B/(d + \Delta) \rfloor$ items, except perhaps the last, each item with size at least ε^2 and at most 1. Since the group G_1 has at most $n\varepsilon^3$ items, the number of bins in the shelf packing \mathcal{P}_1 can be bounded as follows.

$$\begin{aligned} |\mathcal{P}_1| &\leq \left\lceil \frac{n\varepsilon^3}{\lfloor B/(d + \Delta) \rfloor} \right\rceil = \left\lceil \frac{n\varepsilon^3}{\lfloor B/(d + 1) \rfloor} \right\rceil \\ &\leq 2 \frac{n\varepsilon^3}{B/(d + 1)} + 1 \leq 2 \frac{\varepsilon s(L)}{B/(d + 1)} + 1 \\ &\leq 4\varepsilon \text{OPT}(I) + 1, \end{aligned} \quad (12)$$

where the inequality (12) is valid from (11). \square

3.2.2 Packing the Small Items

In this section we present an algorithm to pack the small items. If we only consider the big items, at least one of the packings generated by the algorithm A_{LR} has basically the same configuration of an optimal packing. That is, one of the generated packings has approximately the same number of bins and approximately the same shelves (including empty shelves that are used only for small items) of an optimal packing. Therefore the algorithm can guess how the small items are packed into the shelves of this packing, leaving only a small fraction of small items to be packed in new extra bins. Notice that a first approach to deal with the CCSBP problem, would be to produce the packing of the big items and then try to pack small items greedily. In the classic bin packing problem this approach works, since after packing the small items in the bins, each bin is filled by at least $(1 - \epsilon)$ of its capacity, except perhaps the last bin. In the CCSBP problem this strategy may not work, since after packing the small items, the packing could use more shelves. This way, each bin would not be filled with items by at least $(1 - \epsilon)$ of its capacity, since each bin also contains shelf divisions. To pack small items in the shelves generated by the algorithm A_{LR} we use a linear programming strategy. This approach has an easier and clearer analysis leading to the APTAS.

The algorithm $ASBP''_{\epsilon}$ uses a subroutine denoted by SMALL to pack small items (size less than ϵ^2) into a given packing of big items. Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a shelf packing of a list of items L and assume that we have to pack a set S of small items, with size at most ϵ^2 , into \mathcal{P} . The packing of the small items is obtained from a solution of a linear program. Let $N_1^{ic}, \dots, N_{n_{ic}}^{ic}$ be the shelves of class c in the bin P_i of the packing \mathcal{P} . For each shelf N_j^{ic} , define a non-negative variable x_j^{ic} . The variable x_j^{ic} indicates the total size of small items of class c that is to be packed in the shelf N_j^{ic} . Consider the following linear program denoted by LPS:

$$\begin{aligned} \max \quad & \sum_{i=1}^k \sum_{c=1}^C \sum_{j=1}^{n_{ic}} x_j^{ic} \\ & s(N_j^{ic}) + x_j^{ic} \leq \Delta \quad \forall i \in [k], c \in [C], j \in [n_{ic}], \quad (1) \\ & \sum_{c=1}^C \sum_{j=1}^{n_{ic}} (s(N_j^{ic}) + x_j^{ic} + d) \leq B \quad \forall i \in [k], \quad (2) \\ & \sum_{i=1}^t \sum_{j=1}^{n_{ic}} x_j^{ic} \leq s(S_c) \quad \forall c \in [C], \quad (3) \\ & x_j^{ic} \geq 0 \quad \forall i \in [k], c \in [C], j \in [n_{ic}] \quad (4) \end{aligned}$$

where S_c is the set of small items of class c in S .

Constraint (1) guarantees that the amount of space used in each shelf is at

most Δ and constraint (2) guarantees that the amount of space used in each bin is at most B . Constraint (3) guarantees that variables x_j^{ic} are not greater than the total size of small items.

Given a packing \mathcal{P} , and a set S of small items, the algorithm SMALL first solves the linear program LPS, and then packs small items in the following way: For each variable x_j^{ic} it packs, while possible, the small items of class c into the shelf N_j^{ic} , so that the total size of the packed small items is at most x_j^{ic} . The possible remaining small items are grouped by classes and packed using the algorithm SFF into new bins. The complexity time of algorithm SMALL is polynomial in n , since the linear program LPS can be solved in polynomial time and the algorithm SFF also has polynomial time.

The following lemma is valid for the algorithm SMALL.

Lemma 3.6 *Let \mathcal{P} be a shelf packing of a list of items L , where each bin of \mathcal{P} has at most $\frac{2}{\varepsilon} + 2$ shelves of a same class, G be the set of items in L with size at least ε^2 and S be the set $L \setminus G$. Let G' be a list of items with $G' \preceq G$ and $\hat{\mathcal{P}}$ be a packing of the items $G' \cup S$ obtained from \mathcal{P} as follows:*

- (1) *Let \mathcal{P}_1 be the packing obtained from \mathcal{P} removing the items of S .*
- (2) *Let \mathcal{P}_2 be the packing of G' using the algorithm A_R over the pair (\mathcal{P}_1, G') .*
- (3) *Let $\hat{\mathcal{P}}$ be the packing obtained applying the algorithm SMALL over the pair (\mathcal{P}_2, S) .*

Then, we have $|\hat{\mathcal{P}}| \leq (1 + 8C\varepsilon)|\mathcal{P}| + C + 1$.

PROOF. Notice that $|\mathcal{P}_2| = |\mathcal{P}|$ and for each shelf N_j in a bin of \mathcal{P} , its corresponding shelf N'_j in \mathcal{P}_2 is such that $s(N'_j) \leq s(N_j)$. If $|\hat{\mathcal{P}}| = |\mathcal{P}|$ then the lemma follows. So assume that the algorithm SMALL uses additional bins to pack the items of S .

Given a bin E , denote by $ns(E)$ the number of shelves in E , $ns_c(E)$ the number of shelves of class c in E , $ss(E)$ the total size of small items in E and $ss_c(E)$ the total size of small items of class c in E .

Consider the linear program LPS. An optimum solution for LPS leads to an optimal fractional packing \mathcal{P}^* of the small items such that $|\mathcal{P}^*| = |\mathcal{P}|$. Consider a bin P_i^* of \mathcal{P}^* and \hat{P}_i the corresponding bin in $\hat{\mathcal{P}}$. We first prove that the following inequality is valid,

$$ss(P_i^*) - ss(\hat{P}_i) \leq 4C\varepsilon. \tag{13}$$

To prove (13), notice that $ns_c(P_i^*) = ns_c(\hat{P}_i)$ for each class c . Given a shelf N_j^{ic} and the corresponding variable x_j^{ic} , the algorithm SMALL packs a set of

items T_j^{ic} in N_j^{ic} such that $x_j^{ic} - s(T_j^{ic}) \leq \varepsilon^2$ since a small item has size at most ε^2 . Since each bin in \mathcal{P}^* has at most $\frac{2}{\varepsilon} + 2$ shelves of a same class, we have for each class $c \in [C]$

$$\begin{aligned} ss_c(\hat{P}_i) &\geq \sum_{j=1}^{n_{ic}} (x_j^{ic} - \varepsilon^2) = ss_c(P_i^*) - ns_c(P_i^*)\varepsilon^2 \\ &\geq ss_c(P_i^*) - \left(\frac{2}{\varepsilon} + 2\right)\varepsilon^2 \geq ss_c(P_i^*) - 4\varepsilon. \end{aligned}$$

Since the above inequalities are valid for each class we can conclude the proof of (13). From (13), we know that the total size of small items packed in additional bins by SMALL with the algorithm SFF, is at most $4C\varepsilon|\mathcal{P}^*| = 4C\varepsilon|\mathcal{P}|$. Denote by $\hat{\mathcal{Q}}$ the set of additional bins. Each shelf generated by the algorithm SFF is filled by at least $\Delta - \varepsilon^2$, except perhaps in C shelves. Therefore, the number of shelves in $\hat{\mathcal{Q}}$ is at most $\left\lceil \frac{4C\varepsilon|\mathcal{P}|}{\Delta - \varepsilon^2} \right\rceil + C \leq 8C\varepsilon|\mathcal{P}| + C + 1$. Since each additional bin has at least one shelf, the number of bins in $\hat{\mathcal{Q}}$ is at most $8C\varepsilon|\mathcal{P}| + C + 1$. Therefore, the number of bins in $\hat{\mathcal{P}}$ is at most $(1 + 8C\varepsilon)|\mathcal{P}| + C + 1$. \square

3.2.3 Analysis of the Algorithm $ASBP''_\varepsilon$

In this section we conclude the analysis of the algorithm $ASBP''_\varepsilon$. First, let T_{LR} and T_S be the time complexities of algorithms A_{LR} and SMALL, respectively. Notice that the time complexity of algorithm $ASBP''_\varepsilon$ is dominated by steps 2–4, that have time complexity $O(T_{LR} + T_{LR}T_S)$. Since the time complexity of algorithms A_{LR} and SMALL is polynomial for fixed ε , the time complexity of algorithm $ASBP''_\varepsilon$ is also polynomial. The following lemma concludes the analysis of the algorithm $ASBP''_\varepsilon$.

Lemma 3.7 *The algorithm $ASBP''_\varepsilon$, is an APTAS for the CCSBP problem when the given instance I is such that $\Delta = 1$, $\varepsilon \leq (d + \Delta)/B$ and the number of different classes is bounded by some constant C .*

PROOF. Given an instance $I = (L, s, c, d, \Delta, B)$, with $\Delta = 1$, let G be the set of items in L with size at least ε^2 and S the set $L \setminus G$.

The items in G are packed by the algorithm A_{LR} . It first partitions G into lists G_c for each class c and then it partitions each list G_c into groups $G_c^1 \succeq G_c^2 \succeq \dots \succeq G_c^{k_c}$. From Lemma 3.5 the following inequality is valid for the list $G^1 = \cup_{c=1}^C G_c^1$.

$$|\mathcal{P}_1| \leq 4\varepsilon \text{OPT}(I) + 1. \tag{14}$$

The packing of the items in $G_1^2 \parallel \dots \parallel G_1^{k_1} \parallel \dots \parallel G_C^2 \parallel \dots \parallel G_C^{k_C}$ is obtained from the set of all possible packings of $\overline{G_1^1} \parallel \dots \parallel \overline{G_1^{k_1-1}} \parallel \dots \parallel \overline{G_C^1} \parallel \dots \parallel \overline{G_C^{k_C-1}}$. Notice that

$$\overline{G_1^1} \parallel \dots \parallel \overline{G_1^{k_1-1}} \parallel \dots \parallel \overline{G_C^1} \parallel \dots \parallel \overline{G_C^{k_C-1}} \succeq G_1^2 \parallel \dots \parallel G_1^{k_1} \parallel \dots \parallel G_C^2 \parallel \dots \parallel G_C^{k_C}.$$

Let \mathcal{O} be an optimum shelf packing of I , \mathcal{O}_1 the packing obtained from \mathcal{O} without the items of S but with the possible empty shelves and \mathcal{O}_2 the packing of \mathcal{O}_1 rounding down each item size to the corresponding item in

$$\overline{G_1^1} \parallel \dots \parallel \overline{G_1^{k_1-1}}, \dots, \overline{G_C^1} \parallel \dots \parallel \overline{G_C^{k_C-1}}.$$

Clearly, $\mathcal{O}_2 \in \mathbb{P}$, where \mathbb{P} is the set of packings generated by the algorithm A_{ALL} . Let $\hat{\mathcal{O}}$ be a packing obtained from the algorithm A_{R} over the pair

$$(\mathcal{O}_2, G_1^2 \parallel \dots \parallel G_1^{k_1}, \dots, G_C^2 \parallel \dots \parallel G_C^{k_C}).$$

If \mathcal{Q} is a packing obtained applying the algorithm SMALL over the pair $(\hat{\mathcal{O}}, S)$, we have from Lemma 3.6 the following result.

$$|\mathcal{Q}| \leq (1 + 8C\varepsilon)|\mathcal{O}| + C + 1 = (1 + 8C\varepsilon)\text{OPT}(I) + C + 1 \quad (15)$$

Since the algorithm ASBP'_ε obtains a packing \mathcal{P} that uses at most the number of bins in $\mathcal{P}_1 \cup \mathcal{Q}$, the theorem follows from inequalities (14) and (15). \square

From lemmas 3.2 and 3.7, the following statement holds.

Theorem 3.8 *The algorithm ASBP'_ε is an APTAS for the CCSBP problem.*

4 Concluding Remarks

In this paper we consider the CCSBP problem, a class constrained bin packing problem with non-null shelf divisions. Although this problem has many practical applications, to our knowledge, this is the first paper to present approximation results for it. We first presented hybrid versions of the First Fit (Decreasing) and Best Fit (Decreasing) algorithms for the bin packing problem to the CCSBP problem. When the number of different classes of items is bounded by a constant C , we prove that the versions of the First Fit and Best Fit have asymptotic performance bound 3.4 and the versions of the First Fit Decreasing

and Best Fit Decreasing have asymptotic performance bound 2.445. We also presented an APTAS for this same case whose running time is

$$O(n^{O(2/\varepsilon)O(1/\varepsilon^2)^{C/\varepsilon^3}}).$$

This algorithm is more of theoretical (rather than practical) interest since it has a high running time (yet polynomial). When the number of classes is not bounded by a constant we show that the algorithm SFFD has absolute performance bound 3.

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