Approximation Algorithms for Circle Packing

Flávio K. Miyazawa
UNICAMP

São Paulo School of Advanced Science on
Algorithms, Combinatorics and Optimization

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Contents

Colaborators and references

Main references

- F. K. Miyazawa, L. L. C. Pedrosa, R. C. S. Schouery, M. Sviridenko, Y. Wakabayashi. Polynomial-Time Approximation Schemes for Circle and Other Packing Problems. *Algorithmica*. To appear.
- P. H. Hokama, F. K. Miyazawa and R. C. S. Schouery. A bounded space algorithm for online circle packing. *Information Processing Letters*, 116, p. 337-342, 2016.

This work also obtained collaboration from Lehilton L. C. Pedrosa, Maxim Sviridenko, Rafael C. S. Schouery, Pedro H. Hokama and Yoshiko Wakabayashi.

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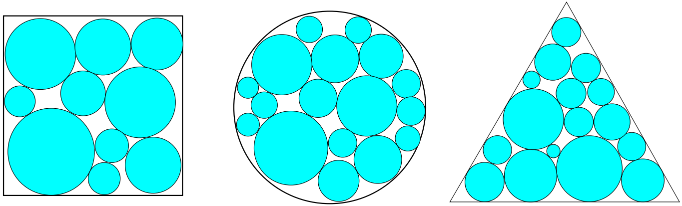
Circle Packing Problems

Packings

Given:

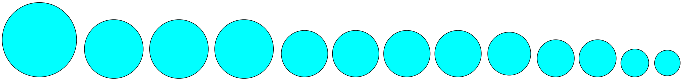
- A list of geometrical items L and bins $\mathcal{B}$
- Obtain a *good* packing of items in L into bin $B \in \mathcal{B}$
- The inner region of two packed items cannot overlap
- Each packed item must be totally contained in the bin



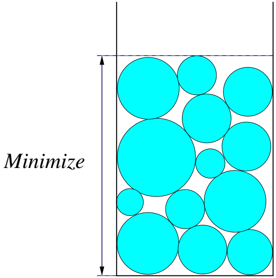
Problems

Circle Strip Packing

- Input: List of circles $L = (c_1, \ldots, c_n)$



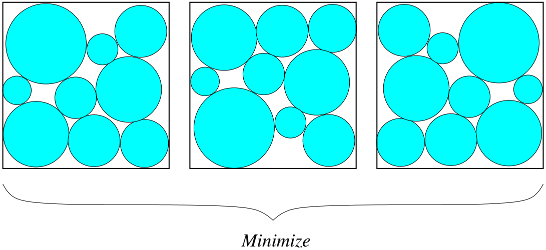
- Output: Packing of L into a rectangle of width 1 and minimum height.



Problems

Circle Bin Packing

- Input: List of circles $L = (c_1, \ldots, c_n)$
- Output: Packing of L into the minimum number of unit bins.



Some Applications

- Cutting and Packing of circular items
- Transportation of tubes, cilinders,...
- Cable assembly/allocations
- Tree plantation
- Origami design
- Marketing
- Cylinder pallet assembly
- ...

Marketing


Get Out There
DIVINE
DAIRY
snap


NISSAN
BURGER KING
LOTUS
STARBUCKS COFFEE
SAAB
FIAT
DELL


Web Images Maps News Shopping Gmail more ▾
Sign in
Google
california driving
Search
Advanced Search
Preferences
Web
Results 1 - 10 of about 1,680,000 for california driving. (0.06 seconds)
California Driving -- A Survivial Guide for Travellers and Tourists
An illustrated survival guide for foreign and domestic tourists and travellers, containing information, tips, warnings, etc, about driving in California.
www.caldrive.com/ - 5k - Cached - Similar pages
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California Driving: Test Your California Driving IQ! So you think you're pretty au fait with California, its drivers, and its culture? ...
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CA DMV Department of Motor Vehicles
Online Guide.
DMV.org/California
$24.95 Online Drivers ED
DMV Authorized Course & Certificate
Summer Special! Enroll now
www.DmvDriversEducation.org
California

Computational Complexity

Demaine, Fekete, Lang'10: To decide if a set of circles can be packed into a square is NP-hard.

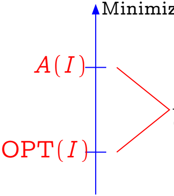
Approximation Algorithms:

- Efficient Algorithms (polynomial time)
- Analysis: How far from the optimum solution value ?
- Compromise:

$$\text{Computational Time} \times \text{Solution Quality}$$

Approximation Algorithms

- $A(I)$ Value of the solution produced by A for instance I
- $\mathrm{OPT}(I)$ Value of an optimum solution of I
- A has approximation factor α if



$$\frac{A(I)}{\mathrm{OPT}(I)} \leq \alpha \quad \text{for any instance } I$$

- A has asymptotic approximation factor α if

$$A(I) \leq \alpha\, \mathrm{OPT}(I) + \beta \qquad \text{for any instance } I,$$

for some constant β

Approximation Algorithms

For Minimization Problems

PTAS: Polynomial Time Approximation Scheme

- A family of polynomial time algorithms A_ε, $\varepsilon > 0$, is a polynomial time approximation scheme if

$$A_\varepsilon(I) \leq (1+\varepsilon)\,\mathrm{OPT}(I), \qquad \text{for any instance } I$$

APTAS: Asymptotic Polynomial Time Approximation Scheme

- A family of algorithms A_ε, $\varepsilon > 0$ is an asymptotic polynomial time approximation scheme if

$$A_\varepsilon(I) \leq (1+\varepsilon)\,\mathrm{OPT}(I) + \beta_\varepsilon, \qquad \text{for any instance } I$$

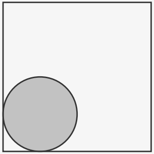
where β_ε is a constant that depends only on ε

Online Packing Algorithms

- Incoming items appears one after the other, sequentially
- An incoming item must be packed when it arrives, without the knowledge of further items
- Once an item is packed, it cannot be repacked again.

Online Circle Bin Packing

Example



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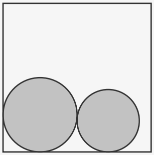
Online Circle Bin Packing

Example



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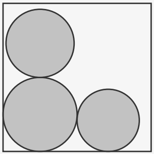
Online Circle Bin Packing

Example



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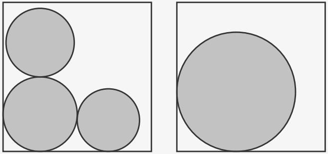
Online Circle Bin Packing

Example



Preliminaries

Some Notation

- If $f : D \rightarrow \mathbb{R}$ is a numerical function, we may write
  - f_e and $f(e)$, indistictly
  - $f(S)$ as the value $\sum_{e \in D} f(e)$, there is no explicit definition
- If c is a circle and $L = (c_1, \ldots, c_n)$ a list of circles, then
  - c is also used to denote its radius
  - $\hat{c}$ is the square with side lengths $2c$
  - $\hat{L}$ is the list $\hat{L} = (\hat{c}_1, \ldots, \hat{c}_n)$
  - $\max(L)$ is the maximum radius of a circle in L
  - $\text{Area}(L)$ is the total area of the circles in L
  - $\overline{L}$ is the list with $|L|$ equal circles with radius $\max(L)$
  - $\mathcal{C}$ is the set containing all lists of circles for the input problem

Preliminaries

If $\mathcal{E}$ is a packing or other geometrical composition,
$\mathcal{E}$ may be considered as the solid structure

- $\text{width}(\mathcal{E})$ is the width of $\mathcal{E}$
- $\text{height}(\mathcal{E})$ is the height of $\mathcal{E}$

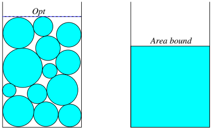
First considerations

- We consider a more general computational model
- Possible to operate over polynomial solutions

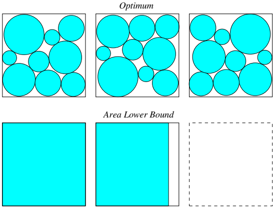
Basic Algorithms

Area Lower Bound

- Circle Strip Packing with bin width 1



- Circle Bin Packing with unit square bins



Area based algorithms

Let

$\mathcal{C}$ the set containing all lists of circles, and

$\mathcal{Q}$ the set containing all lists of squares

next algorithm is a circle version $\mathbb{C}\mathcal{A}$ from square packing algorithm $\mathcal{A}$

$\mathbb{C}\mathcal{A}(L)$

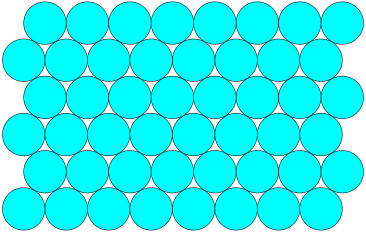
1. Let $\widehat{\mathcal{P}} \leftarrow \mathcal{A}(\widehat{L})$.
2. Let $\mathcal{P}$ the packing $\widehat{\mathcal{P}}$ replacing $\hat{c}_i$ by c_i.
3. Return $\mathcal{P}$.

Lemma. If $\mathcal{A}$ is a square packing algorithm and α, β are constants, st. $\mathcal{A}(S) \leq \alpha \text{Area}(S) + \beta$, for any $S \in \mathcal{Q}$

$$\mathbb{C}\mathcal{A}(L) \leq \alpha \frac{4}{\pi} \text{Area}(L) + \beta, \text{ for any } L \in \mathcal{C}$$

Area based algorithms

Best possible density



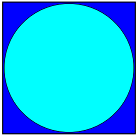
Hexagonal Packing

- Density $\frac{\pi}{\sqrt{12}} \approx 0.9069$

Lemma. There is no algorithm with approximation factor, based only on area arguments, better than $\frac{\sqrt{12}}{\pi} \approx 1.10266$

Using square packing algorithms

- Round each circle c_i to a square $\hat{c}_i$:

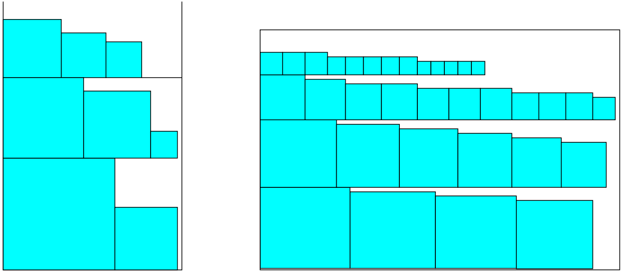


- Use square packing algorithms
- Area increasing: $\frac{Area(\hat{c}_i)}{Area(c_i)} = \frac{4}{\pi} \approx 1.27324$
- Let $\hat{L} = (\hat{c}_1, \ldots, \hat{c}_n)$ the list L rounding each circle to a square
- Bounding the optimum with the area:

$$\text{Area}(\hat{L}) = \frac{4}{\pi}\text{Area}(L) \leq \frac{4}{\pi}\text{OPT}(L)$$

Using square packing algorithms

Shelf Packing:

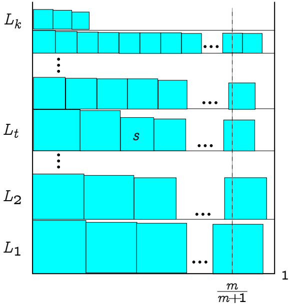


- Items are packed over shelves (of zero thickness)
- side by side in a leftmost way
- Items in a same shelf are packed at the same height.
- Item s can be packed in a shelf S if $\mathrm{width}(s) + \mathrm{width}(S) \leq 1$

Using square packing algorithms

NFDH^s(L) # for Strip Packing

1. Sort $L = (s_1, \ldots, s_n)$ st. $s_1 \geq \cdots \geq s_n$
2. For $i \leftarrow 1$ to n:
3. Pack s_i into the last shelf, if possible
4. otherwise, pack s_i in a new shelf on top of the previous shelf or bin's bottom (in case there is no previous shelf)



Lemma. If L has only squares with side lengths at most $1/m$

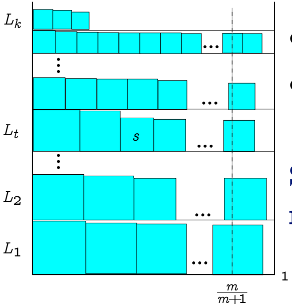
$$\text{NFDH}^s(L) \leq \frac{m+1}{m}\text{Area}(L) + \frac{1}{m}$$

Using square packing algorithms

Sketch.

Each of the first $k-1$ shelves have width filled by at least $\frac{m}{m+1}$

Let L_t the first shelf having square with side $\leq 1/(m+1)$



- $L_1,\ldots,L_{t-1}$: m squares of side $> \frac{1}{m+1}$, each.
- $L_t,\ldots,L_k$: width filled $> 1 - \frac{1}{m+1} = \frac{m}{m+1}$, otherwise receive another item.

Sliding up squares in L_i can cover all rectangular region of shelf L_{i+1}, up to width $\frac{m}{m+1}$.

$$(\mathrm{NFDH}^{\mathrm{S}}(L) - \mathit{height}(L_1))\frac{m}{m+1} \leq \mathrm{Area}(L)$$

So,

$$\mathrm{NFDH}^{\mathrm{S}}(L) \leq \frac{m+1}{m}\mathrm{Area}(L) + \mathit{height}(L_1) \leq \frac{m+1}{m}\mathrm{Area}(L) + \frac{1}{m}$$

Using square packing algorithms

Let

$\mathcal{C}_m$ the set of lists with small circles (diam. $\leq 1/m$, m integer)

Corollary. If $L \in \mathcal{C}_m$, then

$$\mathbb{C}\mathrm{NFDH}^{\mathrm{s}}(L) \leq \frac{m+1}{m}\frac{4}{\pi}\,\mathrm{OPT}(L) + \frac{1}{m} \quad \forall L$$

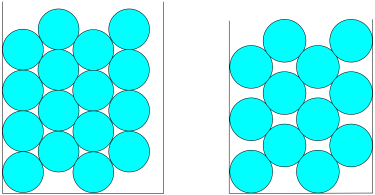
Corollary. If L is a list of circles then

$$\mathbb{C}\mathrm{NFDH}^{\mathrm{s}}(L) \leq 2.548\,\mathrm{OPT}(L) + 1$$

Rounding circles to circles

Algorithm EqualCircles(L) # all circles in L have a same size

1. Let $\mathcal{P}'$ and $\mathcal{P}''$ packings of L as below



2. Return packing $\mathcal{P} \in \{\mathcal{P}', \mathcal{P}''\}$ with minimum height.

Lemma. If all circles of L have radius r, then

$$\text{EqualCircles}(L) \leq 1.654\text{Area}(L) + 2r.$$

Rounding circles to circles

Idea: Circles with close radius are rounded up to the same radius

Given $L \in \mathcal{C}$, $\overline{L}$ is the list with $|L|$ circles with radius $\max(L)$

Algorithm $\mathcal{A}_\varepsilon(L)$

1. $\delta \leftarrow \frac{\varepsilon}{6}$.
2. For $i \geq 0$ do
3. $\quad L_i \leftarrow \{r \in L : \frac{1/2}{(1+\delta)^{i+1}} < r \leq \frac{1/2}{(1+\delta)^{i}}\}$.
4. $\quad \mathcal{P}_i \leftarrow \text{EqualCircles}(\overline{L}_i)$.
5. $\mathcal{P} \leftarrow \mathcal{P}_0 \| \mathcal{P}_1 \| \mathcal{P}_2 \| \ldots$ # concatenation of packings
6. Return $\mathcal{P}$.

Theorem. Given $\varepsilon > 0$, we have

$$\mathcal{A}_\varepsilon(L) \leq (1.654 + \varepsilon)\,\text{Area}(L) + C_\varepsilon, \text{ for any } L \in \mathcal{C}$$

Online Circle Strip Packing

Online Packing

- Incoming items appears one after the other, sequentially
- An incoming item must be packed when it arrives, without the knowledge of further items
- Once an item is packed, it cannot be repacked again.

Baker, Schwarz'83: For $0 < p < 1$, there exists online algorithm $\mathbb{C}\text{NFS}_p$ s.t.,

$$\mathbb{C}\text{NFS}_p(L) \leq \frac{2.548}{p}\,\text{OPT}(L) + \frac{1}{p(1-p)}, \text{ for any } L \in \mathcal{C}$$

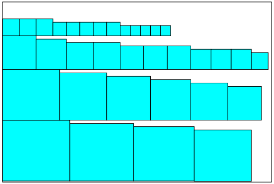
CIRCLE BIN PACKING

Rounding to squares

Adaptation of the strip packing version NFDH^s.

NFDH^b(L) # For the bin packing version

1. Sort $L = (s_1, \ldots, s_n)$ st. $s_1 \geq \cdots \geq s_n$
2. For $i \leftarrow 1$ to n:
3. Pack s_i in the last shelf (of the last bin), if possible
4. otherwise, pack s_i in a new shelf at the top of the previous shelf, if possible
5. otherwise, pack s_i in a new shelf of a new bin.



Rounding to squares

Meir, Moser'68. If all squares of L have side lengths at most $\frac{1}{m}$

$$\mathrm{NFDH}^{\mathrm{b}}(L) \leq \left(\frac{m+1}{m}\right)^2 \mathrm{Area}(L) + \frac{m+2}{m}$$

Proof: Exercise (analogous to the proof of $\mathrm{NFDH}^{\mathrm{s}}$)

Corollary. For any list $L \in \mathcal{C}$ with diameters at most $\frac{1}{m}$

$$\mathbb{C}\mathrm{NFDH}^{\mathrm{b}}(L) \leq \frac{4}{\pi}\left(\frac{m+1}{m}\right)^2 \mathrm{Area}(L) + \frac{m+2}{m}$$

Corollary. For any list $L \in \mathcal{C}$

$$\mathbb{C}\mathrm{NFDH}^{\mathrm{b}}(L) \leq 5.1\mathrm{OPT}(L) + 3$$

Corollary. As radius of circles decrease, the density of the packing is improved and $\mathbb{C}\mathrm{NFDH}^{\mathrm{b}}$ goes to $4/\pi \approx 1.27324$.

Bounded Space Online Bin Packing

Bounded Space Online Bin Packing

- Algorithms must be online
- At any moment, bins are classified as *open* or *closed*
- Only open bins can receive new items
- A bin starts open and once it became closed, it cannot be open again.
- The number of open bins is bounded by a constant

Bounded Space Online Bin Packing

Related results with asymptotic approximation:

- Lee and Lee: Algorithm with factor
  1.69103 for 1-dimensional items and
  showed that no algorithm can have better performance

- Epstein, van Stee'07: Algorithms with factors
  2.3722 for packing squares and
  3.0672 for packing cubes.

Bounded Space Online Bin Packing

We will see

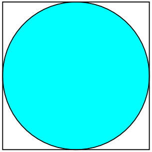
- Algorithm with asymptotic approximation factor 2.44
- Lower bound of 2.29

Techniques

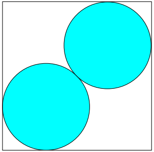
- Weighting system to obtain approximation factors
- Specific algorithms to deal with big and small circles
- Grouping circles to consider as equal circles
- Geometric Partition to combine items of the same type

Packing equal circles

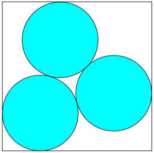
Find the largest ρ^* st. k circles of radius ρ^* can be packed in a unit square



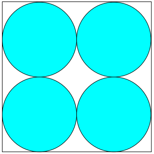
$\rho_1^* = 0.5$



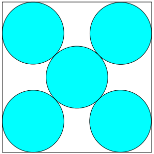
$\rho_2^* = 0.2928$



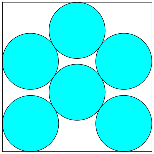
$\rho_3^* = 0.2543$



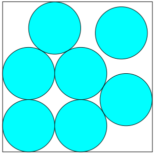
$\rho_4^* = 0.1963$



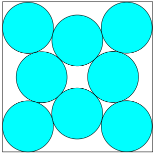
$\rho_5^* = 0.2071$



$\rho_6^* = 0.1876$



$\rho_7^* = 0.1744$



$\rho_8^* = 0.1705$

Previous results: It is known the exact values of ρ_n^*, for $n \leq 30$ and good lower bounds for many.

Packing big circles

Round up big circles to the nearest value of ρ
(to bound the number of different circles)

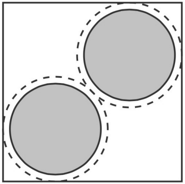
- A circle is big if its radius is larger than $1/M$
- Let ρ_i be the value of ρ_i^*, when it is known, otherwise, the best known lower bound.
- Let K be such that $\rho_{K+1} \leq 1/M < \rho_K$

A circle r is of type i if:

- $\rho_{i+1} < r \leq \rho_i$ (for $1 \leq i < K$)
- $1/M < r \leq \rho_K$ (for $i = K$)

Packing big circles

- For $1 \leq i \leq K$, a *c-bin* of type i is a circular bin of radius ρ_i
- Circles of type i are packed in a c-bin of type i
- Packing in a c-bins of type 2



Algorithm - Part 1

To pack a big circle c of type i :

if there is no empty c-bin of type i

close the current bin of type i (if any)

open a new bin of type i containing i c-bins of type i

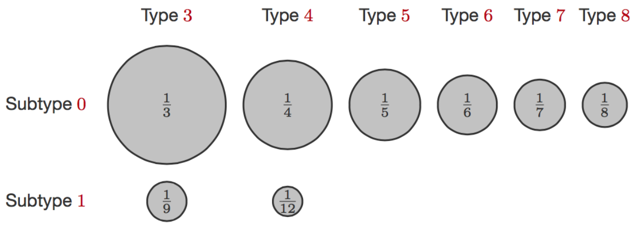
Pack c into a empty c-bin of type i

Small circles

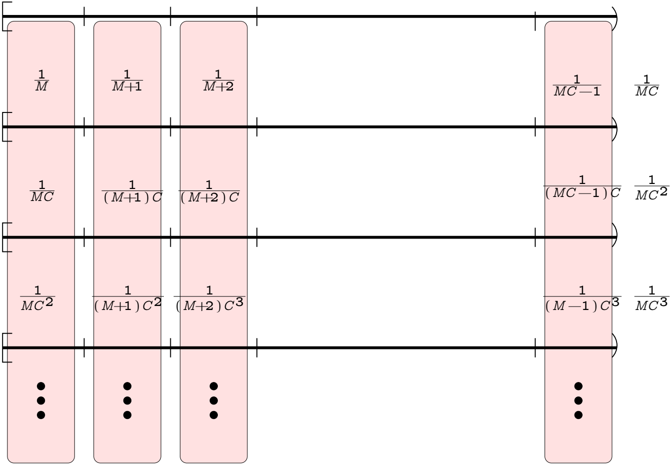
Let $C > 0$ be an integer multiple of 3

A small circle of radius r is of type i, subtype k if

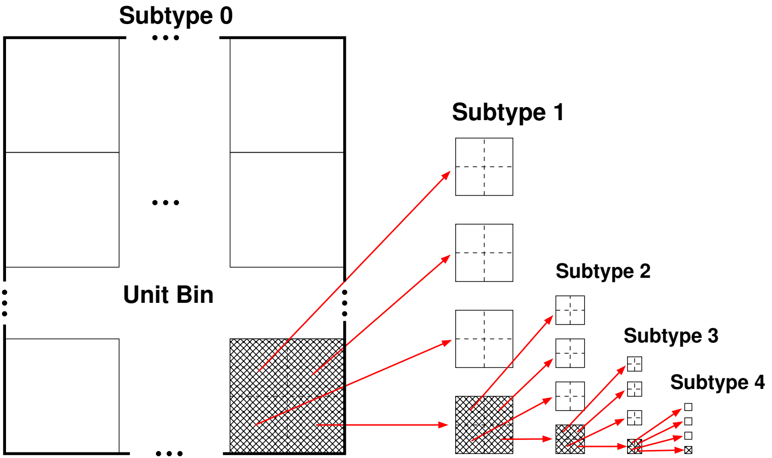
- $1/(i+1) < C^k r \leq 1/i$
- where k is the largest integer such that $C^k r \leq 1/M$
- and the circle is said to be of type (i, k)



Small circles



Idea: Geometric Subdivision



Subdivisions within a same type: Example with squares

Sub-bins (h-bins and t-bins)

Idea: Round/Pack small circles into hexagonal bins

h-bin of type (i, k):

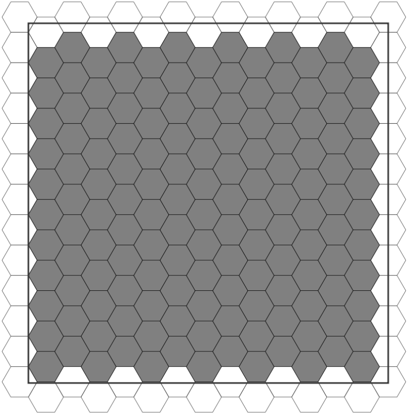
- hexagonal bin of side length $2/(\sqrt{3}\, C^k i)$
- Receives a small circle of type (i, k)

t-bin of type (i, k):

- trapezoidal bin obtained by the subdivision of a h-bin of type (i, k) in the center

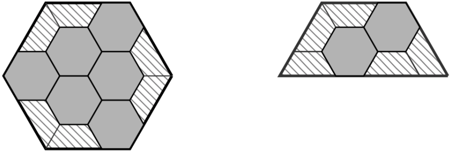
Subdividing a square into h-bins

Subdividing a square into h-Bins

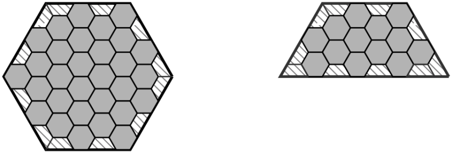


Partitioning sub-bins

For all $M \leq i < CM$ and $k \geq 0$, if C is multiple of 3 then, it is possible to partition an h-bin or an t-bin) of type (i, k) into h-bins and t-bins of type $(i, k + 1)$.



$C = 3$



$C = 6$

Algorithm - Part 2

When a small circle c of type (i, k) arrives:

if there is no empty h-bin of type (i, k) or an empty sub-bin of type (i, k') with $k' < k$

close the current bin of type i (if any)

open a bin of type i subdividing into h-bins of type $(i, 0)$

while there is no h-bin of type (i, k)

let k' the largest number such that $k' < k$ and there exists an en

of type (i, k')

if there exists an empty t-bin of type (i, k')

B tal t-bin

else

let B an h-bin of type (i, k')

particionate B in sub-bins of type $(i, k' + 1)$

packs c into a h-bin of type (i, k)

Analysis by weighting function

Given algorithm $\mathcal{A}$ and weight function $w\colon L \to \mathbb{R}_{\geq 0}$ st.

- $\mathcal{A}$ produce bins with average weight at least 1
  I.e., $w(L)/\mathcal{A}(L) \geq 1$ and therefore

$$\mathcal{A}(L) \leq w(L), \quad \text{for any instance } L$$

- Find $\alpha \geq$ *maximum bin weight*. I.e.,

$$\alpha \geq \sup\{w(S) : S \subseteq L \text{ and } \exists \text{ packing of } S \text{ in one bin}\}$$

  Optimum uses at least $\frac{w(L)}{\alpha}$ bins: $w(L) \leq \alpha\,\mathrm{OPT}$

Algorithm $\mathcal{A}$ has approximation factor α:

$$\mathcal{A}(L) \leq w(L) \leq \alpha\,\mathrm{OPT}(L)$$

Removing *few* bins, before average leads to asymptotic factor

Circle weights

How to obtain average weight ≥ 1 ?

- Algorithm produce bins with weight ≥ 1

$$w(c) = \begin{cases} 1/i & \text{if } c \text{ is big and type } i \\ \text{Area}(c)/\gamma & \text{if } c \text{ is a small circle,} \end{cases}$$

where γ is area density or lower bound, for bins with small

- If B is closed type i bin (big items) then
B has i circles of weight $1/i$ and $w(B) = 1$.

- If B is closed bin for small circles, then
Area(B) is also its density and Area$(B) \geq \gamma$. So

$$w(B) = \sum_{c \in B} w(c) = \sum_{c \in B} \frac{\text{Area}(c)}{\gamma} \geq 1$$

Circle weights

γ: lower bound for area covered in closed bins by small items

The non-covered regions are due to:

- $\mathcal{L}_B$: upper bound for the non-covered region due to the shape and partial intersection of hexagons with the border of the square bin, and is at most $5.89/M$
- $\mathcal{L}_F$: upper bound for to the set of non-covered hexagons when a bin is closed: $2\sqrt{3}C^2/(M^2(C^2-1))$
- $\mathcal{L}_H$: loss factor due to the *rounding* of circles into hexagons: $\frac{\pi}{\sqrt{12}}\frac{M^2}{(M+1)^2}$

That is

$$\gamma = (1 - \mathcal{L}_B - \mathcal{L}_F)\,\mathcal{L}_H$$

Computing β

Value of β is obtained via Mixed Integer Programming:

- x_i: number of circles of type i
- y: area of small circles

$$
\begin{aligned}
\text{maximize } & \frac{y}{\alpha} + \sum_{i=1}^{K} \frac{x_i}{i} \\
\text{subject to } & \quad y + \sum_{i=1}^{K} \pi \rho_{i+1}^2 x_i \leq 1 \\
& \qquad x_i \in \mathbb{Z}_+ \qquad \forall\, 1 \leq i \leq K \\
& \qquad y \geq 0
\end{aligned}
$$

Computing β

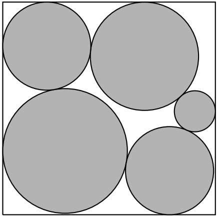
On the other hand, we do not know if a solution can indeed be packed in one bin

- Using Constraint Programing to verify if a solution can be packed in only one bin, with time limit
- If it is not possible, we add a constraint in the model to avoid such solution

For $M = 59$ and $K = 992$, the value of β is 2.4394

- But we do not know if the solution can in fact be packed in only one bin

Lower bound for any competitive factor



- 1 circle of type 1
- 1 circle of type 2
- 2 circle of type 4
- 1 circle of type 25

  (the area covered by the above circles: 0.77139)
- remaining space is completed with sand
  (very small circles, non-necessarily equal)

Lower Bound

Consider N copies of the lower bound pattern with circles sorted by radius

An online bounded space algorithm B uses:

- at least $N - B$ bins for the circles of type 1
- at least $N/2 - 2B$ bins for the circles of type 2
- at least $2N/4 - 2B$ bins for the circles of type4
- at least $N/25 - 2B$ bins for the circles of type25

At least $2.04N - 7B$ for the circles

Lower bound

Let $S = (1 - 0.77139 - \varepsilon)$ the remaining area used by sand

- The best way to obtain a dense packing of equal circles is the hexagonal packing
  - with densities $\pi/\sqrt{12}$
- Even for very small circles, the algorithm cannot do better than the hexagonal packing
- The algorithm uses at least $S\sqrt{12}/\pi - 2kB$ bins
  - k is the number of different radius

The algorithm uses at least $2.2920N - \delta N - O(1)$ bins, that tends to $2.2920 - \delta$ when N goes to infinity

An offline algorithm uses at most N bins

Exercises

- Obtain bounded online approximation algorithms to pack items into bins, each one could be one of the following: equilateral triangles, squares, circles, hexagons, etc.
- For the previous exercise, consider the three-dimensional or d-dimensional case.

Approximation Schemes

We will see ideas to obtain

- APTAS for circle bin packing with resource augmentation
- APTAS for the circle strip packing

Techniques

- Solving packings for non small items (appliable to many shapes)
- Linear Rounding technique
- Improved approach to combine small and large items

Related works

For cubes and rectangles

- An asymptotic 1.405-approximation(Bansal and Khan)
- APTAS for d-dimensional cubes (Bansal et al.)
- APTAS for 3-dimensional strip packing (Bansal et al.)

Common approach to obtain PTAS in packing problems

Main idea

- Separate small and large items
- use a PTAS for large items
- use simple algorithm pack small items in free space

Area occupation for small items

- square packing with side lengths are at most ε
  NFDH can cover $1 + O(\varepsilon)$ of the bin area
- equal circle packing with radii at most ε
  at most a factor of the bin area

- More complex way to divide and define items as: small, medium and large

Subroutines for special cases:

- Optimum algorithm when $|L|$ is constant
- Optimum algorithm when $\min(L) \geq \varepsilon$ and $\mathrm{diff}(L)$ is constant,
  where $\mathrm{diff}(L)$ is the number of different items in L.
- PTAS to obtain a solution for relatively large items

Optimum packing when $|L|$ is constant

One Bin Problem: Given list of circles $Q = (r_1, \ldots, r_n)$, obtain a packing of L into a unit square.

Formulation:
Obtain positions (x_i, y_i) for circle r_i, for $i = 1, \ldots, n$, respecting packing constraints:

$$(P) \qquad \begin{array}{cl} (x_i - x_j)^2 + (y_i - y_j)^2 \geq (r_i + r_j)^2 & \text{for } 1 \leq i < j \leq n, \\ r_i \leq x_i \leq 1 - r_i & \text{for } 1 \leq i \leq n, \text{ and} \\ r_i \leq y_i \leq 1 - r_i & \text{for } 1 \leq i \leq n. \end{array}$$

$$(\exists x_1)(\exists y_1)\ldots(\exists x_n)(\exists y_n) \bigwedge_{i=0}^{s} f_i(x_1, y_1, ..., x_n, y_n) \geq 0.$$

Optimum packing when $|L|$ is constant

One Bin Problem

Resolution of system of polynomials can be solved by standard algebraic quantifier elimination (Grigore'v and Vorobjov Jr.)

Concerning algebraic quantifier elimination algorithms:

- System resolution may obtain irrational points

Computational model

- Use a stronger model of computation
- Obtain rational solutions, within an error
  circles may intersect or
  stay outside the bin within an small error

For rational solutions within an error

- obtained solution may be invalid packing
- use *resource augmentation*: Modify solution within an error to a packing, but into bin of dimensions $(1, 1+\gamma)$.
- Idea: shift items by small steps possibly extending the solution to the augmented border

Optimum packing when $|L|$ is constant

Multiples Bins Problem

Given list of circles $Q = (r_1, \ldots, r_n)$, obtain a packing of L into the minimum number of unit squares.

Obtain an optimum packing

- enumerate all packable partitions of L into one bin
- choose one with small number of parts.

Packing without small circles and constant different radius

Let

- δ be a lower bound for the radius of each circle in L, a constant
- k the number of different radius in L

Idea

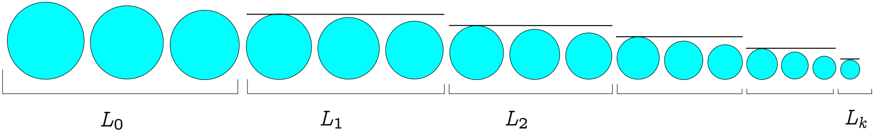
1. Number of circles in a bin is bounded by $1/\text{Area}(\delta)$
2. There are k type possibilities for each circle
3. Obtain all possibles packings into one bin (patterns): $P_1, \ldots, P_t$
4. Enumerate all posible packings for L.
   Each pattern is used at most n times: $O(n^t)$

Packing without small circles

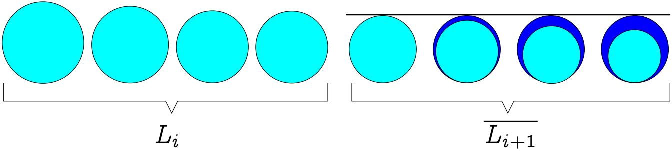
Lemma. $\exists$ PTAS when $\min(L) \geq \delta$, for constant $\delta > 0$.

Linear Rounding:

1. Sort L in non-increasing order of radius and divide in $k+1$ sublists, such that $L = L_0 \| L_1 \| \dots \| L_k$ and $k = \lfloor n\varepsilon \text{Area}(\delta) \rfloor$

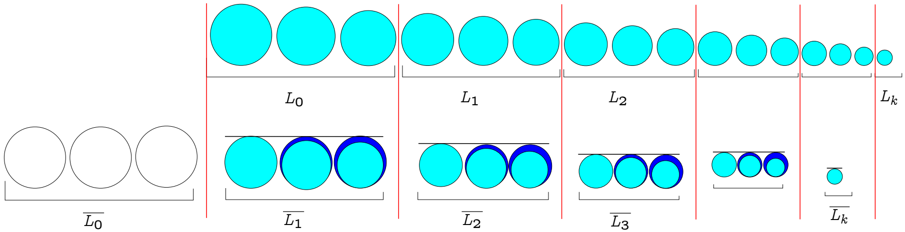


2. Let $\overline{L_i}$ the sublist L_i rounding up each circle to the largest in the corresponding sublist, $i = 1, \dots, k$.



Packing without small circles

Packing of $L_1, \ldots, L_k$:



$$\overline{L_{i+1}} \preceq L_i$$

4. We have: $\overline{L_1} \| \overline{L_2} \| \ldots \| \overline{L_k} \preceq L_0 \| L_1 \| \ldots \| L_{k-1} \preceq L$

So, $\mathrm{OPT}(\overline{L_1} \| \overline{L_2} \| \ldots \| \overline{L_k}) \leq \mathrm{OPT}(L_0 \| L_1 \| \ldots \| L_{k-1}) \leq \mathrm{OPT}(L)$

where $k = \frac{n}{\lfloor n\varepsilon \mathrm{Area}(\delta) \rfloor} + 1 = O(1/(\varepsilon \mathrm{Area}(\delta)))$

Packing without small circles

Packing of L_0:

1. Build packing $\mathcal{P}_0$ placing each circle in a new bin.

$$
\begin{aligned}
|\mathcal{P}_0| &= \lfloor n\varepsilon' \text{Area}(\delta) \rfloor \leq \varepsilon'(n\text{Area}(\delta)) \\
&\leq \varepsilon' \text{Area}(L) \\
&\leq \varepsilon' \text{OPT}(L)
\end{aligned}
$$

Packing without small circles

Algorithm $\mathcal{A}_\varepsilon$

1. $\varepsilon' \leftarrow \varepsilon/2$
2. If L uses at most k/ε' bins, pack optimally.
3. Otherwise:
4. $\mathcal{P}_0 \leftarrow$ pack L_0 using at most ε' OPT(L) bins.
5. $\mathcal{P}_{1k} \leftarrow$ pack $\overline{L_1} \| \overline{L_2} \| \ldots \| \overline{L_k}$ with up to $(1+\varepsilon')\text{OPT}(L)$ bins

$\mathcal{A}_\varepsilon$ is PTAS:

$$\begin{aligned}
\mathcal{A}_\varepsilon(L) &= |\mathcal{P}_0| + |\mathcal{P}_{1k}| \\
&= \varepsilon'\text{OPT}(L) + (1+\varepsilon')\text{OPT}(L) \\
&= (1+\varepsilon)\text{OPT}(L)
\end{aligned}$$

Aplication: APTAS for bin packing

Bin Packing: Given list $L = (s_1, \ldots, s_n)$, where $0 < s_i \leq 1$, obtain a packing of L into the minimum number of bins of size 1.

Changes in

- resolution of the problem with constant sizes
- replace Area$(\cdot)$ by the function $s(\cdot)$.

PTAS for bin packing:

1. Let $B = \{i \in L : s_i \geq \varepsilon\}$ # Big items
2. Let S the remaining set # Small items
3. $\mathcal{P}_B \leftarrow$ PTAS for B
4. Pack items in S:
5. Greedily pack items of S in the remaining space of $\mathcal{P}_B$
6. If needed, use new bins and continue

Aplication: APTAS for bin packing

PTAS for bin packing:

Two cases considering "If needed" new bins to pack S:

- No new bins: continue as PTAS

$$\mathcal{P}_B \leq (1+\varepsilon)\mathrm{OPT}(B) \leq (1+\varepsilon)\mathrm{OPT}(L)$$

- Used new bins: Bins are almost full before obtain new bin
  Proof follows by area arguments

Final packing $\mathcal{P}$ such that $|\mathcal{P}| \leq (1+\varepsilon)\mathrm{OPT}(L)+1$

PTAS for circle bin packing

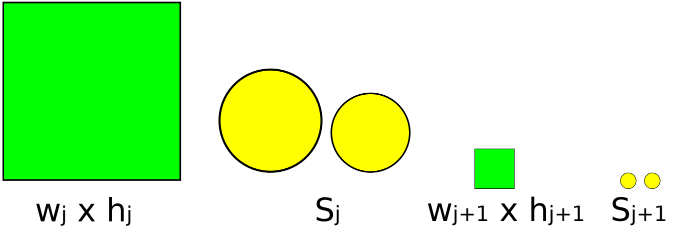
Main steps:

- Get sets from input list: $S_1, S_2, \ldots$
- Sizes in S_{i+1} are much smaller than in S_i
- Use of medium items to obtain gap between sets S_i and S_{i+1}
- Use a separate algorithm to pack medium items
- Apply the method recursively to pack items of S_{i+1} in the remaining space (do not use a simple algorithm)

(3) Packing iteratively

Setting

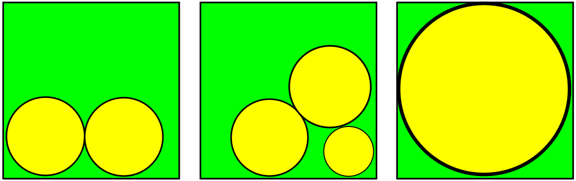
- Items of S_j are packed in bins $w_j \times (1+\gamma)h_j$:
- There are no items of intermediate sizes:
- Bins $w_{j+1} \times (1+\gamma)h_{j+1}$ are <u>much smaller</u> than circles in S_j
- Circles in S_{j+1} are <u>much smaller</u> than bins $w_{j+1} \times (1+\gamma)h_{j+1}$



Improved approach

Packing recursively

- Split the free space left by large items in small sub-bins
- Pack the remaining items recursively



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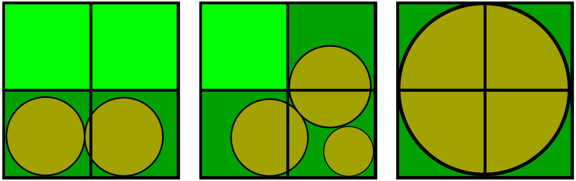

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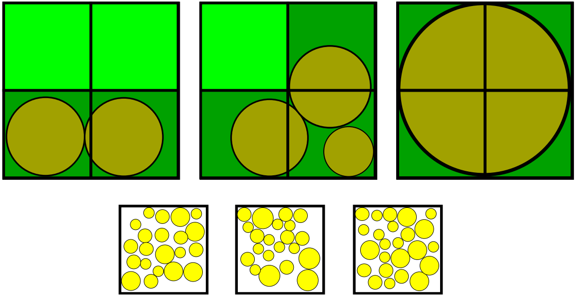

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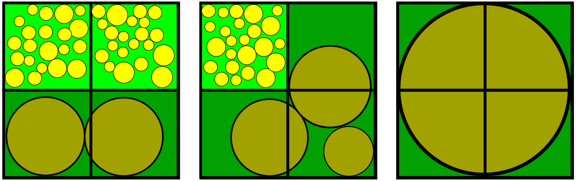


Large, small, and medium items

- Make groups where radii between consecutive decrease $\times\ \varepsilon^2$
- Let $H_j = G_j \cup G_{j+r} \cup G_{j+2r} \cup \ldots$, for $j \le r-1$ and $r = \lceil 1/\varepsilon \rceil + 1$
- Choose H_t s.t. $\text{Area}(H_t) \le \varepsilon \text{Area}(L)$ and remove H_t
- Let $S_0, S_1, \ldots$ union of consecutive groups and pack iteratively

	H_{t+1}		H_0	H_{t-1}	H_t
S_0:			$G_0, \ldots$	G_{t-1}	G_t
S_1:	G_{t+1},	G_{t+2},	$\ldots$	G_{t+r-1}	G_{t+r}
S_2:	G_{t+r+1},	G_{t+r+2},	$\ldots$	G_{t+2r-1}	G_{t+2r}
S_3:	G_{t+2r+1},	G_{t+2r+2},	$\ldots$	G_{t+3r-1}	G_{t+3r}
			$\vdots$		$\vdots$

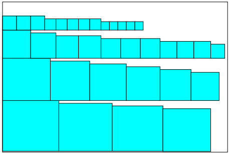
H_t are the medium items and are packed separately!

Iter. #0: large circles: S_0 small circles: $S_1, S_2, S_3, \ldots$

(1) Packing medium items

Use algorithm NFDH^b

- Wrap medium circles in their bounding box
- Use NFDH to pack boxes in unit squares



Analysis

- $\text{Area}(H_t) \leq \varepsilon \text{Area}(L)$
- NFDH^b guarantees occupation of $\frac{\pi}{4}\frac{1}{4} \geq \frac{1}{12}$

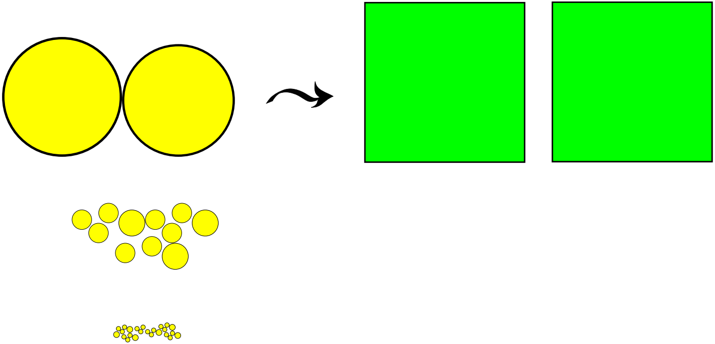
$$\text{NFDH}^b(H_t) \leq 12\varepsilon \text{Area}(L) + 1$$

(2) Packing sets S_j's

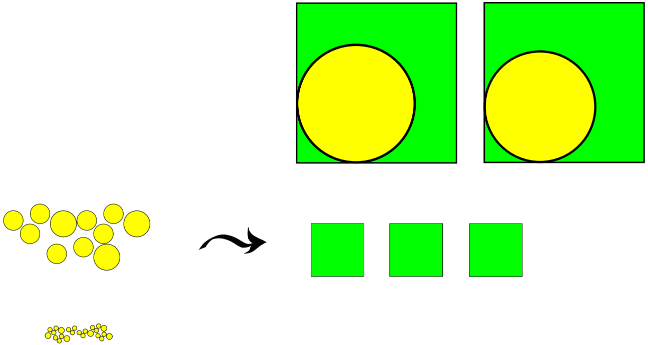
Use PTAS (when items are at least a constant) to pack each S_j

- Item sizes decrease exponentially but also their corresponding bins
- Circles radii in any S_j are at least some δ, relative to its bin
- For each level j, we can maintain constant number of empty bins

Running example



Running example



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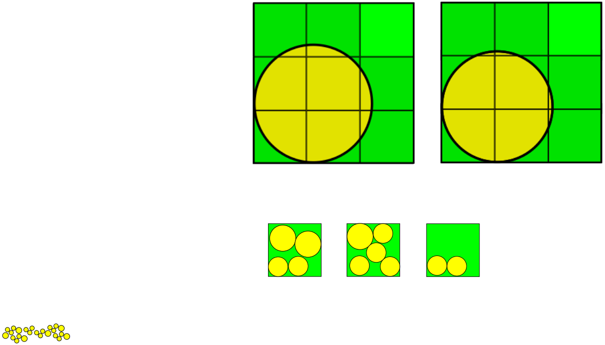
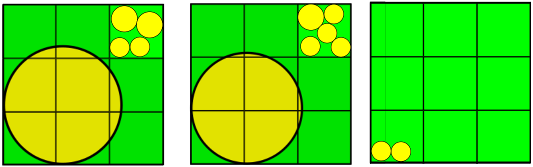


Running example



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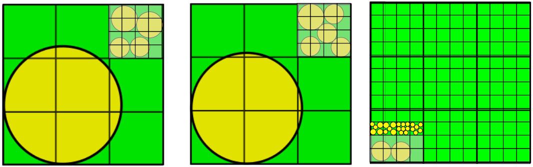


Comparing to the optimum

The generated solution is “almost” optimal, except that it is constrained in two ways.

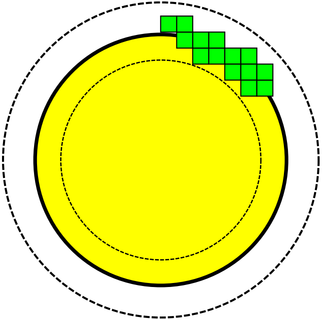
Constraints

A. Sub-bins that partially intersect a larger circle is not used
B. Circles of S_j do not intersect the grid of sides $w_j \times (1+\gamma)h_j$

To calculate the cost of the generated solution, we modify an optimal solution, and bound the wasted space.

(A) Bins that intersect larger circles

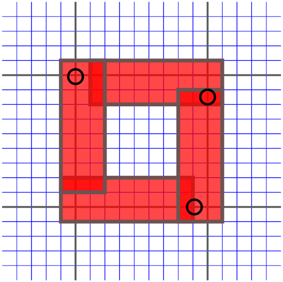
- Bins are much smaller than the circles with partial intersection



wasted space $\leq O(\varepsilon)\text{Area}(L) \leq O(\varepsilon)\text{OPT}$

(B) Circles that intersect grid lines

- circles with intersection not respect bins are much smaller than the bins
- we repack the circles in free space respecting the grid



We do not use a small fraction of bins of an optimal solution.

additional space $\leq O(\varepsilon)\mathrm{OPT}$

Summing everything

An APTAS for circle bin packing (with resource augmentation)

$$\begin{aligned}\text{SOL} &\leq [\text{cost of medium items}] + [\text{cost of modified optimal solutio}\\ &\leq \lceil O(\varepsilon)\text{OPT}\rceil \qquad\qquad\qquad + \lceil \text{OPT} + O(\varepsilon)\text{OPT} + O(\varepsilon)\text{OPT}\rceil\\ &\leq (1+O(\varepsilon))\text{OPT} + 2\end{aligned}$$

Corollary

- An APTAS for circle strip packing.

Summary

Result

- First polynomial-time approximation (scheme) for circle bin packing and circle strip packing

Extensions

- May be generalized to other kinds of items: spheres, ellipsoids, etc.
- Bins of different shapes may also be considered

Open problem

- Is it possible to always obtain a rational solution from the packing decision problem? *(and thus avoid using resource augmentation)*